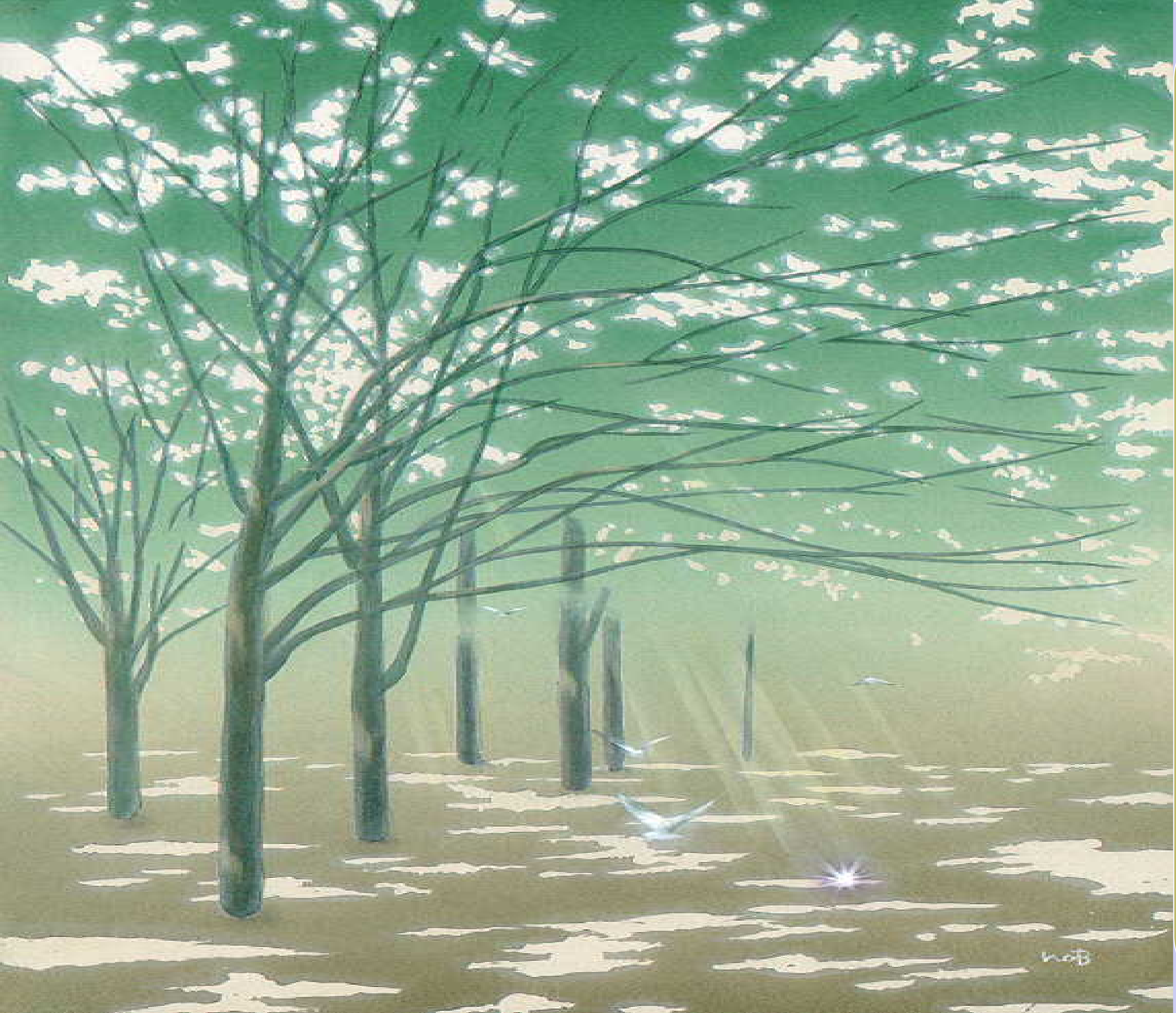

The 2006 International Conference on Design of Experiment and Its Applications
July,9-13, 2006 · Nankai University

On the construction of normal mixed difference matrices

Shanqi Pang

Henan Normal University, China

pangshanqi@263.net

On the construction of normal mixed difference matrices (NMDM)

Outline

1 Introduction

Difference matrix was first defined by Bose and Bush (1952), and it is a simple but powerful tool for the construction of orthogonal arrays of strength 2 (Hedayat et al. 1999, Beth et al. 1985). Mixed difference matrices have also been used for construction of orthogonal arrays (see Wang and Wu, 1991; Wang, 1996; Pang et al. 2004b; Seun and Kuhfeld, 2005, etc).

Proposition 1 (Beth et al. 1985) The Kronecker sum of an orthogonal array $L_{\mu p}(p^s)$ and a difference matrix $D(\lambda p, r; p)$

$$L_{\mu p}(p^s) \oplus D(\lambda p, r; p)$$

is an orthogonal array.

By taking $\mu = \lambda = 1$, the Kronecker sum method reduces to the well known construction of Bose and Bush (1952).

Proposition 2 (Wang and Wu, 1991) If A is an orthogonal array $L_N(p_1^{s_1}\cdots p_n^{s_n})$ with the partition $A=[L_N(p_1^{s_1}),\cdots,L_N(p_n^{s_n})]$,

$$B=[D(M,k_1;p_1),\cdots,D(M,k_n;p_n)]\text{(a mixed difference matrix)},$$

where $D(M,k_i;p_i)$ is a difference matrix. Then

$$[L_N(p_1^{s_1})\oplus D(M,k_1;p_1),\cdots,L_N(p_n^{s_n})\oplus D(M,k_n;p_n)]$$

is an orthogonal array.
Let $L_M(q_1^{r_1}\cdots q_m^{r_m})$ be an orthogonal array. Then

$$[0_N\oplus L_M(q_1^{r_1}\cdots q_m^{r_m}),L_N(p_1^{s_1})\oplus D(M,k_1;p_1),\cdots,L_N(p_n^{s_n})\oplus D(M,k_n;p_n)]$$

is also an orthogonal array.

Proposition 3 (Wang, 1996) Let $D_0 = D(n, k_0; p_1p_2)$, $D_i = D(n, k_i; p_i)$ $(i = 1, 2)$ be difference matrices. Let $D_i^{'}(i = 1, 2)$ be the matrix obtained by taking modulus p_i operation entry-wise on D_0. Suppose $[D_i^{'}, D_i](i = 1, 2)$ is an $n \times (k_0 + k_i)$ difference matrix having p_i levels. Denote $D = [D_0, D_1, D_2]$. Let $C_0 = [0, 1, \ldots, p_1p_2 - 1]^T$ and $C_i(i = 1, 2)$ be obtained by taking modulus p_i operation entry-wise on C_0. Thus, $C = [C_0, C_1, C_2]$ is a partition matrix in which the three columns have p_1p_2, p_1, p_2 levels, respectively. Denote an existing orthogonal array of size n by E, then

$$M = [C_0 \oplus D_0, C_1 \oplus D_1, C_2 \oplus D_2, 0_{p_1p_2} \oplus E]$$

constitutes an orthogonal array, where $0_{p_1p_2}$ is a $p_1p_2 \times 1$ vector of zeros.
$D = [D_0, D_1, D_2]$ is a special mixed difference matrix

Example 1.

$$C = (C_0, C_1, C_2) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 0 \\ 3 & 0 & 1 \\ 4 & 1 & 0 \\ 5 & 2 & 1 \end{pmatrix} = (C_0, L_6(3^1 2^1)).$$

A special mixed difference matrix $D = [D_0(12, 5; 6), D_1(12, 3; 3), D_2(12, 6; 2)]$

$$\begin{pmatrix}
00000 & 000 & 000000 \\
13240 & 221 & 001111 \\
20152 & 011 & 010111 \\
31542 & 000 & 110001 \\
43521 & 102 & 011010 \\
55311 & 120 & 000100 \\
02323 & 212 & 101101 \\
12405 & 102 & 110110 \\
25234 & 011 & 111100 \\
34114 & 212 & 101010 \\
41035 & 221 & 100011 \\
54433 & 120 & 011001
\end{pmatrix}$$

Normal mixed difference matrix is a generalization of the special mixed difference matrix in the above. It is introduced and used for construction of mixed-level orthogonal arrays by Pang et al. (2004b). However, it has not received much attention to construct mixed difference matrices, especially normal mixed difference matrix. Hence, this paper presents some methods for constructing normal mixed difference matrix.

Definition 1. (Pang et al. 2004b) Let $L_q = L_q(s_1 \cdots s_m) = (C_1, \ldots, C_m)$ be an orthogonal array where C_l is a vector with entries from an additive group G_l of order s_l for any l $(l = 1, \ldots, m)$. The array L_q is said to be **normal** if the set $G = \{A_1, \ldots, A_q : A_i$ is the i-th row of $L_q\}$ constitutes an additive subgroup of order q, where $G \subset G_1 \times \cdots \times G_m := \{(x_1, \ldots, x_m); x_l \in G_l,\ l = 1, 2, \ldots, m\}$ with the usual addition, i.e., for any $x, y \in G, x = (x_1, x_2, \ldots, x_m), y = (y_1, y_2, \ldots, y_m)$, we have

$$x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_m + y_m).$$

Proposition 4. (Pang et al. 2004b) There is a map $\phi(A_i) = i - 1$ which is a group isomorphism between $G = \{A_1, A_2 \ldots, A_q\}$ and $G_0 = \{0, 1, \ldots, q-1\}$. And the map

$$\phi_l(i-1) = c_{il} \tag{1}$$

is a group homomorphism from $G_0 = \{0, 1, \ldots, q-1\}$ to G_l for $l = 1, 2, \ldots, m$. If $C_0 = (0, 1, \ldots, q-1)^T$, then $\phi_l(C_0) = C_l$ holds.

Definition 2. Suppose that the array L_q is normal, and that the maps $\phi_1, \ldots, \phi_m$ are defined in (1). Let $D_0 = D(n, k_0; q)$ be a difference matrix based on G_0, and let D_l be a difference matrix based on G_l, $l = 1, 2, \ldots, m$. If the matrices

$$[\phi_1(D_0), D_1], [\phi_2(D_0), D_2], \ldots, [\phi_m(D_0), D_m]$$

are difference matrices based on $G_1, G_2, \ldots, G_m$, respectively, then the matrix

$$[D_0, D_1, D_2, \ldots, D_m]$$

is called a normal mixed difference matrix (NMDM).

Example 2. Orthogonal array $L_6(2\cdot 3)$ and $L_4(2^3)$ are normal.

$$C=(C_0,C_1,C_2,C_3)=\begin{pmatrix}0&0&0&0\\1&0&1&1\\2&1&0&1\\3&1&1&0\end{pmatrix}$$

$$=[C_0,\phi_1(C_0),\phi_2(C_0),\phi_3(C_0)]=[C_0,L_4(2^3)]$$

$+$	000	011	101	110
000	000	011	101	110
011	011	000	110	101
101	101	110	000	011
110	110	101	011	000

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

$(G,+)$ and $(G_0,\oplus)$ are isomorphic, Where $G_0=\{0,1,2,3\}$.

Proposition 5. (Pang et al.,2004b) Let L_n be an existing orthogonal arrays, and let $C_0 = (0, 1, \ldots, q-1)^T$. Suppose that $[D_0, D_1, D_2, \ldots, D_m]$ is a normal mixed difference matrix based on the normal orthogonal array L_q. Then the matrix

$$[C_0 \oplus D_0, \phi_1(C_0) \oplus D_1, \ldots, \phi_m(C_0) \oplus D_m, 0_q \oplus L_n]$$

constitutes an orthogonal array,

2 Construction method for normal mixed difference matrices

Lemma 1. Suppose that $D(\lambda p, m; p)$ is a $\lambda p \times m$ matrix with entries from a Galois field of order p, $GF(p)$, and that γ is a column of orthogonal array $L_n(p^s)$. If $D(\lambda p, m; p) \oplus \gamma$ is also an orthogonal array, then $D(\lambda p, m; p)$ is a difference matrix.

Theorem 1. Suppose that $L_p(s_1 \cdots s_n) = (C_1, \ldots, C_n)$ is a normal orthogonal array, and that $L = [D(m,k;p) \oplus (p), D(m,k_1;s_1) \oplus C_1, \ldots, D(m,k_n;s_n) \oplus C_n]$ is also an orthogonal array. Then

$$D = [D(m,k;p), D(m,k_1;s_1), \ldots, D(m,k_n;s_n)]$$

is a normal mixed difference matrix.

Proof. Since $L_p(s_1 \cdots s_n) = (C_1, \ldots, C_n)$ is a normal orthogonal array, we have a group homomorphism ϕ_l from $G_0 = \{0, 1, \ldots, p-1\}$ to G_l. Set $C_0 = (p) = (0, 1, \ldots, p-1)^T$, then $\phi_l(C_0) = C_l$. From the expansive replacement method in Hedayat et al (1999), if the levels $0, 1, \ldots, p-1$ in $[D(m, k; p) \oplus (p), D(m, k_l; s_l) \oplus C_l]$ are replaced with $A_1, \ldots, A_p$, respectively, where A_i is the ith row of L_p, we can obtain a mixed orthogonal array with the levels $s_1, \cdots, s_n$. Pick out all the s_l-level columns, we can get an s_l-level orthogonal array, which can be written as

$$[\phi_l(D(m, k; p)), D(m, k_l; s_l)] \oplus C_l.$$

It follows from Lemma 1 that $[\phi_l(D(m, k; p)), D(m, k_l; s_l)]$ is a difference matrix, $l = 1, \ldots, n$.

We have $D = [D(m, k; p), D(m, k_1; s_1), \ldots, D(m, k_n; s_n)]$ is a normal mixed difference matrix based on $L_p(s_1 \cdots s_n)$.

Theorem 2. Suppose that $L_p(s_1 \cdots s_n) = (C_1, \ldots, C_n)$ is a normal orthogonal array, and that $D(m, k; p)$ is a difference matrix. If we partition $D(m, k; p)$ into $[D(m, k; p) = [D(m, k - r; p), D(m, r; p)]$, then $[D(m, k - r; p), \phi_1(D(m, r; p)), \ldots, \phi_n(D(m, r; p))]$ is a normal mixed difference matrix, where ϕ_i is as in (1).

Lemma 2. (Zhang et al. 2002) Suppose that both

$$D(r,r;r) = (d_{ij})_{r\times r} = (d_1,\ldots,d_r)$$

and

$$D(r+1,r+1;p) = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & (a_{ij})_{r\times r} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & a_1 & \cdots & a_r \end{pmatrix}$$

are difference matrices with entries from two additive groups $G_r = \{0,1,\ldots,r-1\}$ and $G_p = \{0,1,\ldots,p-1\}$, respectively. For any $d_{ij} \in G_r$, define a permutation matrix $\sigma(d_{ij})$ as follows

$$\sigma(d_{ij})\cdot(r) = d_{ij} + (r). \tag{2}$$

Set $F = (\sigma(d_{ij})A)_{1\le i\le r, 1\le j\le r}$. Then the following array

$$D(r(r+1),r(r+1);p) = \begin{pmatrix} 0 & A\oplus 0_r^T \\ A\oplus 0_r & F \end{pmatrix}.$$

is a difference matrix.

Lemma 3. If $D = [D(m, k; p), D(m, k_1; s_1), \ldots, D(m, k_n; s_n)]$ is a normal mixed difference matrix based on a normal orthogonal array $L_p(s_1 \cdots s_n) = (C_1, \ldots, C_n)$, then D is still an normal mixed difference matrix after performing the following operations:
(1) adding an i ($i \in \{0, 1, \ldots, p-1\}$) to any column of $D(m, k; p)$ or adding an i_j ($i_j \in G_j$) to any column of $D(m, k_j; s_j)$.
(2) adding an i ($i \in \{0, 1, \ldots, p-1\}$) to a row of $D(m, k; p)$ and adding $\phi_j(i)$ to the same row of $D(m, k_j; s_j)$ for $j = 1, \ldots, n$.

Theorem 3. Under the conditions of Lemma 2, suppose that $[D(r+1,k+1;p), D_1(r+1,k_1;s_1),\ldots,D_m(r+1,k_m;s_m)]$ is a normal mixed difference matrix based on a normal orthogonal array $L_p(s_1\cdots s_m)=(C_1,\ldots,C_n)$ and that $D(r,r;r)$ is a difference matrix. Then we can obtain a larger normal mixed difference matrix

$$[D(r(r+1),k(k+1);p), D_1(r(r+1),k_1(r+k+1);s_1),\ldots,D_m(r(r+1),k_m(r+k+1);s_m)]$$

based on the array $L_p(s_1\cdots s_m)$.

Proof. From Lemma 3, we assume that

$$[D(r+1,k+1;p),D(r+1,k_t;s_t)]$$

$$=\begin{pmatrix}0 & 0\\ 0 & H\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & (h_{ij})_{r\times r_t}\end{pmatrix}=\begin{pmatrix}0 & 0 & \cdots & 0\\ 0 & h_1 & \cdots & h_{r_t}\end{pmatrix},$$

where $r_t = k + k_t$ and $H = (A_{r\times k}, B_{r\times k_t})$. And set $F = (\sigma(d_{ij})H)_{1\le i\le r, 1\le j\le r_t}$, where $\sigma(d_{ij})$ is as in (2). Then we construct the following matrix K as follows

$$K=\begin{pmatrix}0 & H\oplus 0_{r_t}^T\\ H\oplus 0_r & F\end{pmatrix}.$$

And in the K, set $K_0=\begin{pmatrix}0\\ H\oplus 0_r\end{pmatrix}$ and $K_j=\begin{pmatrix}h_j\oplus 0_{r_t}^T\\ \sigma(d_{1j})H\\ \cdots\\ \sigma(d_{rj})H\end{pmatrix}$ for $j=1,\ldots,r_t$.

Moreover, if h_j is a p-level column i.e. a column of $A_{r\times k}$, we take K_j as follows

$$K_j = \begin{pmatrix} h_j \oplus 0_k^T, \phi_t(h_j) \oplus 0_{k_t}^T \\ \sigma(d_{1j})(A, B) \\ \cdots \\ \sigma(d_{rj})(A, B) \end{pmatrix}.$$

If h_j is a s_t-level column i.e. a column of $B_{r\times k_t}$, we take K_j as follows

$$K_j = \begin{pmatrix} h_j \oplus 0_{r_t}^T \\ \sigma(d_{1j})(\phi_t(A), B) \\ \cdots \\ \sigma(d_{rj})(\phi_t(A), B) \end{pmatrix}.$$

Through some column permutation, K can be written as

$$K = [D(r(r+1), k(k+1); p), D_t(r(r+1), k_t(r+k+1); s_t)].$$

It easily follows from Lemma 2 that all the p-level columns of K constitute a difference matrix $D(r(r+1), k(k+1); p)$.

Now we prove that $[\phi_t(D(r(r+1), k(k+1); p)), D_t(r(r+1), k_t(r+k+1); s_t)]$ is a difference matrix based on the group G_t.

Let $[\phi_t(D(r+1, k+1; p)), D_t(r+1, k_t; s_t)] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \phi_t(A) & B \end{pmatrix}$.

By using $D(r, r; r)$ and Lemma 3, we can construct an s_t-level difference matrix

$$K_{\phi_t} = [D(r(r+1), k(k+1) + k_t(r+k+1); s_t)]$$
$$= \begin{pmatrix} 0 & (\phi_t(A), B) \oplus 0_{r_t}^T \\ (\phi_t(A), B) \oplus 0_r & \sigma(d_{ij})(\phi_t(A), B) \end{pmatrix}.$$

On the other hand, through some column permutation, we just have that K_{ϕ_t} can be written as

$$K_{\phi_t} = [\phi_t(D(r(r+1), k(k+1); p)), D_t(r(r+1), k_t(r+k+1); s_t)].$$

Hence, To take $t = 1, \ldots, m$, respectively, we can get a normal mixed difference matrix

$$[D(r(r+1), k(k+1); p), D_1(r(r+1), k_1(r+k+1); s_1), \ldots, D_m(r(r+1), k_m(r+k+1); s_m)]$$

based on the normal orthogonal array $L_p(s_1 \cdots s_m)$.

When the difference matrix $D(r,r,,r)$ in Theorem 3 does not exist but an orthogonal array exists, we state the following theorem.

Theorem 4. Under the conditions of Lemma 2, suppose that $[D(r+1,k+1;p), D_1(r+1,k_1;s_1),\ldots,D_m(r+1,k_m;s_m)]$ is a normal mixed difference matrix based on a normal orthogonal array $L_p(s_1\cdots s_m) = (C_1,\ldots,C_m)$ and that there exists an orthogonal array $L_{r^2}(r^{x+1}) = [(r)\otimes 1_r, Q_1(1_r\otimes(r)),\cdots,Q_x(1_r\otimes(r))]$ where $Q_j = diag(\sigma_{1j},\cdots,\sigma_{rj})$ is a permutation matrix satisfying $Q_j((r)\otimes 1_r) = (r)\otimes 1_r$. Then we can obtain a normal mixed difference matrix

$$[D(r(r+1),k(k+1);p), D_1(r(r+1),y_1;s_1),\ldots,D_m(r(r+1),y_m;s_m)]$$

based on the normal orthogonal array $L_p(s_1\cdots s_m)$, where

$$y_t = \begin{cases} k_t(2k+k_t+1) & \text{if } x \geq k+k_t \\ k_t(k+1)+(x-k)(k+k_t) & \text{if } k \leq x < k+k_t \end{cases}$$

If $x<k$, the above normal mixed difference matrix becomes

$$[D(r(r+1),k(x+1);p), D_1(r(r+1),k_1(x+1);s_1),\ldots,D_m(r(r+1),k_m(x+1);s_m)].$$

3 Examples

Example 3. Construction of two normal mixed difference matrices

$$[D_0(132, 30; 6), D_1(132, 42; 3), D_2(132, 102; 2)]$$

and

$$[D_0(132, 20; 6), D_1(132, 112; 3), D_2(132, 112; 2)].$$

By Lemma 3, we can make the normal mixed difference matrix $[D_0(12, 6; 6), D_1(12, 3; 3), D_2(12, 6; 2)]$ in Wang (1996) have the following form

$$\begin{pmatrix} 0 & 0 \\ 0 & H \end{pmatrix}.$$

Then using the difference matrix $D(11, 11; 11)$ and Theorem 3, we can construct a normal mixed difference matrix $[D_0(132, 30; 6), D_1(132, 42; 3), D_2(132, 102; 2)]$.

Similarly by using $[D_0(12, 5; 6), D_1(12, 7; 3), D_2(12, 7; 2)]$ in Wang (1996), we can obtain a larger normal mixed difference matrix $[D_0(132, 20; 6), D_1(132, 112; 3), D_2(132, 112; 2)]$.

Remark In theorem 3, if $r(r+1)-1$ is a prime or prime power, then we can use the method again to construct a new normal mixed difference matrix.

For instance, by continuing to use difference matrix $D(131, 131; 131)$ in Example 3 and Theorem 3, we can construct a normal mixed difference matrix $[D_0(131 \times 132, 19 \times 20; 6), D_1(131 \times 132, 112 \times (131+20); 3), D_2(131 \times 132, 112 \times (131+20); 2)] = [D_0(17292, 380; 6), D_1(17292, 16912; 3), D_2(17292, 16912; 2)]$.

Example 4. Construction of normal mixed difference matrix $[D_0(23 \times 24, 380; 4), D_1(23 \times 24, 172; 2), D_2(23 \times 24, 172; 2), D_3(23 \times 24, 172; 2)]$

By Lemma 3, we can make the $[D_0(24, 20; 4), D_1(24, 4; 2), D_2(24, 4; 2), D_3(24, 4; 2)]$ in Pang et al.(2004b) have the following form

$$\begin{pmatrix} 0 & 0 \\ 0 & H \end{pmatrix}.$$

Then using the difference matrix $D(23, 23; 23)$ and Theorem 3, we can construct a normal mixed difference matrix $[D_0(23 \times 24, 380; 4), D_1(23 \times 24, 172; 2), D_2(23 \times 24, 172; 2), D_3(23 \times 24, 172; 2)]$.

4 Discussion

How can we generalize the normal orthogonal array to any orthogonal array? $L_p(s_1 \cdots s_n) = (C_1, \ldots, C_n)$ is an orthogonal array, then there exists a map ϕ_l such that $\phi_l(C_0) = C_l$. Can we find a mixed difference matrix $D = [D_0, D_1, \ldots, D_n]$ such that $[C_0 \oplus D_0, C_1 \oplus D_1, \ldots, C_n \oplus D_n]$ constitute an orthogonal array? How?

References

[1] Bose, R.C., Bush, K. A., Orthogonal arrays of strength two and three. Ann. Math. Stat. 1952, 23: 508-524.

[2] Beth, T., Jungnickel, D. and Lenz, H. *Design theory.* Bibliographishes institute, Mannheinu-Wien-Zürich 1985.

[3] Hedayat, A.S., Sloane, N.J.A., Stufken, J., Orthogonal arrays: Theory and Applications, Springer, New York, 1999.

[4] Pang, S.Q.,Zhang, Y.S.,Liu S.Y., Further Results on the Orthogonal Arrays Obtained by Generalized Hadamard Product, Statist. Probab. Lett., 2004a, 68(1): 17-25.

[5] Pang, S.Q.,Zhang, Y.S.,Liu S.Y., Normal Mixed Difference Matrix and the Construction of Orthogonal Arrays. Statist. Probab. Lett.,2004b, 69(4): 431-437.

[6] Wang, J.C., Mixed difference matrices and the construction of orthogonal arrays. Statist. Probab. Lett., 1996, 28: 121-126.

[7] Wang, J.C., Wu, C.F.J. An approach to the construction of asymmetrical orthogonal arrays. J. Amer. Statist. Assoc., 1991 6: 450-456.

[8] Chung-yi Suen and Warren F. Kuhfeld (2005). On the construction of mixed orthogonal arrays of strength two. J. Statist. Plann. Inference, 133, 555-560.

[9] Wu,C.F.J., Zhang, R., Wang, R. Construction of asymmetrical orthogonal array of the type OA$(s^k, s^m(s_1^r)^{n_1} \cdots (s_t^r)^{n_t})$. Statist. Sinica., 1992, 1: 203-219.

[10] Zhang, Y.S., Duan, L., Lu, Y.Q. Construction of Generalized Hadamard Matrices $D(r^m(r+1), r^m(r+1); p)$. J. of Statist. Plann. Inference, 2002, 104:239-258.

[11] Zhang, Y.S., Pang, S.Q., Wang, Y.P. Orthogonal arrays obtained by generalized Hadamard product. Discrete Math., 2001, 238: 151-170.

Acknowledgement
The work is joint with Professor Yingshan Zhang.

Thank you!