Quasi Monte Carlo Simulation of Stochastic String Model

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Outline

- Quasi Monte Carlo Method
- Simulation of Brownian Sheet
- Stochastic String Model
- Interest Rate Option Pricing

Introduction

- High-dimensional integral problem:
  $\mu = \int_{C^d} f(\mathbf{x})d\mathbf{x}, \mathbf{x} = (x_1, ..., x_d)$, where $C^d = [0,1)^d$.
- Monte Carlo(MC) methods can solve the integration problem with the convergence rate of $\sigma(f)O(n^{-1/2})$.
- Quasi-Monte Carlo(QMC) methods use low discrepancy sequences (or quasi-random numbers).
- Popular low discrepancy sequences include: Halton sequence, Sobol' sequence, Faure sequence and GFaure sequence.

Figure ?? offers a visual comparison of pseudo-random numbers and quasi-random numbers.

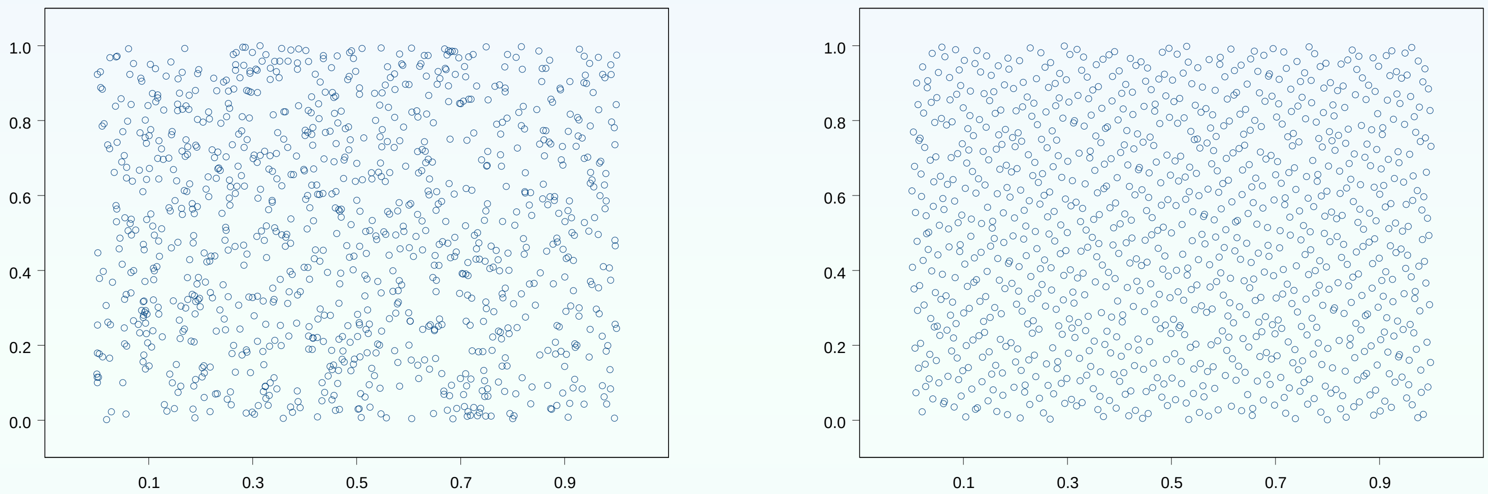

Figure 1: Left: Scatter Plot of 1000 pseudo-random numbers; Right: Scatter Plot of 1000 Sobol' numbers

Halton Sequence

- The one-dimensional Halton sequence is generated by using a prime p and expanding integers 0,1,2, ... into base p notation. The nth term of the sequence is defined as:

$$z_n = \frac{a_0}{p} + \frac{a_1}{p^2} + \frac{a_2}{p^3} ... + \frac{a_m}{p^{m+1}},$$

where the a_i's are integers from the base p expansion of $n-1$

$$[n-1]_p = a_m a_{m-1} ... a_1 a_0,$$

with $0 \leq a_i < p$. For example, assume the base $p = 2$, then the one-dimensional Halton sequence is:

$$0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{5}{8}, \frac{3}{8}, \frac{7}{8}, \frac{1}{16}, \frac{7}{16} ....$$

Sobol' Sequence

- The Sobol' sequence is generated with the first 2^m $(m = 0, 1, 2...)$terms of each dimension representing a permutation of the Halton sequence's corresponding terms with $p = 2$.
- For example, the first 10 points of the two-dimension Sobol' sequence are:$(0, 0)$, $(\frac{1}{2}, \frac{1}{2})$, $(\frac{3}{4}, \frac{1}{4})$, $(\frac{1}{4}, \frac{3}{4})$, $(\frac{3}{8}, \frac{3}{8})$, $(\frac{7}{8}, \frac{7}{8})$, $(\frac{5}{8}, \frac{1}{8})$, $(\frac{1}{8}, \frac{5}{8})$, $(\frac{3}{16}, \frac{5}{16})$, $(\frac{11}{16}, \frac{13}{16})$.

QMC Sequence Advantages

- Niederreiter proved that Halton, Sobol' and Faure sequences' discrepancy D_n has the property,

$$D_n \leq c_d \frac{(\log n)^d}{n} + \mathcal{O}(\frac{(\log n)^{d-1}}{n}),$$

  where the constant c_d depends on d only.

- Applying QMC methods can achieve the error rate with upper bound of $O(\frac{1}{n}(\log n)^d)$, which is better than MC method.

- It is straightforward to combine QMC methods with other variance reduction techniques to achieve better computational efficiency.

Karhounen-Loéve Expansion for Brownian Sheet

- Brownian sheet $W(t,x)$ is a Gaussian random field with mean zero and covariance

$$Cov[W(t,x), W(s,y)] = min(s,t)\, min(x,y)$$

- Its covariance kernel has eigenvalues and eigenfunctions:

$$\lambda_{ij} = \frac{2}{\pi^2\,(i-1/2)\,(j-1/2)}$$

$$\psi_{ij}(t,x) = sin\left\{\left(i-\frac{1}{2}\right)\pi\, t\right\} sin\left\{\left(j-\frac{1}{2}\right)\pi\, x\right\}, i,j = 1,2,\cdots.$$

- Karhounen-Loéve expansion:

$$W(t,x) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \lambda_{ij}\, \psi_{ij}(s,t)\, Z_{ij},$$

where Z_{ij} are i.i.d. standard normal random variables.

Simulation of Brownian Sheet via K-L Expansion

- In the Karhounen-Loéve expansion we select k terms with the largest k eigenvalues. Denote $\mathcal{C}_k$ the corresponding k indices (i, j).
- Use QMC to simulate k normal r.v. $\hat{Z}_{ij}$, $(i, j) \in \mathcal{C}_k$.
- Simulate Brownian sheet by

$$\hat{W}(t, x) = \sum_{(i,j)\in\mathcal{C}_k} \lambda_{ij}\, \psi_{ij}(s, t)\, \hat{Z}_{ij}$$

Top Eigenvalues in Karhounen-Loéve Expansion

The 20 largest eigenvalues λ_{ij} are listed in Table **??**.

Table 1: Indices (i, j) for the top 20 λ_{ij}

k	(i,j) pairs	k	(i,j) pairs
k=1	(1,1)	k=11	(1,11), (11,1), (2,4), (4,2)
k=2	(1,2), (2,1)	k=12	(1,12), (12,1)
k=3	(1,3), (3,1)	k=13	(1,13), (13,1), (3,3)
k=4	(1,4), (4,1)	k=14	(1,14), (14,1), (2,5), (5,2)
k=5	(1,5), (5,1), (2,2)	k=15	(1,15), (15,1)
k=6	(1,6), (6,1)	k=16	(1,16), (16,1)
k=7	(1,7), (7,1)	k=17	(1,17), (17,1), (2,6), (6,2)
k=8	(1,8), (8,1), (2,3), (3,2)	k=18	(1,18), (18,1), (3,4), (4,3)
k=9	(1,9), (9,1)	k=19	(1,19), (19,1)
k=10	(1,10), (10,1)	k=20	(1,20), (20,1), (2,7), (7,2)

Theory for the QMC Simulation

Theorem. Suppose H is a bounded smooth functional mapping from Brownian sheet to $\mathbb{R}^d$. Then $H(\hat{W})$ is a QMC simulation of $H(W)$ and with $k \sim n^{-1/4} (\log n)^{d/4}$,

$$D_n[H(\hat{W}), H(W)] \sim n^{-3/4} (\log n)^{3\,d/4}$$

Stochastic String Model

- Santa-Clara and Sornette introduced a new class of interest rate Models with stochastic string shocks.
- $P(t, s)$ denotes the price of the s maturity asset at time t.
- Instantaneous forward rates at time t for all times to maturity $x > 0$ satisfy,

$$f(t, x) = -\frac{\partial \log P(t, t + x)}{\partial x}.$$

- Forward rates are modeled with stochastic string shock model:

$$\begin{aligned} d_t f(t, x) &= [\frac{\partial f(t, x)}{\partial x} + \sigma(t, x) \int_0^x c(x, y)\, \sigma(t, y)\, dy]dt \\ &\quad + \sigma(t, x)\, d_t Z(t, x), \end{aligned}$$

where $Z(t, x)$ is a stochastic string.

Stochastic String

- Stochastic String $Z(t,x)$:

$$Z(t,x) = Z(0,x) + \int_0^t du \int_0^{h(x)} \frac{1}{\sqrt{h(x)}} \delta(u,v)\, dv$$
$$= Z(0,x) + \frac{W(t,h(x))}{\sqrt{h(x)}},$$

where $W(t,x)$ is a standard Brownian sheet.

- **Example**. Brownian sheet $Z(t,x) = W(t,x)$ with

$$c(x,y) = \frac{min(x,y)}{max(x,y)}.$$

Stochastic String

- **Example**. Ornstein-Uhlenbeck sheet

$$Z(t,x) = Z(0,x) + e^{-\lambda\, x} \int_0^x e^{\lambda\, v} dv \int_0^t \delta(u,v)\, du,$$

$$cov[Z(t,x), Z(s,y)] = min(t,s)\, exp(-\lambda\, |x-y|),$$

$$c(x,y) = exp(-\lambda\, |x-y|), \qquad h(x) = e^{2\lambda x}.$$

- **Example**. String with term structure of correlations:

$$Z(t,x) = Z(0,x) + e^{-\lambda\, \sqrt{x}} \int_0^x e^{\lambda\, \sqrt{v}} dv \int_0^t \delta(u,v)\, du,$$

$$c(x,y) = exp(-\lambda\, |\sqrt{x} - \sqrt{y}|).$$

Simulate Stochastic String

- Fix an integer m, let $(t_\ell, x_r) = (\ell/n, r/n)$, $\ell, r = 1, \cdots, m$.
- Simulate Brownian sheet $W(t,x))$ and evaluate its values at (t_ℓ, x_r).
- Simulate stochastic string $Z(t,x)$ by

$$Z(t_\ell, x_r) = Z(0, x_r) + \frac{1}{h(x_r)} \sum_{j=1}^{r} h(x_j) \left[W(t_\ell, x_j) - W(t_\ell, x_{j-1}) \right]$$

Simulation from Stochastic String Model

- Start with initial forward rate $f(0, x_r)$,
- Calculate $f(t_\ell, x_r)$ by

$$\begin{aligned} f(t_\ell, x_r) \quad &= \quad f(t_{\ell-1}, x_r) + [f(t_{\ell-1}, x_{r+1}) - f(t_{\ell-1}, x_r)] \\ &+ \sigma(t_{\ell-1}, x_r) \left(\int_0^x c(x_r, y)\, \sigma(t_{\ell-1}, y)\, dy \right) / m \\ &+ \sigma(t_{\ell-1}, x_r)\, [Z(t_\ell, x_r) - Z(t_{\ell-1}, x_r)] \end{aligned}$$

- $\sigma(t, x)$ is volatility function

$$\sigma(t, x) = \sigma\, exp\, (-\gamma\, x)$$

Option pricing of 1999 long bond futures contract

- The long bond futures contract is traded on the Chicago Board of Trade.
- On the delivery day, the seller can choose from many treasury bonds with at least 15 years to maturity. This is called delivery option.
- Seller will receive k times the futures price plus the accrued interest.
- We ignore timing option.

Table 2: Implied Forward Interest Rates

Date	Forward	Date	Forward	Date	Forward	Date	Forward
10/15/98	5.1572%	10/15/06	5.9618%	10/15/14	5.7862%	10/15/22	5.7862%
04/15/99	4.5420%	04/15/07	5.9858%	04/15/15	5.7862%	04/15/23	5.7862%
10/15/99	4.4722%	10/15/07	6.2611%	10/15/15	5.7862%	10/15/23	5.7862%
04/15/00	4.2391%	04/15/08	6.2820%	04/15/16	5.7862%	04/15/24	5.7862%
10/15/00	5.0690%	10/15/08	5.9657%	10/15/16	5.7862%	10/15/24	5.7862%
04/15/01	5.2236%	04/15/09	5.9615%	04/15/17	5.7862%	04/15/25	5.7862%
10/15/01	4.9391%	10/15/09	5.9516%	10/15/17	5.7862%	10/15/25	5.7862%
04/15/02	4.9687%	04/15/10	5.9384%	04/15/18	5.7862%	04/15/26	5.7862%
10/15/02	5.6378%	10/15/10	5.9203%	10/15/18	5.7862%	10/15/26	5.7862%
04/15/03	5.7631%	04/15/11	5.9005%	04/15/19	5.7862%	04/15/27	5.7862%
10/15/03	5.4199%	10/15/11	5.8746%	10/15/19	5.7862%	10/15/27	5.7862%
04/15/04	5.4611%	04/15/12	5.8474%	04/15/20	5.7862%	04/15/28	5.7862%
10/15/04	5.5992%	10/15/12	5.8177%	10/15/20	5.7862%	10/15/28	5.7862%
04/15/05	5.6378%	04/15/13	5.7862%	04/15/21	5.7862%	04/15/29	5.7862%
10/15/05	5.7792%	10/15/13	5.7862%	10/15/21	5.7862%	10/15/29	5.7862%
04/15/06	5.8102%	04/15/14	5.7862%	04/15/22	5.7862%	04/15/30	5.7862%

Table 3: Deliverable Bonds

Bond No	Maturity	Coupon	Conversion Factor	Market Price
1	15-Feb-15	11.250	1.2879	167.9688
2	15-Nov-15	9.875	1.1701	152.5000
3	15-Aug-15	10.625	1.2361	161.1563
4	15-Feb-16	9.250	1.1140	145.2188
5	15-May-17	8.750	1.0709	140.3125
6	15-Aug-17	8.875	1.0830	141.9688
7	15-May-16	7.250	0.9310	122.1563
8	15-Nov-16	7.500	0.9533	125.2500
9	15-May-18	9.125	1.1089	145.9063
10	15-Nov-18	9.000	1.0979	144.9688
11	15-Feb-19	8.875	1.0859	143.8125
12	15-Aug-19	8.125	1.0122	134.6875
13	15-May-20	8.750	1.0757	143.3438
14	15-Feb-20	8.500	1.0500	139.9063
15	15-Aug-20	8.750	1.0758	143.5938
16	15-May-21	8.125	1.0128	136.1563
17	15-Feb-21	7.875	0.9870	132.7813
18	15-Aug-21	8.125	1.0127	136.3750
19	15-Nov-21	8.000	1.0000	135.0000
20	15-Nov-22	7.625	0.9605	130.6875
...	...	...	...	...

Table 4: Simulated Option Prices

Model	Futures price of today's CTD	Futures Price	Delivery Option Value
Brownian motion	124.4917	124.4083	0.0833
O-U sheet	123.9798	123.4682	0.5116
Subexp. Corr.	124.5605	124.1593	0.4012