Fractional Factorial Designs With Admissible Sets of Clear Two-Factor Interactions

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An Experiment on How Time to Cycle up a Hill Is Affected by Seven Variables

(Box, Hunter, and Hunter, 1978)

Variables **(Factors)**	Variable Settings **(Levels)** $-$	$+$
A: handlebars	up	down
B: generator	off	on
C: seat height	up	down
D: breakfast	yes	no
E: raincoat	on	off
F: gear	low	medium
G: tire pressure	high	low

Response variable: time (seconds) to complete a trial bicycle run up to a hill between fixed marks

An Eight-Run Experimental Design With Seven Factors: A 2^{7-4} Design

run	$\mathbf{I}$	A 1	B 2	C 3	D 12	E 13	F 23	G 123	time (sec)
1	$+$	$-$	$-$	$-$	$+$	$+$	$+$	$-$	69
2	$+$	$-$	$-$	$+$	$+$	$-$	$-$	$+$	52
3	$+$	$-$	$+$	$-$	$-$	$+$	$-$	$+$	60
4	$+$	$-$	$+$	$+$	$-$	$-$	$+$	$-$	83
5	$+$	$+$	$-$	$-$	$-$	$-$	$+$	$+$	71
6	$+$	$+$	$-$	$+$	$-$	$+$	$-$	$-$	50
7	$+$	$+$	$+$	$-$	$+$	$-$	$-$	$-$	59
8	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	88

Four independent defining words:

$$\mathbf{I} = ABD = ACE = BCF = ABCG$$

A 2^{m-p} Fractional Factorial Design

- m two-level factors
- 2^{m-p} runs
- determined by its defining contrast subgroup, which consists of $2^p - 1$ defining words generated by p independent defining words

Example: For a 2^{8-3} design with

$$\mathbf{I} = ABCF = ABDG = ACDEH,$$

its defining contrast subgroup is

$$\begin{aligned}\mathbf{I} &= ABCF = ABDG = CDFG = ACDEH \\ &= BDEFH = BCEGH = AEFGH.\end{aligned}$$

The number of letters in a word is its length. **Resolution:** the length of the shortest word in the defining contrast subgroup (Box and Hunter, 1961)

MaxC2 Designs

- **Clear main effect**: a main effect not aliased with any other main effect or any two-factor interaction (2fi)

- **Clear 2fi**: a 2fi not aliased with any main effect or any other 2fi

- For economical reasons, resolution IV designs are often used for estimating main effects and some 2fi's.

- **MaxC2 designs:** resolution IV designs with the maximum number of clear 2fi's among all 2^{m-p} designs with maximum resolution IV

- Wu and Hamada (2000) recommend that MaxC2 designs be the best among designs with maximum resolution IV.

Clear Compromise Plans

- Suppose that the m factors are divided into two groups: G_1 of size m_1 and G_2 of size $m_2 = m - m_1$.

- $G_1 \times G_1$: the set of 2fi's among the factors in G_1 (similarly, $G_2 \times G_2$)

  $G_1 \times G_2$: the set of 2fi's between the factors in G_1 and those in G_2

- Clear compromise plans (Ke, Tang, and Wu, 2005) are resolution IV designs with all the specified 2fi's (in one of the following sets) being clear:

  (1) $G_1 \times G_1$ (class I)

  (2) $G_1 \times G_1$ and $G_2 \times G_2$ (class II)

  (3) $G_1 \times G_1$ and $G_1 \times G_2$ (class III)

  (4) $G_1 \times G_2$ (class IV)

Designs With Admissible Sets of Clear 2fi's

- For a 2^{m-p} resolution IV design d, its clear 2fi's can be represented by a linear graph $G(d)$:
  - Each vertex represents a factor.
  - Each line connecting two vertices represents a clear 2fi between the two vertices (factors).

- $G(d)$ is called admissible if it is not a real subgraph of $G(d^{'})$ for any other 2^{m-p} resolution IV design $d^{'}$.

- If $G(d)$ is admissible, then d is called *admissible*.

Admissible Designs

- Admissible designs simplify the search for resolution IV designs that satisfy certain requirements for some 2fi's to be clear.
  - The search for clear compromise plans can be carried out within the class of admissible designs. This greatly reduces the computational burden.

- The graphs representing the clear 2fi's of these admissible designs give the structures of the clear 2fi's. This is useful for practitioners to arrange for factors in an experiment.

Notation and Basic Facts

- $k = m - p$

- $M(k)$: the maximum value of m for which there exists a 2^{m-p} design of resolution at least V

  (e.g., $M(5) = 6$ and $M(6) = 8$, Draper and Lin, 1990)

- There exists a 2^{m-p} design of resolution IV with clear 2fi's if and only if $m \leq 2^{k-2} + 1$ (Chen and Hedayat, 1998).

- It suffices to consider $k \geq 5$ and

$$M(k) + 1 \leq m \leq 2^{k-2} + 1.$$

Number of Admissible Designs and Graphs

Design	N	N_{des}	N_{graph}
2^{7-2}	3	1	1
2^{8-3}	4	1	1
2^{9-4}	5	1	1
2^{9-3}	12	1	1
2^{10-4}	24	4	3
2^{11-5}	34	7	6
2^{12-6}	43	12	9
2^{13-7}	47	10	6
2^{14-8}	49	9	5
2^{15-9}	44	5	4
2^{16-10}	48	1	1
2^{17-11}	40	1	1
2^{12-5}	249	3	1
2^{13-6}	623	5	3
2^{14-7}	1535	30	15
2^{15-8}	3522	140	99
2^{16-9}	7500	682	584

N: # of non-isomorphic resolution IV designs; N_{des}: # of admissible designs; N_{graph}: # of admissible graphs

Complete Catalog of 32-Run Admissible Designs

Design	Additional Columns	C2	C
7_2.1	7 27	*(8, 16, 27)	15
8_3.1	7 11 29	*(16, 29)	13
9_4.1	7 11 13 30	*(16, 30)	15

- Columns: in the sense of Chen, Sun, and Wu (1993)

- C2: clear 2fi's

- C: the number of clear 2fi's

- *(8, 16, 27): the set of 2fi's involving at least one of the factors (columns) 8, 16, or 27

All 2^{9-3} and 2^{10-4} Admissible Designs

- 9_3.1: Additional columns (AC)=$\{7, 27, 45\}$; C=30; C2: *(8, 16, 32, 27, 45)

- 10_4.1: AC=$\{7, 27, 43, 53\}$; C=33; C2: *(8, 53), (1, 2, 4, 7)×(16, 32, 27, 43)

- 10_4.2: AC=$\{7, 11, 29, 51\}$; C=30; C2: *(16, 32, 29, 51)

- 10_4.3: AC=$\{7, 11, 29, 46\}$; C=30; C2: *(16, 32, 29, 46)

- 10_4.4: AC=$\{7, 25, 42, 53\}$; C=27; C2: *53; @((4, 7), (16, 25), (32, 42)); 8×(4, 7); 2×(16, 25); 1×(32, 42)

Note: @$(G_1, G_2, G_3) \equiv (G_1 \times G_2) \bigcup (G_2 \times G3) \bigcup (G_3 \times G_1)$

All 2^{13-6} Admissible Designs

- 13_6.1: AC=$\{7, 27, 43, 85, 102, 120\}$; C=66;
  C2: *(8, 64, 85, 102, 120);
  (1, 2, 4, 7)×(16, 32, 27, 43)

- 13_6.2: AC=$\{7, 27, 43, 53, 78, 120\}$; C=66;
  C2: *(8, 64, 53, 78, 120);
  (1, 2, 4, 7)×(16, 32, 27, 43)

- 13_6.3: AC=$\{7, 27, 45, 78, 121, 122\}$; C=63;
  C2: *(8, 16, 32, 64, 27, 45, 78)

- 13_6.4: AC=$\{7, 25, 42, 77, 118, 120\}$; C=60;
  C2: *(64, 77, 118, 120);
  @((4, 7), (16, 25), (32, 42));
  8×(4, 7); 2×(16, 25); 1×(32, 42)

- 13_6.5: AC=$\{7, 25, 42, 53, 78, 120\}$; C=60;
  C2: *(64, 53, 78, 120);
  @((4, 7), (16, 25), (32, 42));
  8×(4, 7); 2×(16, 25); 1×(32, 42)

Some Observations

- MaxC2 designs are always admissible.
- In general, minimum aberration designs (Fries and Hunter, 1980) are inadmissible unless they are MaxC2 designs.
- For 2^{7-2}, 2^{8-3}, 2^{9-4}, 2^{9-3}, 2^{16-10}, 2^{17-11}, 2^{32-25}, and 2^{33-26} designs, MaxC2 designs are unique and are the only admissible designs.
- All three admissible 2^{12-5} designs are MaxC2 designs and have the same linear graph of clear 2fi's.
- It is easy to identify from the catalogs of admissible designs those that have the same linear graph.

Applications

- Use a 64-run design to study 10 factors A to J.

  To estimate all the main effects and 2fi's among factors A, B, C, and D:

  - Select the design 10_4.2 and assign factors A, B, C, and D to the columns 16, 32, 29, and 51.

- Use a 128-run design to study 13 factors A to M.

  To estimate all the main effects, any 2fi's that contain at least one of the factors from $\{A, B, C, D, E\}$, and any 2fi's in $\{F, G, H, I\} \times \{J, K, L, M\}$:

  - Select the design 13_6.1 and assign factors A to M to the columns 8, 64, 85, 102, 120, 1, 2, 4, 7, 16, 32, 27, and 43, respectively.