A GENERAL MINIMUM LOWER-ORDER CONFOUNDING CRITERION FOR TWO-LEVEL REGULAR DESIGNS

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Abstract

Based on the hierarchical ordering principle of factorial effects in experimental design, we propose an aliased effect-number pattern (AENP) as a criterion to judge a two-level regular design; such a pattern contains the basic information of non-aliased effects as well as effects aliased at varying degrees in a design. A design that sequentially maximizes the numbers in the AENP is called a general minimum lower-order confounding (GMLOC) design. We call the new criterion a GMLOC criterion. As the word-length pattern, as the core of the minimum aberration (MA) criterion, is a function of the AENP, the MA criterion can be treated as a special case of the new criterion. The same also holds for the clear effects criterion under the hierarchical ordering principle. Furthermore, since the estimation capacity of a design can be calculated as a function of the new pattern, this criterion can then be treated as

a special case of the GMLOC criterion as well. From the new pattern, certain ties between the MA and clear effects criteria are revealed. In addition, we introduce in this paper a concept of estimation ability for regular designs, and infer that a GMLOC design is simply a design with the best estimation ability. At last, a simple algorithm for computing the AENP is provided. All the GMLOC designs for 16 and 32 runs and some comparisons with MA designs are tabulated in the Appendix.

1. Introduction (Motivation)

The purpose of experiments: **to estimate more effects and related models.**

The most important principal: **the effect hierarchy principal**

Two-level regular designs are most useful.

Four most popular good criteria:

(a) the maximum resolution criterion (Box and Hunter (1961)) (defect: may there are many designs with maximum resolution, but it can not distinguish which one is better.)

(b) the minimum aberration (MA) criterion (Fries and Hunter (1980))(defect: sometimes it is fail to detect good designs under the effect hierarchy principal)

(c) clear effects criterion (Wu and Chen (1992)) (defect: it can only be used when there exist designs having clear effects; may there are many equally good designs under the clear effects criterion, but it can not distinguish which one is better.)

(d) the criterion of estimation capacity (Sun (1993), Cheng and Mukerjee (1998)) (defect: to estimate possible most models involving all the main effects and some special 2fi's, need a strong assumption that all other 2fi's not involved in the models but aliasing the 2fi's in the models are absent or negligible.)

Especially, for MA and clear criteria, both usually give the same optimal designs, but sometimes the optimal designs obtained by the two criteria are conflict. The following is a famous example (Wu & Hamada (2000)):

Example 1. Consider the two 2^{9-4} designs:

$$d_1 : I = 1236 = 1247 = 1258 = 13459, \;\; d_2 : I = 1236 = 1247 = 1348 = 23459,$$

WLPs of d_1 and d_2 are (0,0,0,6,8,0,0,1,0) and (0,0,0,7,7,0,0,0,1) respectively. Under MA d_1 is better, but under clear d_2 is better since d_2 has 15 clear 2fi's and d_1 only has 8 (They all have 9 clear ME).

Questions:

- What relationships are there between the criteria?
- Why the criteria in which the start point seems the same, especially for MA and clear, at most cases give the same optimal designs, but sometimes give conflict results?
- Why do the existing good criteria have own defect?
- What is the basic information being contained in the defining contrast subgroup $\boldsymbol{G}$?
- Is there a criterion which more reasonably reflect the effect hierarchy principal?

In this paper we try to answer these questions.

2. A New Aliasing Pattern and Minimum Lower-Order Confounding Criterion

First we need to explore further the basic information hidden in the subgroup G. Consider a description of i-order effect being aliased by j-order effects for any i,j.

Two basic elements should be considered:

1. For a given i-order effect, how severe it is aliased by j-order effects. If the i-order effect is aliased by k j-order effects simultaneously, it is said that the i-order effect is aliased by j-order effects at degree k. Especially, if $k = 0$, then it is said that the i-order effect is not aliased by j-order effects.

2. In a design, how many i-order effects are aliased by j-order effects at a given degree k.

We use the notation ${}_i^{\#}C_j^{(k)}$ to denote the number of i-order effects aliased by j-order effects at degree k. Thus, for a design, we have a set

$$\{{}_i^{\#}C_j^{(k)}, i, j = 0, 1, \ldots, n, k = 0, 1, \ldots, K_j\}, \tag{1}$$

where $K_j = \binom{n}{j}$ and use the set to reflect the whole confounding between effects.

Obviously, the numbers in (1) are not equally important, we need arrange them in a particular order.

First, we should rank the numbers ${}_i^{\#}C_j^{(k)}$ from degree $\mathbf{0}$ to the most severe degree in the following vector:

$${}_i^{\#}C_j = ({}_i^{\#}C_j^{(0)}, {}_i^{\#}C_j^{(1)}, \ldots, {}_i^{\#}C_j^{(K_j)}). \tag{2}$$

Actually, the vector indicates a distribution of the total number of i-order effects aliased by j-order effects on the degrees $k = 0, 1, \ldots, K_j$.

Now we consider the rank of the different vectors ${}^{\#}_{i}\boldsymbol{C}_{j}$'s. First we drop ${}^{\#}_{0}\boldsymbol{C}_{0}$, ${}^{\#}_{0}\boldsymbol{C}_{1}$ and ${}^{\#}_{1}\boldsymbol{C}_{0}$. And then put ${}^{\#}_{1}\boldsymbol{C}_{1}$ at the first place. Next, consider the vectors related two-factor interactions. If two-factor interactions are not negligible, then should rank the vectors ${}^{\#}_{2}\boldsymbol{C}_{0}$, ${}^{\#}_{1}\boldsymbol{C}_{2}$, ${}^{\#}_{2}\boldsymbol{C}_{1}$ and ${}^{\#}_{2}\boldsymbol{C}_{2}$ in order as $({}^{\#}_{2}\boldsymbol{C}_{0}, {}^{\#}_{1}\boldsymbol{C}_{2}, {}^{\#}_{2}\boldsymbol{C}_{1}, {}^{\#}_{2}\boldsymbol{C}_{2})$. Similarly, if the three-order effects are not negligible, rank the vectors ${}^{\#}_{3}\boldsymbol{C}_{0}$, ${}^{\#}_{1}\boldsymbol{C}_{3}$, ${}^{\#}_{3}\boldsymbol{C}_{1}$, ${}^{\#}_{2}\boldsymbol{C}_{3}$, ${}^{\#}_{3}\boldsymbol{C}_{2}$ and ${}^{\#}_{3}\boldsymbol{C}_{3}$ in order as $({}^{\#}_{3}\boldsymbol{C}_{0}, {}^{\#}_{1}\boldsymbol{C}_{3}, {}^{\#}_{3}\boldsymbol{C}_{1}, {}^{\#}_{2}\boldsymbol{C}_{3}, {}^{\#}_{3}\boldsymbol{C}_{2}, {}^{\#}_{3}\boldsymbol{C}_{3})$ and so on.

The general rule is: (i) if $\max(\boldsymbol{i}, \boldsymbol{j}) < \max(\boldsymbol{s}, \boldsymbol{t})$ then ${}^{\#}_{i}\boldsymbol{C}_{j}$ is at before of ${}^{\#}_{s}\boldsymbol{C}_{t}$, (ii) if $\boldsymbol{i} + \boldsymbol{j} < \boldsymbol{s} + \boldsymbol{t}$ then ${}^{\#}_{i}\boldsymbol{C}_{j}$ is at before of ${}^{\#}_{s}\boldsymbol{C}_{t}$, and (iii) if $\boldsymbol{i} + \boldsymbol{j} = \boldsymbol{s} + \boldsymbol{t}$ and $\boldsymbol{i} < \boldsymbol{s}$ then ${}^{\#}_{i}\boldsymbol{C}_{j}$ is at before of ${}^{\#}_{s}\boldsymbol{C}_{t}$. Therefore, according to the principle that lower-order effects are more important than higher-order effects, finally we rank the numbers in set (1) in the following ordering:

$$ {}^{\#}\boldsymbol{C} = ({}^{\#}_{1}C_1,\ {}^{\#}_{2}C_0,\ {}^{\#}_{1}C_2,\ {}^{\#}_{2}C_1,\ {}^{\#}_{2}C_2,\ {}^{\#}_{3}C_0,\ {}^{\#}_{1}C_3,\ {}^{\#}_{3}C_1,\ {}^{\#}_{2}C_3,\ {}^{\#}_{3}C_2,\ {}^{\#}_{3}C_3,\ {}^{\#}_{4}C_0,\ {}^{\#}_{1}C_4,\ {}^{\#}_{4}C_1, {}^{\#}_{2}C_4,\ {}^{\#}_{4}C_2,\ {}^{\#}_{3}C_4,\ {}^{\#}_{4}C_3,\ {}^{\#}_{4}C_4, \ldots). \tag{3} $$

We call the ordering (3) an aliased effect-number pattern (AENP), such a pattern as well as set (1) contains the basic information of non-aliased effects as well as effects aliased at varying degrees in a design.

Based on ${}^{\#}\boldsymbol{C}$, we define a general minimum lower-order confounding (GMLOC) criterion as follows. The GMLOC criterion selects designs having GMLOC as the optimal ones.

Definition 1. *Let* ${}^{\#}\boldsymbol{C}_{\boldsymbol{l}}$ *be the* $\boldsymbol{l}$*-th component of* ${}^{\#}\boldsymbol{C}$, *and* ${}^{\#}\boldsymbol{C}(\boldsymbol{d}_1)$ *and* ${}^{\#}\boldsymbol{C}(\boldsymbol{d}_2)$ *the AENPs of two designs* $\boldsymbol{d}_1$ *and* $\boldsymbol{d}_2$. *Suppose* ${}^{\#}\boldsymbol{C}_{\boldsymbol{l}}$ *is the first component such that* ${}^{\#}\boldsymbol{C}_l(\boldsymbol{d}_1)$ *and* ${}^{\#}\boldsymbol{C}_l(\boldsymbol{d}_2)$ *are different from each other. If* ${}^{\#}\boldsymbol{C}_l(\boldsymbol{d}_1) > {}^{\#}\boldsymbol{C}_l(\boldsymbol{d}_2)$, *then* $\boldsymbol{d}_1$ *is said to have less general lower-order confounding (GLOC) than* $\boldsymbol{d}_2$. *A design* $\boldsymbol{d}$ *is said to have GMLOC if no other design has less GLOC than* $\boldsymbol{d}$.

Example 2: For the following two $\mathbf{2^{8-3}}$ designs:

$$\boldsymbol{d}_3 : \boldsymbol{I} = \mathbf{1236} = \mathbf{1247} = \mathbf{1358}, \quad \boldsymbol{d}_4 : \boldsymbol{I} = \mathbf{1236} = \mathbf{1247} = \mathbf{1348},$$

we have ${}^{\#}_{1}\boldsymbol{C}_1(\boldsymbol{d}_3) = {}^{\#}_{1}\boldsymbol{C}_1(\boldsymbol{d}_4) = (\mathbf{8}, \mathbf{0}, \ldots, \mathbf{0})$, ${}^{\#}_{2}\boldsymbol{C}_0(\boldsymbol{d}_3) = {}^{\#}_{2}\boldsymbol{C}_0(\boldsymbol{d}_4) = (\mathbf{8})$, ${}^{\#}_{1}\boldsymbol{C}_2(\boldsymbol{d}_3) = {}^{\#}_{1}\boldsymbol{C}_2(\boldsymbol{d}_4) = (\mathbf{8}, \mathbf{0}, \ldots, \mathbf{0})$ and ${}^{\#}_{2}\boldsymbol{C}_1(\boldsymbol{d}_3) = {}^{\#}_{2}\boldsymbol{C}_1(\boldsymbol{d}_4) = (\mathbf{28}, \mathbf{0}, \ldots, \mathbf{0})$. But ${}^{\#}_{2}\boldsymbol{C}_2(\boldsymbol{d}_4) = (\mathbf{7}, \mathbf{0}, \mathbf{21}, \mathbf{0}, \ldots, \mathbf{0})$ and ${}^{\#}_{2}\boldsymbol{C}_2(\boldsymbol{d}_3) = (\mathbf{4}, \mathbf{18}, \mathbf{6}, \mathbf{0}, \ldots, \mathbf{0})$. Therefore, by the first non-equal numbers ${}^{\#}_{2}\boldsymbol{C}_2^{(0)}(\boldsymbol{d}_4) = \mathbf{7} > {}^{\#}_{2}\boldsymbol{C}_2^{(0)}(\boldsymbol{d}_3) = \mathbf{4}$, it is said that $\boldsymbol{d}_4$ has less GLOC than $\boldsymbol{d}_3$.

A GMLOC design is just one which sequentially maximizes the components ${}^{\#}_{\boldsymbol{i}}\boldsymbol{C}_{\boldsymbol{j}}^{(\boldsymbol{k})}$'s of ${}^{\#}\boldsymbol{C}$ in (3).

We have directly the following theorem from Definition 1:

Theorem 1. *A GMLOC* $\mathbf{2}^{\boldsymbol{n}-\boldsymbol{m}}$ *design must be one with maximum resolution in all* $\mathbf{2}^{\boldsymbol{n}-\boldsymbol{m}}$ *designs.*

3. Relations with Minimum Aberration Criteria

The word-length pattern (WLP) of a regular design is denoted by

$$\boldsymbol{W} = (\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{A}_3, \boldsymbol{A}_4, \ldots, \boldsymbol{A}_n). \tag{4}$$

We firstly have the following relation between the WLP and the AENP:

Theorem 2. *For any* $\mathbf{2}^{\boldsymbol{n-m}}$ *design, its WLP (4) is a function of* $\{{}^{\#}_{i}\boldsymbol{C}^{(\boldsymbol{k})}_{\boldsymbol{j}}, \boldsymbol{i}, \boldsymbol{j} = \mathbf{0}, \mathbf{1}, \ldots, \boldsymbol{n}, \boldsymbol{k} = \mathbf{1}, \ldots, \boldsymbol{K_j}\}$ *in the following three forms:*

1. $\boldsymbol{A_i} = {}^{\#}_{i}\boldsymbol{C}^{(\mathbf{1})}_{\mathbf{0}}, \boldsymbol{i} = \mathbf{1}, \ldots, \boldsymbol{n};$

2. $\boldsymbol{A_j}$ *is a function of* ${}^{\#}_{0}\boldsymbol{C_j}$, $\boldsymbol{j} = \mathbf{1}, \ldots, \boldsymbol{n};$

3. For any $\boldsymbol{i}$, $\boldsymbol{A_i}$ *is a function of* ${}_{\boldsymbol{s}}\boldsymbol{C_t}, \boldsymbol{s}, \boldsymbol{t} = \mathbf{1}, \ldots, \boldsymbol{n}$ *in (6), where* ${}_{\boldsymbol{s}}\boldsymbol{C_t}$ *is a function of* $\{{}^{\#}_{s}\boldsymbol{C}^{(\boldsymbol{k})}_{\boldsymbol{t}}, \boldsymbol{k} = \mathbf{1}, \ldots, \boldsymbol{K_j}\}$ *as in (7), and sequentially minimizing* $\boldsymbol{A_i}$*'s of* $\boldsymbol{W}$ *in (4) is equivalent to sequentially minimizing* ${}_{\boldsymbol{s}}\boldsymbol{C_t}$*'s of* $\boldsymbol{C}$ *in (6).*

The 1 and 2 are obvious, only consider 3. Considering the 2^{n-m} designs with resolution at least III, Zhang and Park (2000) defined ${}_iC_j$ as the number of alias relations of i- and j-order effects in a design and obtained a general formula for calculating ${}_iC_j$ with $i \le j$ as:

$$ {}_iC_j = \sum_{l=0}^{i} \binom{n-(j-i+2l)}{i-l} \binom{j-i+2l}{l} A_{j-i+2l}, \quad i, j = 1, 2, \ldots, n, \tag{5} $$

where $\binom{x}{0} = 1$, $\binom{x}{y} = 0$ for $x < y$ or $x < 0$, and $A_i = 0$ for $i \le 2$ or $i > n$.

And they proved that sequentially minimizing the sequence

$$ C = ({}_1C_1,\ {}_1C_2,\ {}_2C_2,\ {}_1C_3,\ {}_2C_3,\ {}_3C_3,\ {}_1C_4,\ {}_2C_4,\ {}_3C_4,\ {}_4C_4,\ \ldots) \tag{6} $$

is equivalent to sequentially minimizing the sequence (4).

By the definition of ${}_iC_j$ and comparing with the definition of alias sets for a regular design, it is easy to see that

$$ {}_iC_j = \begin{cases} \sum_{k=1}^{K_i} k \cdot {}_i^{\#}C_i^{(k)}/2, & \text{if } i = j, \\ \sum_{k=1}^{K_j} k \cdot {}_i^{\#}C_j^{(k)}, & \text{if } i \neq j. \end{cases} \tag{7} $$

Corollary 1. *The designs with different WLPs must have different AENPs.*

But the inverse of the corollary does not hold. It means that the designs with different AENPs may have the same WLP. Let us to see the following example.

Example 3. Consider the two 2^{12-7} designs:

$d_5 : I = 126 = 137 = 238 = 12349 = 1235t_0 = 45t_1 = 12345t_2,$
$d_6 : I = 126 = 137 = 248 = 349 = 125t_0 = 135t_1 = 145t_2,$

The WLPs of d_5 and d_6 are the same: $W = (0, 0, 8, 15, 24, 32, 24, 15, 8, 0, 0, 1)$. But the AENPs of d_5 and d_6 are different and the first different components of them are ${}_2^{\#}C_2^{(1)}(d_5) = 60$ and ${}_2^{\#}C_2^{(1)}(d_6) = 54$.

So, the AENP is a more refined pattern than WLP to judge designs.

From Theorem 2, we can see that the MA only use the information of $\{{}_i^{\#}C_j^{(k)}, i, j = 0, 1, \ldots, n, k = 1, \ldots, K_j\}$, but not $\{{}_i^{\#}C_j^{(0)}, i, j = 0, 1, \ldots, n, \}$.

We note that ${}_i^{\#}C_j^{(0)} + \sum_{k=1}^{K_j} {}_i^{\#}C_j^{(k)} = \binom{n}{i}$. Although ${}_i^{\#}C_j^{(0)}$ can determine the sum $\sum_{k=1}^{K_j} {}_i^{\#}C_j^{(k)}$, but can not determine the vector $({}_i^{\#}C_j^{(1)}, \ldots, {}^{\#}C_j^{(K_j)})$ and ${}_iC_j = \sum_{k=1}^{K_j} k \cdot {}_i^{\#}C_j^{(k)}$. Therefore, it is possible that for two designs d and d' with ${}_i^{\#}C_j^{(0)}(d) > {}_i^{\#}C_j^{(0)}(d')$, having $\sum_{k=1}^{K_j} {}_i^{\#}C_j^{(k)}(d) < \sum_{k=1}^{K_j} {}_i^{\#}C_j^{(k)}(d')$, but still have ${}_iC_j = \sum_{k=1}^{K_j} k \cdot {}_i^{\#}C_j^{(k)} > {}_iC_j = \sum_{k=1}^{K_j} k \cdot {}_i^{\#}C_j^{(k)}$.

Designs d_1 and d_2 in Example 1 just are this case. They have ${}_1^{\#}C_2(d_1) = {}_1^{\#}C_2(d_2) = (9, 0, \ldots)$, ${}_2^{\#}C_1(d_1) = {}_2^{\#}C_1(d_2) = (36, 0, \ldots)$, ${}_2^{\#}C_2(d_1) = (8, 24, 0, 4)$ and ${}_2^{\#}C_2(d_2) = (15, 0, 21)$. We have ${}_2^{\#}C_2^{(0)}(d_1) = 8 < {}_2^{\#}C_2^{(0)}(d_2) = 15$, but still ${}_2C_2(d_1) = 1 \times 24 + 3 \times 4 = 36 < {}_2C_2(d_2) = 2 \times 21 = 41$. Thus, by sequentially minimizing (6) the MA criterion infers that

d_1 is better than d_2 and d_1 is a MA design. Actually, under the effect hierarchical principle d_2 is better than d_1, since d_2 has 15 clear 2fis but d_1 only has 8 ones except for both have all 9 clear main effects.

Only using a part of information in the AENP can be as a reasonable explanation why sometimes the MA criterion is fail to detect optimal designs under the effect hierarchy principal.

From equation (7), one can find that ${}_iC_j$ is a linear function of the components of ${}^{\#}_{i}C_j$ with k as the weigh of ${}^{\#}_{i}C_j^{(k)}$. And a design which sequentially maximizes the components of ${}^{\#}_{i}C_j$ tends to minimize ${}_iC_j$. This is why in most of cases the optimal designs under MA and GMLOC criteria are consistent. However, there are quite a few optimal designs under the two criteria differ from each other, since they are established on different bases. One more example is shown below.

Example 4. Consider the three $\mathbf{2^{13-7}}$ designs with 64 runs:

$\boldsymbol{d_7: I = 12347 = 34568 = 2459 = 1456t_0 = 256t_1 = 136t_2 = 235t_3,}$

$\boldsymbol{d_8: I = 12347 = 3458 = 2459 = 356t_0 = 256t_1 = 456t_2 = 346t_3,}$

$\boldsymbol{d_9: I = 12347 = 34568 = 2459 = 1456t_0 = 246t_1 = 12356t_2 = 256t_3,}$

The WLPs of $\boldsymbol{d_7}$, $\boldsymbol{d_8}$ and $\boldsymbol{d_9}$ are respectively

$$\boldsymbol{d_7: (0, 14, 28, 24, 24, 17, 12, 8, 0, 0, 0),\ d_8: (0, 26, 12, 24, 28, 13, 20, 0, 4, 0, 0),}$$

$$\boldsymbol{d_9: (0, 14, 33, 16, 16, 33, 14, 0, 0, 0, 1)}$$

and the most important part of their AENP are shown in the following table:

	$\boldsymbol{d_7}$		$\boldsymbol{d_8}$		$\boldsymbol{d_9}$	
$\boldsymbol{{}^{\#}_{i}C_j}$	$\boldsymbol{j=1}$	$\boldsymbol{j=2}$	$\boldsymbol{j=1}$	$\boldsymbol{j=2}$	$\boldsymbol{j=1}$	$\boldsymbol{j=2}$
$\boldsymbol{i=1}$	**13**	**13**	**13**	**13**	**13**	**13**
$\boldsymbol{i=2}$	**78**	$\boldsymbol{20, 36, 18, 4}$	**78**	$\boldsymbol{23, 0, 24, 16, 15}$	**78**	$\boldsymbol{36, 0, 42}$

According to the MA criterion, the optimality order is $\boldsymbol{d_7}$,$\boldsymbol{d_9}$ and $\boldsymbol{d_8}$. But, from their AENP above, it is easy to know that they all have 13 clear ME, $\boldsymbol{d_7}$ only has 20 clear 2fi's, $\boldsymbol{d_8}$ has 23 clear 2fi's and $\boldsymbol{d_9}$ has 36 clear 2fi's. Therefore, according the GMLOC and clear effects criteria their order of optimality is $\boldsymbol{d_9}$, $\boldsymbol{d_8}$ and $\boldsymbol{d_7}$. The MA criterion fails also to detect the best design in this case.

4. Relations with Clear Effects Criterion

We first note some results related the clear effects criterion.

LEMMA **1.** *When* $\mathbf{2}^{\boldsymbol{n-m-1}} < \boldsymbol{n} < \mathbf{2}^{\boldsymbol{n-m}} - \mathbf{1}$, *there exist only the designs with resolution* $\boldsymbol{R} \leq \mathbf{III}$, *and for any* $\mathbf{2}^{\boldsymbol{n-m}}$ *design with resolution* $\mathbf{III}$, *it has no any clear main effect and any clear two-factor interaction.*

LEMMA **2.** *When* $\mathbf{2}^{\boldsymbol{n-m-2}} + \mathbf{1} < \boldsymbol{n} \leq \mathbf{2}^{\boldsymbol{n-m-1}}$, *there exist resolution* IV *designs, but any such resolution* IV $\mathbf{2}^{\boldsymbol{n-m}}$ *design does not contain any clear two-factor interaction. If a* $\mathbf{2}^{\boldsymbol{n-(n-k)}}$ *design contains clear two-factor interaction for* $\mathbf{2}^{\boldsymbol{n-m-2}} < \boldsymbol{n} \leq \mathbf{2}^{\boldsymbol{n-m-1}}$, *then its resolution must be less than* IV.

LEMMA **3.** *When* $\boldsymbol{M(n-m) < n \leq 2^{n-m-2}+1}$, *there exist* $\boldsymbol{2^{n-m}}$ *designs with resolution* IV *which contain clear two-factor interactions, where* $\boldsymbol{M(n-m)}$ *is the maximum number of factors that can be accommodated in a* $\boldsymbol{2^{n-m}}$ *design with the maximum resolution at least* V.

LEMMA **4.** *Consider the* $\boldsymbol{2^{n-m}}$ *designs which has resolution at least* III. *Then* $\boldsymbol{{}_1^{\#}C_2^{(0)}}$ *is just the number of clear main effects in a design, and* $\boldsymbol{{}_2^{\#}C_2^{(0)} - {}_1^{\#}C_2^{(1)}}$ *is just the number of clear 2fi's in a design.*

The results of Lemma 1, 2 and 3 come from Chen and Hedayat (1998). Lemma 4 can be easily obtained by the definitions of clear effects and the AENP.

We can immediately obtain the following theorem about the relation between the two criteria by the lemmas and the definition of the GMLOC criterion:

Theorem 3. *The clear effects criterion selects the $\mathbf{2^{n-m}}$ designs which sequentially maximize ${}_{1}^{\#}C_{2}^{(0)}$ and ${}_{2}^{\#}C_{2}^{(0)}$ as the optimal ones when $n \leq \mathbf{2^{n-m-1}}$. For given n and m, if the optimal design under the clear effects criterion exsts, then the GMLOC criterion must be the best one of optimal clear effects criterion designs, where the meaning of ' the best' is under the GMLOC criterion.*

The clear criterion can not be used in many situations. For example, when $n > 2^{n-m-1}$, Lemma 1 tells us that the existed resolution III designs do not contain any clear ME and 2fi. However, the GMLOC criterion can be used for all kinds of parameters. When $2^{n-m-2}+1 < n \leq 2^{n-m-1}$, all of the resolution IV 2^{n-m} designs existed make no difference under the clear effects criterion. But the GMLOC criterion can discriminate them further.

Example 5. Consider the designs 10-5.1,10-5.2, 10-5.3 and 10-5.4 in Table 6 in the Appendix. Under the clear effects criterion the four designs have no difference, but the GMLOC criterion can distinguish them.

Based on the analysis above, we can conclude that the GMLOC criterion is more refined and reasonable one than the clear effects criterion.

The ties between MA and Clear criteria:

- MA criterion only uses the information of $\{{}_i^{\#}C_j^{(k)}, i, j = 0, 1, \ldots, n, k = 1, \ldots, K_j\}$.
- Clear criterion only uses the information of $\{{}_i^{\#}C_j^{(k)}, i, j = 0, 1, \ldots, n, k = 0\}$.
- They separately use the different part of the information in the set (1).

As mentioned above, the two parts have the relation ${}_i^{\#}C_j^{(0)} + \sum_{k=1}^{K_j} {}_i^{\#}C_j^{(k)} = \binom{n}{j}$ for any i and j. So, the larger the number ${}_i^{\#}C_j^{(0)}$, the lesser the number $\sum_{k=1}^{K_j} {}_i^{\#}C_j^{(k)}$. At most cases, when ${}_i^{\#}C_j^{(0)}$ is large, it tends that the weighed sum ${}_iC_j = \sum_{k=1}^{K_j} k \cdot {}_i^{\#}C_j^{(k)}$ is small. So sequentially maximizing the sequence $({}_1^{\#}C_2^{(0)}, {}_2^{\#}C_2^{(0)}, \ldots)$ tends to sequentially minimize the sequence (6). So, it is just the reason why for the two criteria, at most cases they give the same optimal designs.

But the number ${}_{i}^{\#}C_{j}^{(0)}$ can not affirmatively determine the number ${}_{i}C_{j} = \sum_{k=1}^{K_j} k \cdot {}_{i}^{\#}C_{j}^{(k)}$ which is different from the sum $\sum_{k=1}^{K_j} {}_{i}^{\#}C_{j}^{(k)}$. Therefore, sometimes the conflict results will appear as the mentioned example in the previous section. It can explain why sometimes they may give conflict results on optimal designs.

5. Relations with Maximum Estimation Capacity Criterion

Cheng and Mukerjee (1998) and Cheng, Steinberg and Sun (1999) discussed the estimation capacity of a design. Let $E_r(d)$ denote the number of models containing all the main effects and $r (1 \leq r \leq n(n-1)/2)$ 2fi's which can be estimated by the design d.

A design which maximized $E_r(d)$ for all r is said to have maximum estimation capacity (MEC).

Clearly, there are ${}_2^{\#}C_2^{(k)}/(k+1)$ alias sets containing $k+1$ 2fi's and ${}_1^{\#}C_2^{(k+1)}/(k+1)$ alias sets containing $k+1$ 2fi's and one main effect. An alias set contains at most $l = \min\{\lfloor n/2 \rfloor, 2^m\}$ 2fi's. Then all of the alias sets containing 2fi's but none of the main effects can be partitioned into l classes and the i-th class includes the alias sets containing $i+1$ 2fi's for $i = 0, 1, \ldots, l-1$. Let $\mathcal{C}_i$ be the i-th class. Then $|\mathcal{C}_i| = ({}_2^{\#}C_2^{(i)} - {}_1^{\#}C_2^{(i+1)})/(i+1)$, where $|\cdot|$ denote the cardinality of a set. Note that there maybe exist $|\mathcal{C}_i| = 0$ for some i. By the definition of $E_r(d)$, it is easy to get the following theorem:

Theorem 4. *$E_r(d)$ can be expressed as a function of ${}_2^{\#}C_2$ and ${}_1^{\#}C_2$ as follows:*

$$E_r(d) = \begin{cases} \sum \cdots \sum_{r_0+\cdots+r_{l-1}=r} \prod_{i=0}^{l-1} \binom{|\mathcal{C}_i|}{r_i} (i+1)^{r_i}, & \text{if } r \leq f, \\ 0, & \text{otherwise}. \end{cases}$$

where $0 \leq r_i \leq |\mathcal{C}_i|$, $f = 2^{n-m} - 1 - n$.

This theorem shows the relation between the MEC and GMLOC criteria. Hence the MEC criterion can be treated to optimize a function of the AENP.

In the notations in Cheng and Mukerjee (1998), by a lemma about upper weakly majorized, note that $\sum_{i=n+1}^{n+f} m_i(d) = \sum_{i=0}^{l-1} |\mathcal{C}_i|(i+1)$ and $\sum_{i=n+1}^{n+f} m_i^2(d) = \sum_{i=0}^{l-1} |\mathcal{C}_i|(i+1)^2$. Then a design d which maximizes $\sum_{i=0}^{l-1} |\mathcal{C}_i|(i+1)$ and minimizes $\sum_{i=0}^{l-1} |\mathcal{C}_i|(i+1)^2$ tends to behave well under the MEC criterion.

6. Maximum Estimation Ability

The optimal designs under the MEC criterion can estimate as many as possible models involving all main effects and some 2fi's with the assumption that all other 2fi's not involving in the model are negligible. Such an assumption is too strong to be justified. To avoid the strong assumption, we introduce the notion of estimation ability to choose designs with slight aliasing between the 2fi's.

Now let us consider the classes $\mathcal{C}_i$ for $i = 0, 1, \ldots, l-1$. Note that there are $i+1$ 2fi's in each alias set of the class $\mathcal{C}_i$. Hence, the smaller i means the slighter aliasing between the 2fi's in $\mathcal{C}_i$. Any model involving all the main effects and $r \leq |\mathcal{C}_0|$ 2fi's can be estimated without bias under the weaker assumption of absence of interactions involving at least three factors. And any model involving all the main effects and $|\mathcal{C}_0| < r \leq |\mathcal{C}_0| + |\mathcal{C}_1|$ 2fi's can be estimated under the following weak assumption: the $|\mathcal{C}_1|$ 2fi's in the alias sets of $\mathcal{C}_1$ are absence and the interactions involving at least three factors are negligible. Similarly, any model involving all the main effects and $\sum_{i=0}^{j} |\mathcal{C}_i| < r \leq \sum_{i=0}^{j+1} |\mathcal{C}_i|$ $(j = 0, 1, \ldots, l-1)$ 2fi's can be estimated under the following week assumption: the $i|\mathcal{C}_i|$ 2fi's in the alias sets of $\mathcal{C}_i$ for $i = 0, \ldots, j$ are absence and the interactions involving at least three factors are negligible. A model involving only the 2fi's in $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_i$ is called the i-class model in following. A good design should sequentially maximize $|\mathcal{C}_i|$ for $i = 0, 1, \ldots, l-1$ since such a design can be used to estimate the main effects and 2fi's with the slightest alias between the 2fi's. A design sequentially maximizing $|\mathcal{C}_i|$ for $i = 0, 1, \ldots, l-1$ is said to be a design with maximum estimation ability. The criterion selecting such designs as the optimal ones is called maximum estimation ability (MEA) criterion.

The optimal designs under the MEA criterion can estimate the model involving all main effects and some 2fi's with the slightest confounding between the 2fi's. If the experimenter want to de-alias the confounding between the 2fi's, he/she needs to do only a few follow-up experiments.

Note that $|\mathcal{C}_i| = ({}_2^{\#}C_2^{(i)} - {}_1^{\#}C_2^{(i+1)})/(i+1)$. For given ${}_1^{\#}C_2$ and ${}_2^{\#}C_1$, sequentially maximizing the components of ${}_2^{\#}C_2$ is equivalent to sequentially maximizing $|\mathcal{C}_i|$ for $i = 0, 1, \ldots, l-1$. Hence a GMLOC design sequentially maximizes the estimation ability of i-class models for $i = 0, 1, \ldots, l-1$. Under the effect hierarchy principle, the ability of the main effects to be estimated is first concerned, so a good design must sequentially maximize ${}_1^{\#}C_2$ and ${}_2^{\#}C_1$. Therefore, in any case, a GMLOC design can sequentially maximize the estimation ability of i-class models for $i = 0, 1, \ldots, l-1$.

7. Algorithm for AENP and GMLOC Designs with 16- and 32-run

Let us give a simple algorithm for computing the AENP through an example.

Consider a 2^{n-m} regular design d.

• Use a $2^m \times n$ matrix D to express its defining contrast subgroup G, where the entry (i, j) of D equals 1 if the i-th word in G contains letter j and 0 otherwise, and call it the defining structure matrix (or defining pencil matrix) of d. The first row of D corresponds to the element I in G, and every other row corresponds a word in G.

• Let S denote the set of all effects of n factors in d, where a k-order effect $i_1 \cdots i_k$ of d is expressed as an n-dimensional row vector with the i_1-th, $\ldots$, i_k-th entries ones and zeros otherwise. The effects in $\boldsymbol{G}$ or the defining structure matrix $\boldsymbol{D}$ are those which are aliased with $\boldsymbol{I}$, the total mean effect. Let the column sum (in ordinary addition) of $\boldsymbol{D}$ be as its marginal column.

The algorithm of computing ${}^{\#}_{i}\boldsymbol{C}_j(\boldsymbol{d})$ can be described as follows:

Step 1. Set $\boldsymbol{S}_0$ to be the empty set. And set ${}^{\#}_{i}\boldsymbol{C}_j^{(\boldsymbol{k})} = \mathbf{0}$ for all $\boldsymbol{i}, \boldsymbol{j} = \mathbf{0}, \mathbf{1}, \ldots, \boldsymbol{n}, \boldsymbol{k} = \mathbf{0}, \mathbf{1}, \ldots, \boldsymbol{K_j}$.

Step 2. Let $\boldsymbol{S} = \boldsymbol{S} \setminus \boldsymbol{S}_0$. Selecting one vector (i.e. a effect) from $\boldsymbol{S}$ (can be from lower order to higher order). Adding (in module 2) it to the every row of $\boldsymbol{D}$, we obtain the aliased-effect matrix $\boldsymbol{D}'$ of the selected effect. From the matrix $\boldsymbol{D}'$, we can get the alias set $\boldsymbol{T}$ to which the selected effect belongs (also the element of $\boldsymbol{T}$ is expressed as an $\boldsymbol{n}$-dimensional vector). And then set $\boldsymbol{S}_0 = \boldsymbol{S}_0 \cup \boldsymbol{T}$ and $\boldsymbol{i} = \boldsymbol{j} = \mathbf{0}$.

Step 3. Let p_i and q_j be the numbers of i- and j-order effects in T respectively (just count the numbers of i's and j's at the marginal column by the D' in the table respectively). We set ${}_i^{\#}C_j^{(q_j)} = {}_i^{\#}C_j^{(q_j)} + p_i$ if $i \neq j$ or ${}_i^{\#}C_j^{(q_j-1)} = {}_i^{\#}C_j^{(q_j-1)} + p_i$ if $i = j$. Then repeat this step for all cases: $1 \leq i + j \leq n, i, j = 1, \ldots, n.$

Step 4. Stop if $|S_0| = 2^n$ and go to Step 2 otherwise, where $|\cdot|$ is the cardinality of a set.

Let us consider the design d_3 in Example 2. The defining contrast subgroup G of d_3 is

$$\{I, 1236, 1247, 1358, 2568, 3467, 145678, 234578\},$$

and its defining structure matrix D is given in Table 1.

Table 1. Defining structure matrix of $\boldsymbol{d_3}$

0	0	0	0	0	0	0	0	0
1	1	1	0	0	1	0	0	4
1	1	0	1	0	0	1	0	4
1	0	1	0	1	0	0	1	4
0	1	0	0	1	1	0	1	4
0	0	1	1	0	1	1	0	4
1	0	0	1	1	1	1	1	6
0	1	1	1	1	0	1	1	6

The marginal column is just the distribution of word-length's in the subgroup $\boldsymbol{G}$.

For the example, in step 1 the $\boldsymbol{S}$ is the set of all effects of 5 factors. For simplicity, we only consider to calculate ${}_{\boldsymbol{2}}^{\#}\boldsymbol{C}_{\boldsymbol{j}}^{(\boldsymbol{k})}$'s of $\boldsymbol{d_3}$.

At step 2, say, we select vector $(\mathbf{1}, \mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$ (2fi 12) from $\boldsymbol{S}$. Adding it to every row of $\boldsymbol{D}$ in Table 1, we obtain the aliased-effect matrix $\boldsymbol{D'}$ and its marginal column of the 2fi 12 of $\boldsymbol{d_3}$ which is shown in Table 2.

Table 2. Aliased-effect matrix of the 2fi 12 of d_3

1	1	0	0	0	0	0	0	2
0	0	1	0	0	1	0	0	2
0	0	0	1	0	0	1	0	2
0	1	1	0	1	0	0	1	4
1	0	0	0	1	1	0	1	4
1	1	1	1	0	1	1	0	6
0	1	0	1	1	1	1	1	6
1	0	1	1	1	0	1	1	6

The marginal column is the distribution of effect orders in the alias set containing 2fi 12.

From D' we obtain the alias set T of the 2fi 12:

$$T = \{12, 36, 47, 2358, 1568, 123467, 245678, 134578\}.$$

Set $S_0 = S_0 \cup T$ (i.e. add the row vectors of D' into S_0) and $i = j = 0$.

In step 3, in this case we take $i = j = 2$ and have $p_2 = q_2 = 3$ (the number of 2's at the marginal column in Table is 3), and then set ${}^{\#}_{2}C_2^{(2)} = {}^{\#}_{2}C_2^{(2)} + 3$. Considering $i = 2, j = 4$, have $p_2 = 3, q_4 = 2$, and set ${}^{\#}_{2}C_4^{(2)} = {}^{\#}_{2}C_4^{(2)} + 3$. Considering $i = 2$ and $j = 6$, have

$\boldsymbol{p_2 = 3, q_6 = 3}$, and set ${}^{\#}_{2}\boldsymbol{C}_{6}^{(3)} = {}^{\#}_{2}\boldsymbol{C}_{6}^{(3)} + \boldsymbol{3}$. No change for other ${}^{\#}_{2}\boldsymbol{C}_{j}^{(k)}$'s.

In step 4, in this case only calculate the number of 2fi's in $\boldsymbol{S_0}$, the number 3 is less than $\binom{5}{2} = \mathbf{10}$, then go to step 2. Set $\boldsymbol{S} = \boldsymbol{S} \setminus \boldsymbol{S_0}$. Select one 2fi belonging to $\boldsymbol{S}$, say 45, and add $(\mathbf{0,0,0,1,1,0,0,0})$ to the rows of Table 1. we obtain the aliased-effect matrix of the 2fi 45 of $\boldsymbol{d_3}$ which is shown in Table 3.

Table 3. Aliased-effect matrix of the 2fi 45 of $\boldsymbol{d_3}$

0	0	0	1	1	0	0	0	2
1	1	1	1	1	1	0	0	6
1	1	0	0	1	0	1	0	4
1	0	1	1	0	0	0	1	4
0	1	0	1	0	1	0	1	4
0	0	1	0	1	1	1	0	4
1	0	0	0	0	1	1	1	4
0	1	1	0	0	0	1	1	4

The marginal column is the distribution of effect orders in the alias set containing 2fi 45.

From Table 3, we can get the alias set containing 2fi 45:

$$\boldsymbol{T = \{45, 123456, 1257, 1348, 2468, 3567, 1678, 2378\}.}$$

Let $\boldsymbol{S_0 = S_0 \cup T}$. Since $\boldsymbol{p_2 = q_2 = 1}$ (the number of $\mathbf{2}$'s at the marginal column in Table 3 is 1), set ${}_{\mathbf{2}}^{\#}\boldsymbol{C}_{\mathbf{2}}^{(0)} = {}_{\mathbf{2}}^{\#}\boldsymbol{C}_{\mathbf{2}}^{(0)} + \mathbf{1}$. In same way to consider $\boldsymbol{i = 2, j = 4}$ and $\boldsymbol{i = 2, j = 6}$.

Repeat the procedure above, we can get ${}_{\mathbf{2}}^{\#}\boldsymbol{C_2(d_3)} = \mathbf{(4, 18, 6, 0 \ldots, 0)}$ and ${}_{\mathbf{2}}^{\#}\boldsymbol{C_j^{(k)}, j} = \mathbf{1, 2, 3, \ldots,} \boldsymbol{k} = \mathbf{0, 1, 2 \ldots}$ of $\boldsymbol{d_3}$.

8. Simplification of the AENP and Its More Usage

The AENP seems complicated. Actually, from the view of application, the most important part of the AENP is only the small part on top left corner:

- If we only consider the designs in which three and higher order interactions are negligible, then we can only concern the $\mathbf{2 \times 2}$ sub-matrix $({}_i^{\#}C_j)$ with $i, j = \mathbf{1, 2}$, which usually can discriminate different designs.

- If we consider the designs in which four and higher order interactions are negligible, then we can only concern the $\mathbf{3 \times 3}$ sub-matrix $({}_i^{\#}C_j)$ with $i, j = \mathbf{1, 2, 3}$.

- From the small sub-matrix we can get all the information about the numbers of clear main effects and two-factor interactions and how severe confounded between the lower-order effects.

Example 6. Let us consider 2^{9-4} designs d_1, d_2 in Example 1 and d_{10} (in Table 14):

$d_1 : I = 1236 = 1247 = 1258 = 13459, \quad d_2 : I = 1236 = 1247 = 1348 = 23459,$
$d_{10} : I = 1236 = 2347 = 1348 = 1249.$

All the main effects of d_1, d_2 and d_{10} are clear and the number of clear 2fi's of d_1, d_2 and d_{10} are 8, 15 and 8, respectively. The WLPs of the three designs are (0,0,0,6,8,0,0,1,0), (0,0,0,7,7,0,0,0,1) and (0,0,0,14,0,0,...) respectively. d_1 is an MA design, and d_2 has the most clear 2fi's. Note that

$$
\begin{aligned}
&{}_1^{\#}C_1(d_1) = {}_1^{\#}C_1(d_2) = {}_1^{\#}C_1(d_{10}) = (9, 0, \ldots, 0),\\
&{}_2^{\#}C_0(d_1) = {}_2^{\#}C_0(d_2) = {}_1^{\#}C_1(d_{10}) = 36,\\
&{}_1^{\#}C_2(d_1) = {}_1^{\#}C_2(d_2) = {}_1^{\#}C_1(d_{10}) = (9, 0, \ldots, 0),\\
&{}_2^{\#}C_1(d_1) = {}_2^{\#}C_1(d_2) = {}_1^{\#}C_1(d_{10}) = (36, 0, \ldots, 0),\\
&{}_2^{\#}C_2(d_1) = (8, 24, 0, 4, 0, \ldots, 0),\ {}_2^{\#}C_2(d_2) = (15, 0, 21, 0, \ldots, 0),\\
&{}_2^{\#}C_2(d_{10}) = (8, 0, 0, 28, 0, \ldots, 0).
\end{aligned}
$$

Although according to WLP of MA criterion d_1 is better than d_2 and d_{10}, according to the GMLOC and clear criteria d_2 is obviously better than d_1 and d_{10}. d_1 and d_{10} make no difference under the clear effects criterion. but the GMLOC criterion can discriminate them.

In the tables of the Appendix, we list only the three entries ${}_1^{\#}\boldsymbol{C}_2$, ${}_2^{\#}\boldsymbol{C}_1$, and ${}_2^{\#}\boldsymbol{C}_2$ for every design.

The AENP has more usages:

- Easily to judge if a clear effect is strongly clear.

  For example, from the AENPs of $\boldsymbol{d}_1$ and $\boldsymbol{d}_{10}$, we can easily conclude that none of the eight clear 2fi's of $\boldsymbol{d}_1$ are strongly clear, while the eight clear 2fi's of $\boldsymbol{d}_{10}$ are all strongly clear.

- Can choose different functions of the AENP to get different criteria.

  We find that nearly all the existing criteria can be expressed into a function of the AENP, like MA criterion, clear effects criterion, week MA criterion (Chen and Hedayat (1996)), MMA Criterion (Zhu and Zeng (2005) and so on.

Unlimitedly, for the maximal designs of resolution IV proposed by Chen and Cheng (2006), we find that a $\boldsymbol{2^{n-m}}$ design of resolution IV is maximal if and only if the design satisfies the two conditions: $\boldsymbol{{}_{1}^{\#}C_{2}^{(0)} = n}$ and $\boldsymbol{\sum_{k\geq 1, j\geq 3} {}_{j}^{\#}C_{2}^{(k)} + \binom{n}{2} = 2^n - (n+1)2^m}$.

Appendix

Table 6. 32-run GMLOC designs and comparisons with the MA and clear effects criteria

designs	add. columns	${}^{\#}_{1}C_2; {}^{\#}_{2}C_1; {}^{\#}_{2}C_2$	WLP	Cs	Orders G,M,C
8-3.1	30 7 11	8 ; 28 ; 13,12,3	0, 3, 4, 0	8, 13	1, 1, 1
8-3.2	28 14 22	8 ; 28 ; 7,0,21	0, 7, 0, 0	8, 7	2, 4, 2
8-3.3	28 14 7	8 ; 28 ; 4,18,6	0, 5, 0, 2	8, 4	3, 2, 3
8-3.4	28 14 13	8 ; 28 ; 0,24,0,4	0, 6, 0, 0	8, 0	4, 3, 4
8-3.5	30 7 12	5,3 ; 25,3 ; 16,12	1, 2, 3, 1	5, 13	5, 5, 5
8-3.6	28 14 3	5,3 ; 25,3 ; 13,12,3	1, 3, 2, 0	5, 10	6, 6, 6
8-3.7	30 7 3	3,4,1 ; 22,6 ; 22,6	2, 1, 2, 2	3, 18	7, 7, 7
8-3.8	28 6 3	3,4,1 ; 22,6 ; 16,12	2, 2, 1, 1	3, 12	8, 8, 8
8-3.9	30 7 6	3,4,1 ; 22,6 ; 16,12	2, 2, 2, 0	3, 12	9, 9, 8
8-3.10	28 14 6	3,4,1 ; 22,6 ; 13,12,3	2, 3, 2, 0	3, 9	10,10,10
9-4.1	30 7 11 13	9 ; 36 ; 15,0,21	0, 7, 7, 0	9, 15	1, 2, 1
9-4.2	30 7 11 19	9 ; 36 ; 8,24,0,4	0, 6, 8, 0	9, 8	2, 1, 2
9-4.3	28 14 22 26	9 ; 36 ; 8,$\mathbf{0}^2$,28	0, 14, 0, 0	9, 8	3, 5, 2
9-4.4	28 14 13 7	9 ; 36 ; 2,12,18,4	0, 10, 0, 4	9, 2	4, 4, 4
9-4.5	28 14 7 19	9 ; 36 ; 0,18,18	0, 9, 0, 6	9, 0	5, 3, 5
9-4.6	28 14 22 3	6,3 ; 33,3 ; 15,0,21	1, 7, 4, 0	6, 12	6, 7, 6
9-4.7	30 7 11 24	6,3 ; 33,3 ; 12,18,6	1, 5, 6, 2	6, 9	7, 6, 7
9-4.8	30 7 11 6	4,4,1 ; 30,6 ; 15,18,3	2, 4, 6, 2	4, 11	8, 9, 8
9-4.9	28 14 7 3	4,4,1 ; 30,6 ; 12,18,6	2, 5, 4, 2	4, 8	9, 10,9
9-4.10	28 14 7 10	4,4,1 ; 30,6 ; 12,18,6	2, 5, 5, 2	4, 8	10,11,9

Table 6. 32-run GMLOC designs and comparisons with the MA and clear effects criteria (continued)

designs	add. columns	${}^{\#}_{1}C_2$; ${}^{\#}_{2}C_1$; ${}^{\#}_{2}C_2$	WLP	Cs	Orders G,M,C
10-5.1	30 7 11 19 29	10 ; 45 ; 0,40,$\mathbf{0^2}$,5	0, 10, 16, 0	10, 0	1, 1, 1
10-5.2	28 14 22 26 7	10 ; 45 ; 0,16,0,24,5	0, 18, 0, 8	10, 0	2, 4, 1
10-5.3	28 14 7 19 11	10 ; 45 ; 0,6,27,12	0, 16, 0, 12	10, 0	3, 3, 1
10-5.4	28 14 7 19 25	10 ; 45 ; $\mathbf{0^2}$,45	0, 15, 0, 15	10, 0	4, 2, 1
10-5.5	28 14 22 26 3	7,3 ; 42,3 ; 17,$\mathbf{0^2}$,28	1, 14, 7, 0	7, 14	5, 6, 5
10-5.6	30 7 11 19 14	7,3 ; 42,3 ; 11,12,18,4	1, 10, 11, 4	7, 8	6, 5, 6
10-5.7	28 14 22 3 5	5,4,1 ; 39,6 ; 11,12,18,4	2, 10, 8, 4	5, 7	7, 10, 7
10-5.8	28 14 7 19 5	5,4,1 ; 39,6 ; 9,18,18	2, 9, 9, 6	5, 5	8, 9, 8
10-5.9	30 7 11 19 6	5,4,1 ; 39,6 ; 8,30,3,4	2, 8, 12, 4	5, 4	9, 8, 9
10-5.10	30 7 11 24 21	4,6 ; 39,6 ; 12,24,9	2, 7, 12, 7	4, 6	10, 7, 48
11-6. 1	28 14 22 26 7 11	11 ; 55 ; $\mathbf{0^2}$,24,16,15	0, 26, 0, 24	11, 0	1, 2, 1
11-6. 2	28 14 7 19 25 11	11 ; 55 ; $\mathbf{0^2}$,15,40	0, 25, 0, 27	11, 0	2, 1, 1
11-6. 3	28 14 22 26 7 3	6,4,1 ; 49,6 ; 10,16,0,24,5	2, 18, 14, 8	6, 6	3, 5, 3
11-6. 4	28 14 7 19 11 17	6,4,1 ; 49,6 ; 10,6,27,12	2, 16, 16, 12	6, 6	4, 4, 3
11-6. 5	30 7 11 19 29 6	6,4,1 ; 49,6 ; 4,28,18,0,5	2, 14, 22, 8	6, 0	5, 3, 5
11-6. 6	30 7 11 19 6 5	5,0,6 ; 43,12 ; 4,28,18,0,5	4, 14, 16, 8	5, 4	6, 66, 6
11-6. 7	28 14 7 19 11 18	4,6,0,1 ; 46,9 ; 10,6,27,12	3, 16, 12, 12	4, 4	7, 8, 7
11-6. 8	28 14 7 19 11 6	4,6,0,1 ; 46,9 ; 10,6,27,12	3, 16, 13, 12	4, 4	8, 9, 7
11-6. 9	28 14 7 19 25 3	4,6,0,1 ; 46,9 ; 10,0,45	3, 15, 13, 15	4, 4	9, 7, 7
11-6.10	30 7 11 24 21 14	4,5,2 ; 46,9 ; 8,24,15,8	3, 13, 19, 11	4, 3	10, 6, 10

Table 13. 64-run GMLOC designs and comparisons with MA and Clear criteria (continued)

designs	add. columns	${}^{\#}_{1}C_2; {}^{\#}_{2}C_1; {}^{\#}_{2}C_2$	WLP	Cs	Orders G,M,C
13-7.1	60 15 22 39 21 59 19	13; 78; 36,0,42	0, 14, 33, 16	13, 36	1, 2, 1
13-7.2	60 14 22 11 19 7 13	13; 78; 23,0,24,16,15	0, 26, 12, 24	13, 23	2, 37, 2
13-7.3	60 14 22 26 7 11 13	13; 78; 23,0,24,16,15	0, 26, 13, 24	13, 23	3, 38, 2
13-7.4	60 14 22 26 7 11 19	13; 78; 23,0,15,40	0, 25, 13, 27	13, 23	4, 34, 2
13-7.5	60 15 22 39 19 46 21	13; 78; 22,30,18,8	0, 15, 28, 20	13, 22	5, 5, 5
13-7.6	56 11 22 7 35 19 45	13; 78; 21,16,36,0,5	0, 18, 21, 24	13, 21	6, 14, 6
13-7.7	60 15 22 39 19 41 26	13; 78; 20,36,18,4	0, 14, 28, 24	13, 20	7, 1, 7
13-7.8	60 14 19 7 37 26 11	13; 78; 20,18,24,16	0, 19, 19, 25	13, 20	8, 16, 7
13-7.9	60 14 22 38 11 19 25	13; 78; 20,18,24,16	0, 19, 20, 24	13, 20	9, 17, 7
13-7.10	56 28 14 38 50 23 13	13; 78; 20,12,42,4	0, 18, 20, 28	13, 20	10, 13, 7
14-8.1	60 14 22 11 19 7 13 21	14; 91; 25,$\mathbf{0^2}$,48,0,18	0, 39, 16, 48	14, 25	1, 42, 1
14-8.2	60 14 22 26 7 11 19 13	14; 91; 25,$\mathbf{0^2}$,36,30	0, 38, 17, 52	14, 25	2, 40, 1
14-8.3	60 14 19 7 37 26 11 13	14; 91; 19,16,24,12,20	0, 30, 25, 44	14, 19	3, 17, 3
14-8.4	60 14 22 38 11 19 25 7	14; 91; 19,16,15,36,5	0, 29, 26, 46	14, 19	4, 15, 3
14-8.5	56 11 22 7 35 19 45 28	14; 91; 18,16,36,16,5	0, 26, 29, 48	14, 18	5, 11, 5
14-8.6	60 15 22 39 19 46 21 43	14; 91; 16,34,24,12,5	0, 23, 38, 38	14, 16	6, 5, 6
14-8.7	60 14 22 38 58 11 19 25	14; 91; 16,34,24,12,5	0, 23, 40, 36	14, 16	7, 6, 6
14-8.8	56 28 14 38 50 23 13 27	14; 91; 16,28,42,0,5	0, 22, 40, 41	14, 16	8, 2, 6
14-8.9	60 14 22 38 11 19 35 25	14; 91; 16,24,27,24	0, 25, 30, 50	14, 16	9, 9, 6
14-8.10	60 14 22 38 11 19 37 31	14; 91; 16,18,45,12	0, 24, 31, 54	14, 16	10, 7, 6

Table 14. AENPs of $\boldsymbol{d}_1$, $\boldsymbol{d}_2$ and $\boldsymbol{d}_7$ in Example 6

$\#_i C_j^{(k)}(\boldsymbol{d}_1)$	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$	$j=9$
$i=0$	1	1	1	1	$0^6,1$	$0^8,1$	1	1	$0,1$	1
$i=1$	9	9	9	$1,0^2,8$	$0^4,8,0^3,1$	$0^3,8,0^2,1$	$1,0^3,8$	$1,8$	9	$8,1$
$i=2$	36	36	$8,24,0,4$	$4,0,24,0,8$	$4,0^2,8,24$	$0^4,32,0^3,4$	$0^2,24,8,4$	$12,0,24$	$28,8$	36
$i=3$	84	$60,24$	$28,32,24$	$0,24,24,36$	$0^3,32,48,0^3,4$	$4,0^2,24,56$	$4,0,24,32,24$	$32,24,24,0,4$	$52,32$	84
$i=4$	$120,6$	$86,40$	$54,24,48$	$14,0,48,32,32$	$0^2,24,80,0,6,0,16$	$8,0^2,32,72,0,8,0,6$	$22,0,48,24,32$	$38,32,48,0,8$	$96,30$	$118,8$
$i=5$	$118,8$	$96,30$	$38,32,48,0,8$	$22,0,48,24,32$	$8,0^2,32,72,0,8,0,6$	$0^2,24,80,0,6,0,16$	$14,0,48,32,32$	$54,24,48$	$86,40$	$120,6$
$i=6$	84	$52,32$	$32,24,24,0,4$	$4,0,24,32,24$	$4,0^2,24,56$	$0^3,32,48,0^3,4$	$0,24,24,36$	$28,32,24$	$60,24$	84
$i=7$	36	$28,8$	$12,0,24$	$0^2,24,8,4$	$0^4,32,0^3,4$	$4,0^2,8,24$	$4,0,24,0,8$	$8,24,0,4$	36	36
$i=8$	$8,1$	9	$1,8$	$1,0^3,8$	$0^3,8,0^2,1$	$0^4,8,0^3,1$	$1,0^2,8$	9	9	9
$i=9$	1	$0,1$	1	1	$0^8,1$	$0^6,1$	1	1	1	1

$\#_i C_j^{(k)}(\boldsymbol{d}_2)$	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$	$j=9$
$i=0$	1	1	1	1	$0^7,1$	$0^7,1$	1	1	1	$0,1$
$i=1$	9	9	9	$2,0^3,7$	$0^3,7,0^3,2$	$0^3,7,0^3,2$	$2,0^3,7$	9	$0,9$	9
$i=2$	36	36	$15,0,21$	$0,21,0,14,0^3,1$	$1,0^3,35$	$1,0^3,35$	$0,21,0,14,0^3,1$	$0,15,0,21$	36	36
$i=3$	84	$56,28$	$28,49,0,7$	$7,0,42,28,0^2,7$	$7,0^2,28,49$	$7,0^2,28,49$	$0,7,0,42,28,0^2,7$	$28,49,0,7$	$56,28$	84
$i=4$	$119,7$	$91,35$	$42,56,0,28$	$21,28,0,56,21$	$0^2,21,84,0^2,21$	$0^3,21,84,0^2,21$	$21,28,0,56,21$	$42,56,0,28$	$91,35$	$119,7$
$i=5$	$119,7$	$91,35$	$42,56,0,28$	$21,28,0,56,21$	$0^3,21,84,0^2,21$	$0^2,21,84,0^2,21$	$21,28,0,56,21$	$42,56,0,28$	$91,35$	$119,7$
$i=6$	84	$56,28$	$28,49,0,7$	$0,7,0,42,28,0^2,7$	$7,0^2,28,49$	$7,0^2,28,49$	$7,0,42,28,0^2,7$	$28,49,0,7$	$56,28$	84
$i=7$	36	36	$0,15,0,21$	$0,21,0,14,0^3,1$	$1,0^3,35$	$1,0^3,35$	$0,21,0,14,0^3,1$	$15,0,21$	36	36
$i=8$	9	$0,9$	9	$2,0^3,7$	$0^3,7,0^3,2$	$0^3,7,0^3,2$	$2,0^3,7$	9	9	9
$i=9$	$0,1$	1	1	1	$0^7,1$	$0^7,1$	1	1	1	1

$\#_i C_j^{(k)}(\boldsymbol{d}_7)$	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$	$j=9$
$i=0$	1	1	1	1	$0^{14},1$	1	1	1	$0,1$	1
$i=1$	9	9	9	$1,0^6,8$	9	$0^7,8,0^6,1$	9	$1,8$	9	$8,1$
$i=2$	36	36	$8,0^2,28$	36	$0^7,8,28$	36	$0^4,28,0^2,8$	36	$28,8$	36
$i=3$	84	$28,56$	84	$0^3,28,0^2,56$	84	$0^7,56,28$	84	$0,56,0^2,28$	84	84
$i=4$	$112,14$	126	$14,56,0^2,56$	126	$0^6,56,56,0^5,14$	126	$14,0^3,56,0^2,56$	126	$56,70$	126
$i=5$	126	$56,70$	126	$14,0^3,56,0^2,56$	126	$0^6,56,56,0^5,14$	126	$14,56,0^2,56$	126	$112,14$
$i=6$	84	84	$0,56,0^2,28$	84	$0^7,56,28$	84	$0^3,28,0^2,56$	84	$28,56$	84
$i=7$	36	$28,8$	36	$0^4,28,0^2,8$	36	$0^7,8,28$	36	$8,0^2,28$	36	36
$i=8$	$8,1$	9	$1,8$	9	$0^7,8,0^6,1$	9	$1,0^6,8$	9	9	9
$i=9$	1	$0,1$	1	1	1	$0^{14},1$	1	1	1	1

References

Ai, M.Y., Zhang, R.C., 2004. Theory of minimum aberration blocked regular mixed factorial designs. J. Statist. Plann. Inference 126, 305–323.

Ai, M.Y., Zhang, R.C., 2004. s^{n-m} designs containing clear main effects or two-factor interactions, Statist. Probab. letters 69, 151–160.

Ai, M.Y., Zhang, R.C., 2004. Multistratum fractional factorial split-plot designs with minimum aberration and maximum estimation capacity, Statist. Probab. letters, 69, 161–170.

Box, G.E.P., Hunter, J.S., 1961. The 2^{k-p} fractional factorial designs. Technometrics 3, 311–351 and 449–458.

Chen, J., 1992. Some results on 2^{n-k} fractional factorial designs and search for minimum aberration designs. Ann. Statist. 20, 2124–2141.

Chen, H., Cheng, C.S., 2006. Doubling and projection: A method of constructing two-level designs of resolution IV. Ann. Statist. 34, 546–558.

Chen, B.J., Li, P.F., Liu, M.Q., Zhang, R.C., 2005. Some results on blocked regular 2-level fractional factorial designs with clear effects. J. Statist. Plann. Inference, in press.

Chen, H., Hedayat, A.S., 1996. 2^{n-l} designs with weak minimum aberration. Ann. Statist. 24, 2536–2548.

Chen, H., Hedayat, A.S., 1998. 2^{n-m} designs with resolution **III** and **IV** containing clear two-factor interactions. J. Statist. Plann. Inference 75, 147–158.

Chen, J., Wu, C.F.J., 1991. Some results on s^{n-k} fractional factorial designs with minimum aberration or optimal moments. Ann. Statist. 19, 1028–1041.

Cheng, C.S., Mukerjee, R., 1998. Regular fractional factorial designs with minimum aberration and maximum estimation capacity. Ann. Statist. 26, 2289–2300.

Cheng, C.S., Steinberg, D.M., Sun, D.X., 1999. Minimum aberration and model robustness for two-level factorial designs. J. Roy. Statist. Soc. Ser. B 61, 85–93.

Cheng C.S., Tang B., 2005. A general theory of minimum aberration and its applications. Ann. Statist. 33, 944–958.

Franklin, M.F., 1984. Constructing tables of minimum aberration p^{n-m} designs. Technometrics 26, 225–232.

Fries, A., Hunter, W.G., 1980. Minimum aberration 2^{k-p} designs. Technometrics 22, 601–608.

Mukerjee, R., Wu, C.F.J., 2001. Minimum aberration designs for mixed Factorials in terms of complementary sets. Statist. Sinica 11, 225–239.

Suen, C.Y., Chen, H., Wu, C.F.J., 1997. Some identities on q^{n-m} designs with application to minimum aberrations. Ann. Statist. 25, 1176–1188.

Sun, D.X., 1993. Estimation capacity and related topics in experimental designs. PhD dissertation. University of Waterloo, Waterloo.

Tang, B., Ma, F., Ingram, D., Wang, H., 2002. Bounds on the maximum number of clear two-factor interactions for 2^{m-p} designs of resolution III and IV. Canad. J. Statist. 30, 127–136.

Tang, B., Wu, C.F.J., 1996. Characterization of minimum aberration 2^{n-k} designs in terms of their complementary designs. Ann. Statist. 25, 1176–1188.

Wu, C.F.J., Chen, Y., 1992. A graph-aided method for planning two-level experiments when certain interactions are important. Technometrics 34, 162–175.

Wu , C.F.J., Hamada, M., 2000. Experiments: Planning, Analysis, and Parameter Design Optimization. Wiley, Now York.

Wu, H.Q., Wu, C.F.J., 2002. Clear two-factor interaction and minimum aberration. Ann. Statist. 30, 1496–1511.

Yang, G.J., Liu, M.Q., Zhang, R.C., 2005. Weak minimum aberration and maximum number of clear two-factor interactions in $2_{\mathbf{IV}}^{m-p}$ designs. Sci. China Ser. A 48, in press.

Yang, J.F., Li, P.F., Liu, M.Q., Zhang, R.C., 2005. $2^{(n_1-n_2)-(k_1-k_2)}$ fractional factorial split-plot designs containing clear effects. J. Statist. Plann. Inference, in press.

Zhang, R.C., Park, D.K., 2000. Optimal blocking of two-level fractional factorial designs. J. Statist. Plann. Inference 91, 107–121.

Zhang, R.C., Shao, Q., 2001. Minimum aberration $(s^2)s^{n-k}$ designs. Statist. Sinica 11, 213–223.

Zhao, S.L., Zhang, R.C., 2005. 4^m2^n designs with resolution III or IV containing clear two-factor interaction components. Proceedings of the Fifth Eastern Asia Symposium on Statistics and Its Applications, 187–196.

Zhu, Y., Zeng P., 2005. On the coset pattern matrices and minimum M-aberration of 2^{n-p} designs. Statist. Sinica 15, 717–730.