# Orthogonal Arrays Obtained By Generalized Kronecker Product

Yingshan Zhang

Department of Statistics, East China Normal University,

Shanghai,200062, P.R.China. Email: ysh_zhang@163.com

# The generalized Kronecker product

Let $k(x, y)$ be a map from $\Omega_1 \times \Omega_2$ to $V$, where $\Omega_1 \times \Omega_2 = \{(x, y) : x \in \Omega_1, y \in \Omega_2\}$ and $\Omega_1, \Omega_2, V$ are some sets. For two matrices $A = (a_{ij})_{n\times m}$ with entries from $\Omega_1$ and $B = (b_{uv})_{s\times t}$ with entries from $\Omega_2$, define their **generalized Kronecker product**, denoted by $\overset{k}{\otimes}$, as follows

$$A \overset{k}{\otimes} B = (k(a_{ij}, b_{uv}))_{ns\times mt} = (k(a_{ij}, B))_{1\le i\le n, 1\le j\le m}, \qquad (1)$$

where each submatrix $k(a_{ij}, B) = (k(a_{ij}, b_{uv}))_{s\times t}$ of $A \overset{k}{\otimes} B$ is obtained by operating $a_{ij}$ to each entry of $B$ under the map $k(x, y)$.

# The atomic difference matrix

If a difference matrix $D(\lambda p, m; p)$ exists, it can always be constructed so that only one of its rows and one of its columns contain the zero element of $G$. Deleting this column from $D(\lambda p, m; p)$, we obtain a difference matrix, denoted by $D^0(\lambda p, m-1; p)$, called **an atom of difference matrix** $D(\lambda p, m; p)$ or **an atomic difference matrix.** Without loss of generality, the matrix $D(\lambda p, m; p)$ can be written as

$$D(\lambda p, m; p) = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} = (0 \quad D^0(\lambda p, m-1; p)). \qquad (2)$$

The property is important for the following discussions.

# The normal Kronecker sum

If the $\Omega_1, \Omega_2$ and $V$ are additive (or abelian) groups $G_1, G_2$ of order $\lambda p$, $p$ and a row-vector space of $m$-dimensions respectively, and if $k(i,j)$ is the $(ip+j+1)$th row of $D^0(\lambda p, m-1; p) \oplus (p)$(i.e., the usual Kronecker sum $\oplus$ of the atomic difference matrix $D^0(\lambda p, m-1; p)$ and $(p)$ (Shrikhande 1964)), the generalized Kronecker product $\overset{k}{\otimes}$ is really denoted by $(\lambda p) \overset{k}{\otimes} (p) = D^0(\lambda p, m-1; p) \oplus (p)$, namely **normal Kronecker sum.**

Such as $(2) \overset{k}{\otimes} (2) = D^0(2,1;2) \oplus (2)$,

$(4) \overset{k}{\otimes} (2) = D^0(4,3;2) \oplus (2)$, $(3) \overset{k}{\otimes} (3) = D^0(3,2;3) \oplus (3)$,

$(6) \overset{k}{\otimes} (3) = D^0(6,5;3) \oplus (3), \cdots$

# The array product

There is an orthogonal array

$((3)\oplus 0_6, 0_3\oplus(6)) \overset{K}{\otimes} (3) = (((3)\oplus 0_6) \overset{k_1}{\otimes} (3), (0_3\oplus(6)) \overset{k_2}{\otimes} (3))$, if define $K=\{k_1,k_2\}$ and

$$(3)\overset{k_1}{\otimes}(3) = \begin{pmatrix} 0 & 0 \\ 1 & 2 \\ 2 & 1 \end{pmatrix} \oplus (3), \quad (6)\overset{k_2}{\otimes}(3) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 2 \\ 1 & 2 & 0 & 1 & 2 \\ 1 & 0 & 2 & 2 & 1 \\ 2 & 1 & 2 & 1 & 0 \\ 2 & 2 & 1 & 0 & 1 \end{pmatrix} \oplus (3).$$

The array product is an essential operation of the generalized Kronecker product for constructing asymmetrical arrays.

# Matrix Images (MI)

Let $A$ be an orthogonal array of strength 1, i.e.,

$$A = (a_1, \ldots, a_m) = (T_1(0_{r_1} \oplus (p_1)), \ldots, T_m(0_{r_m} \oplus (p_m))),$$

where $r_i p_i = n, T_i$ is a permutation matrix for any $i = 1, \ldots, m$. The following projection matrix,

$$A_j = T_j(P_{r_j} \otimes \tau_{p_j})T_j^T, \tag{3}$$

is called the **matrix image** (MI) of the $j$th column $a_j$ of $A$, denoted by $m(a_j) = A_j$ for $j = 1, \ldots, m$. In general,the MI of a subarray of $A$ is defined as the sum of the MI's of all its columns. In particular, we denote the MI of $A$ by $m(A)$.

# Basic Theorems

- Let $D^0(\lambda p, m-1; p)$ be an atom of difference matrix $D(\lambda p, m; p)$. Then $D^0(\lambda p, m-1; p) \oplus (p)$ is an orthogonal array whose MI is less than or equal to $\tau_{\lambda p} \otimes \tau_p$, where $\tau_{\lambda p} = I_{\lambda p} - P_{\lambda p}$ and $\tau_p = I_p - P_p$.
- Suppose that $L_{n_1} = [L_{n_1}(p_1^{x_1}), \ldots, L_{n_1}(p_s^{x_s})]$ and $L_{n_2} = [L_{n_2}(q_1^{y_1}), \ldots, L_{n_2}(q_t^{y_t})]$ are two orthogonal arrays. Then the array product of $L_{n_1}$ and $L_{n_2}$ , i.e., $L_{n_1} \overset{K}{\otimes} L_{n_2}$, is also orthogonal array whose MI is less than or equal to $m(L_{n_1}) \otimes m(L_{n_2})$.

# Basic Corollaries

- **(Two-factor method )** Let $L_p^1, L_p^2, L_q^1$ and $L_q^2$ be orthogonal arrays.Then $(L_p^1 \oplus 0_q, 0_p \oplus L_q^1, L_p^2 \overset{K}{\otimes} L_q^2)$ is an orthogonal array.
- **(Three-factor method)** Let $n = prq$ and let $L_{pr}, L_{rq}$ and $L_q$ be orthogonal arrays of run sizes $pr, rq$ and $q$, respectively. If there exist orthogonal arrays $L_{pr}^{(-)}$, $L_{pr}^{(=)}$ and $L_{rq}^{(-)}$ such that $m(L_{pr}^{(-)}), m(L_{pr}^{(=)}) \leq \tau_p \otimes I_r$ and $m(L_{rq}^{(-)}) \leq I_r \otimes \tau_q$, then $[L_{pr} \oplus 0_q, 0_p \oplus L_{rq}^{(-)}, L_{pr}^{(=)} \overset{K}{\otimes} L_q]$ and $[L_{pr}^{(-)} \oplus 0_q, 0_p \oplus L_{rq}, L_{pr}^{(=)} \overset{K}{\otimes} L_q]$ are orthogonal arrays.

# Constructions of OA's with Run Size 72

- $L_{72}(\cdots 4^1) = [L_{36}^{(-)}(\cdots) \oplus 0_2, 0_{18} \oplus (4), L_{36}^{(=)}(2^{34}) \overset{k}{\otimes} (2)]$
where $m(L_{36}^{(-)}(\cdots)), m(L_{36}^{(=)}(2^{34})) \leq \tau_{18} \otimes I_2$, such as
$L_{72}(2^{61}3^1 4^1) = [L_{36}^{(-)}(2^{27}3^1) \oplus 0_2, 0_{18} \oplus (4), L_{32}^{(=)}(2^{34}) \oplus (2)]$

- $L_{72}(\cdots 6^1) = [L_{36}^{(-)} \oplus 0_2, 0_{12} \oplus (6), L_{36}^{(=)}(2^{28}) \overset{k}{\otimes} (2)]$where
$m(L_{36}^{(-)}(\cdots)), m(L_{36}^{(=)}(2^{28})) \leq \tau_{12} \otimes I_3$, such as
$L_{72}(2^{28}3^{11}6^1 12^1) =$
$[L_{36}^{(-)}(3^{11}12^1) \oplus 0_2, 0_{12} \oplus (6), L_{36}^{(=)}(2^{28}) \oplus (2)]$

- $L_{72}(\cdots 12^1) = [L_{36}^{(-)} \oplus 0_2, 0_6 \oplus (12), L_{36}^{(=)}(2^{28}) \overset{k}{\otimes} (2)]$where
$m(L_{36}^{(-)}(\cdots)), m(L_{36}^{(=)}(2^{18})) \leq \tau_6 \otimes I_6$, such as
$L_{72}(2^{18}3^7 6^2 12^1) = [L_{36}^{(-)}(3^7 6^2) \oplus 0_2, 0_6 \oplus (12), L_{36}^{(=)}(2^{18}) \oplus (2)]$

# Constructions of OA’s with Run Size 96

- Step 1. There is an orthogonal decomposition of the projection matrix $\tau_{96}$ as follows:

$$\tau_{96} = I_{24} \otimes \tau_4 + \tau_{24} \otimes P_8 =$$

$$\sum_{i=1}^{3} M_i(P_3 \otimes \tau_4 \otimes I_2 \otimes \tau_4) M_i^T + (I_2 \otimes \tau_4 \otimes P_3 \otimes \tau_4 + \tau_{24} \otimes P_4), \quad (4)$$

where $M_i = (I_2 \otimes T_i)K(8, 12)$ for $i = 1, 2, 3$;

$$T_1 = \text{diag}(I_2 \otimes N_2, N_2 \otimes I_2, N_2 \otimes N_2, K(3,3) \otimes I_4),$$

$$T_2 = \text{diag}(N_2 \otimes I_2, N_2 \otimes N_2, I_2 \otimes N_2, [\text{diag}(I_3, N_3, N_3^2)K(3,3)] \otimes$$

$$T_3 = \text{diag}(N_2 \otimes N_2, I_2 \otimes N_2, N_2 \otimes I_2, [\text{diag}(I_3, N_3^2, N_3)K(3,3)] \otimes$$

# Constructions of OA's with Run Size 96

- Step 2. There is an orthogonal array $L_{32}(2^3 4^5)$ such that $m(L_{32}(2^3 4^5)) = \tau_4 \otimes I_2 \otimes \tau_4$,where $L_{32}(2^3 4^5) = [0_2 \oplus (2) \oplus 0_2 \oplus (2) \oplus 0_2, (2) \oplus 0_8 \oplus (2),(2) \oplus (2) \oplus (2) \oplus (2) \oplus (2), D(8,5;4) \oplus (4)],$

$$D(8,5;4) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 3 & 2 \\ 1 & 3 & 3 & 2 & 1 \\ 1 & 2 & 0 & 1 & 3 \\ 2 & 2 & 1 & 3 & 3 \\ 2 & 3 & 2 & 0 & 1 \\ 3 & 1 & 2 & 1 & 2 \\ 3 & 0 & 1 & 2 & 0 \end{pmatrix}.$$

# Constructions of OA's with Run Size 96

- Step 2. There are the following orthogonal arrays $L_{96}^{(-)}(\cdots)$ such that $m(L_{96}^{(-)}(\cdots)) \leq (I_2 \otimes \tau_4 \otimes P_3 \otimes \tau_4 + \tau_{24} \otimes P_4)$.
  1. $L_{96}^{(-)}(4^5 24^1) = [M_0(0_3 \oplus L_{32}(4^5 2^3)), (24) \oplus 0_4]$ where $M_0 = K(8,12)$.
  2. $L_{96}^{(-)}(2^9 4^7 12^1) = [K(8,12)(0_3 \oplus (8) \overset{k}{\otimes} (4)), L_{24}^{(-)}(2^9 12^1) \oplus 0_4]$where$L_{24}(2^{12}12^1) = [(2) \oplus L_4(2^3) \oplus 0_3, (L_{24}^{(-)}(2^9 12^1)]$.
  3. $L_{96}^{(-)}(2^{17} 4^8) = [K(8,12)(0_3 \oplus (8) \overset{k}{\otimes} (4)), L_{24}^{(-)}(2^{17} 4^1) \oplus 0_4]$where$L_{24}(2^{20} 4^1)) = [(2) \oplus L_4(2^3) \oplus 0_3, L_{24}^{(-)}(2^{17} 4^1)]$.
  4. $L_{96}^{(-)}(2^{10} 3^1 4^8) = [K(8,12)(0_3 \oplus (8) \overset{k}{\otimes} (4)), L_{24}^{(-)}(2^{10} 3^1 4^1) \oplus 0_4]$where$L_{24}(2^{13} 3^1 4^1)) = [(2) \oplus L_4(2^3) \oplus 0_3, L_{24}^{(-)}(2^{10} 3^1 4^1)]$.

華東師範大學 East China Normal University

# Constructions of OA's with Run Size 96

- Step 2.There are the following orthogonal arrays $L_{96}^{(-)}(\cdots)$ such that $m(L_{96}^{(-)}(\cdots)) \leq (I_2 \otimes \tau_4 \otimes P_3 \otimes \tau_4 + \tau_{24} \otimes P_4)$.
  - $L_{96}^{(-)}(2^8 6^1 4^8) = [K(8,12)(0_3 \oplus (8) \overset{k}{\otimes} (4)), L_{24}^{(-)}(2^8 4^1 6^1) \oplus 0_4]$where$L_{24}(2^{11}6^1 4^1)) = [(2) \oplus L_4(2^3) \oplus 0_3, L_{24}^{(-)}(2^8 4^1 6^1)]$ and $(8) \overset{k}{\otimes} (4) = D^0(8,7;4) \oplus (4) =$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 & 2 & 3 & 2 \\ 1 & 3 & 2 & 1 & 3 & 0 & 2 \\ 1 & 2 & 3 & 2 & 1 & 3 & 0 \\ 2 & 2 & 0 & 3 & 3 & 1 & 1 \\ 2 & 3 & 1 & 0 & 1 & 2 & 3 \\ 3 & 1 & 2 & 2 & 0 & 1 & 3 \\ 3 & 0 & 3 & 1 & 2 & 2 & 1 \end{pmatrix} \oplus (4).$$

# Constructions of OA's with Run Size 96

- Step 2.There are the following orthogonal arrays $L_{96}^{(-)}(\cdots)$ such that $m(L_{96}^{(-)}(\cdots)) \leq (I_2 \otimes \tau_4 \otimes P_3 \otimes \tau_4 + \tau_{24} \otimes P_4)$.
  - $L_{96}^{(-)}(2^{11}4^4 8^1 12^1) = [L_{96}^{(=)}(2^2 4^4 8^1), L_{24}^{(=)}(2^9 12^1) \oplus 0_4]$where $L_{24}(2^{12}12^1) = [(2) \oplus [0_4, (2) \oplus 0_2, 0_2 \oplus (2)] \oplus 0_3, L_{24}^{(=)}(2^9 12^1)]$ and $L_{96}^{(=)}(2^2 4^4 8^1) =$
    $[((2) \oplus 0_{48}) \diamond (0_2 \oplus (2) \oplus 0_6 \oplus (2) \oplus 0_2) \diamond ((2) \oplus 0_2 \oplus (2) \oplus 0_6 \oplus (2)),$
    $((2) \oplus (2) \oplus 0_{24}) \diamond (0_4 \oplus (2) \oplus 0_3 \oplus (2) \oplus 0_2),$
    $((2) \oplus 0_2 \oplus (2) \oplus 0_{12}) \diamond (0_2 \oplus (2) \oplus 0_{12} \oplus (2)),$
    $((2) \oplus (2) \oplus 0_6 \oplus (2) \oplus (2)) \diamond (0_2 \oplus (2) \oplus (2) \oplus 0_6 \oplus (2)),$
    $((2) \oplus (2) \oplus 0_{12} \oplus (2)) \diamond (0_2 \oplus (2) \oplus (2) \oplus 0_3 \oplus (2) \oplus 0_2),$
    $0_2 \oplus (2) \oplus 0_6 \oplus (2) \oplus (2),\ 0_4 \oplus (2) \oplus 0_3 \oplus (2) \oplus (2)].$

# Constructions of OA's with Run Size 96

- Step 2.There are the following orthogonal arrays $L_{96}^{(-)}(\cdots)$ such that $m(L_{96}^{(-)}(\cdots)) \leq (I_2 \otimes \tau_4 \otimes P_3 \otimes \tau_4 + \tau_{24} \otimes P_4)$.
  - $L_{96}^{(-)}(2^{19}4^5 8^1) = [L_{96}^{(=)}(2^2 4^4 8^1), L_{24}^{(=)}(2^{17}4^1) \oplus 0_4]$where $L_{24}(2^{20}4^1)) = [(2) \oplus [0_4, (2) \oplus 0_2, 0_2 \oplus (2)] \oplus 0_3, L_{24}^{(=)}(2^{17}4^1)]$.
  - $L_{96}^{(-)}(2^{12}3^1 4^5 8^1) = [L_{96}^{(=)}(2^2 4^4 8^1), L_{24}^{(=)}(2^{10}3^1 4^1) \oplus 0_4]$where $L_{24}(2^{13}3^1 4^1)) =$ $[(2) \oplus [0_4, (2) \oplus 0_2, 0_2 \oplus (2)] \oplus 0_3, L_{24}^{(=)}(2^{10}3^1 4^1)]$.
  - $L_{96}^{(-)}(2^{10}4^5 6^1 8^1) = [L_{96}^{(=)}(2^2 4^4 8^1), L_{24}^{(=)}(2^8 4^1 6^1) \oplus 0_4]$where $L_{24}(2^{11}6^1 4^1)) =$ $[(2) \oplus [0_4, (2) \oplus 0_2, 0_2 \oplus (2)] \oplus 0_3, L_{24}^{(=)}(2^8 4^1 6^1)]$.

  The $L_{24}^{(-)}(\cdots)$ and $L_{24}^{(=)}(\cdots)$ due to Zhang et al (2001).

# Constructions of OA's with Run Size 96

- Step 3. We lay out the new orthogonal arrays $L_{96}(2^{12}4^{20}24^1) = [M_1(0_3 \oplus L_{32}(2^3 4^5)), M_2(0_3 \oplus L_{32}(2^3 4^5)), M_3(0_3 \oplus L_{32}(2^3 4^5)), L_{96}^{(-)}(\cdots)]$, such as
  - $L_{96}(2^{12}4^{20}24^1) = [M_1(0_3 \oplus L_{32}(2^3 4^5)), M_2(0_3 \oplus L_{32}(2^3 4^5)), M_3(0_3 \oplus L_{32}(2^3 4^5)), M_0(0_3 \oplus L_{32}(2^3 4^5)), (24) \oplus 0_4]$.
  - $L_{96}(2^{18}4^{22}12^1) = [M_1(0_3 \oplus L_{32}(2^3 4^5)), M_2(0_3 \oplus L_{32}(2^3 4^5)), M_3(0_3 \oplus L_{32}(2^3 4^5)), L_{96}^{(-)}(2^9 4^7 12^1)]$.
  - $L_{96}(2^{26}4^{23}), L_{96}(2^{19}3^1 4^{23}), L_{96}(2^{17}4^{23}6^1), L_{96}(2^{20}4^{19}8^1 12^1), L_{96}(2^{28}4^{20}8^1), L_{96}(2^{21}3^1 4^{20}8^1), L_{96}(2^{19}4^{20}6^1 8^1), \cdots$

# The normal mixed difference matrix

- A new difference matrix $D(24, 20; 4)$ can be drawn out from the orthogonal array $L_{96}(2^{12}4^{20}24^1)$ over above Abelian group $G = \{0, 1, 2, 3\}$ which was observed by Zhang (2003).
- A normal mixed difference matrix

$$[D(24, 20; 4), D_1(12, 4; 2) \oplus 0_2, D_2(12, 4; 2) \oplus 0_2, D_3(12, 4; 2) \oplus (2)]$$

also can be drawn out from the orthogonal array $L_{96}(2^{12}4^{20}24^1)$ over above Abelian group $G = \{0, 1, 2, 3\}$ which was observed by Pang, Zhang and Liu (2004).

# Thanks !