Geometric Meanings of Curvatures in Finsler Geometry

Zhongmin Shen

October 24-28, 2011

Chern Centenial Conference
Chern Institute of Mathematics, Tianjin

Riemannian Metrics and Finsler Metrics

Riemannian metrics are *quadratic* metrics

$$F(x,y) = \sqrt{g_{ij}(x)y^i y^j},$$

where $y = y^i \frac{\partial}{\partial x^i}|_x \in T_x M$.

Finsler metrics are in general *non-quadratic* metrics,

$$F(x,y) = \sqrt{g_{ij}(x,y)y^i y^j},$$

where $g_{ij}(x,y) = \frac{\partial^2}{\partial y^i \partial y^j}[\frac{1}{2}F^2(x,y)]$.

Finsler Metrics

A Finsler metric F on a manifold M is a family of Minkowski norms F_x on tangent spaces T_xM.

Finsler metric: $F : TM \to [0, \infty)$

(i) F is C^∞ on $TM - \{0\}$

(ii) $F_x := F|_{T_xM}$ is a Minkowski norm on T_xM:

$F(x, \lambda y) = \lambda F(x, y)$, $\forall \lambda > 0$,

the fundamental tensor is positive definite,

$$(g_{ij}(x, y)) > 0,$$

where $g_{ij}(x, y) := \frac{\partial^2}{\partial y^i \partial y^j}[\frac{1}{2}F^2(x, y)]$.

F induces an *inner product* on T_xM for each $y \neq 0$.

$$g_y(u, v) := g_{ij}(x, y)u^i v^j.$$

Examples:

Let $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ be a Riemannian metric and $\beta = b_i(x)y^i$ be a 1-form on a manifold M.

(a) (Randers metrics):

$$F := \alpha + \beta.$$

(b) (Square metrics)

$$F := \frac{(\alpha + \beta)^2}{\alpha}.$$

(c) (mth-root metrics)

$$F := \{a_{i_1 \cdots i_m}(x) y^{i_1} \cdots y^{i_m}\}^{1/m}.$$

Geodesics

Geodesics (locally minimizing curves):

$$\ddot{x}^i + 2G^i(x, \dot{x}) = 0,$$

where $G^i = G^i(x, y)$ are given by

$$G^i = \frac{1}{4} g^{il} \{ \frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \} y^j y^k.$$

Fact: *For a Riemannian metric* $g_{ij} = g_{ij}(x)$, *thus*

$$G^i = \frac{1}{2} \Gamma^i_{jk}(x) y^j y^k \quad \text{are quadratic in } y.$$

Berwald Metrics

Finsler metrics are called *Berwald metrics* if

$$G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^iy^j.$$

The following are equivalent.

(a) F is a Berwald metric,

(b) $\exp_x T_xM \to M$ is C^∞ at the origin $0 \in T_xM$,

(c) (T_xM, F_x) are all linearly isometric via parallel translation along geodesics.

Z.I. Szabo proved that any Berwald metric is geodesically equivalent to a Riemannian metric. Then he classified the local structure of Berwald metrics.

Examples of Berwald Metrics

- $F = \alpha + \beta$ is a Berwald metric if and only if $\nabla^{\alpha}\beta = 0$.
- Define $F : T(M_1 \times M_2) \to [0, \infty)$ by

$$F := \sqrt{f([\alpha_1(x_1, y_1)]^2,\ [\alpha_2(x_2, y_2)]^2)},$$

  where $x = (x_1, x_2) \in M$ and $y = y_1 \oplus y_2 \in T_{(x_1,x_2)}(M_1 \times M_2) \cong T_{x_1}M_1 \oplus T_{x_2}M_2$.

Douglas Metrics

Finsler metrics are called *Douglas metrics* if

$$G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^iy^j + P(x,y)y^i.$$

If a Finsler metric has the same geodesics as a Riemannian metric, then it is a Douglas metric.

Examples of Douglas Metrics

Let $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ be a Riemannian metric and $\beta = b_i(x)y^i$ be a 1-form on a manifold M.

- $F = \alpha + \beta$ is a Douglas metric if and only if $d\beta = 0$.
- (B.Li-Y.Shen-Z.S.) $F = (\alpha+\beta)^2/\alpha$ is a Douglas metric if
$$b_{i|j} = 2\tau\{(1 + 2b^2)a_{ij} - 3b_i b_j\},$$
where $\tau = \tau(x)$ is a scalar function.

Riemann Curvature

Extend $y \in T_xM$ to a vector field Y on $U \subset M$ so that *the integral curves of Y are geodesics*. Let

$$\hat{g}_p := g_{ij}(p, Y_p) dx^i \otimes dx^j, \qquad p \in U$$

be a Riemann metric induced by Y. Let $\hat{R}$ denote the Riemann curvature tensor of $\hat{g}$ on U. Set

$$\mathbf{R}_y(\cdot) := \hat{R}(\cdot, y)y : T_xM \to T_xM.$$

Flag Curvature

Flag curvature $\mathbf{K} = \mathbf{K}(P, y)$

$$\mathbf{K} = \frac{g_y(\mathbf{R}_y(u), u)}{g_y(y, y)g_y(u, y) - [g_y(y, u)]^2},$$

where $P = \mathrm{span}\{y, u\} \subset T_xM$.

For a Finsler metric on a manifold of dimension $n \geq 3$,

$$\mathbf{K}(P, y) = \sigma(x) \Longleftrightarrow \quad \mathbf{K}(P, y) = constant.$$

For a Riemannian metric,

$$\mathbf{K}(P, y) = \mathbf{K}(P).$$

Examples: (Funk metric) Let $\phi = \phi(y)$ be a Minkowski norm on R^n and $\Omega := \{\phi(y) < 1\}$. Define $\Theta = \Theta(x, y) > 0$ by

$$x + \frac{y}{\Theta(x, y)} \in \partial\Omega, \qquad y \in T_x\Omega \approx R^n.$$

It satisfies

$$\Theta_{x^i}(x, y) = \Theta(x, y)\Theta_{y^i}(x, y).$$

Θ is projectively flat with $\mathbf{K} = -1/4$.

Using Θ, one can construct two important metrics

(i) (Hilbert) projectively flat with $\mathbf{K} = -1$

$$F = \frac{1}{2}\{\Theta(x, y) + \Theta(x, -y)\},$$

(ii) (Z.S.) projectively flat with $\mathbf{K} = 0$

$$F = \Theta(x, y)\{1 + \Theta_{y^i}(x, y)x^i\}.$$

Finsler Manifolds with $\mathbf{K} \leq 0$

Theorem (Auslander-Cartan-Hadamard) Let (M, F) be a forward complete Finsler manifold. If $\mathbf{K} \leq 0$, then the universal cover $\tilde{M} \approx R^n$.

Ricci Curvature

Ricci curvature $\mathbf{Ric} : TM \to R$,

$$\mathbf{Ric}(x,y) := \sum_{i=1}^{n} g_y(\mathbf{R}_y(e_i), e_i) = R^m{}_m(x,y),$$

where $\{e_i\}$ is a g_y-orthonormal basis for T_xM.

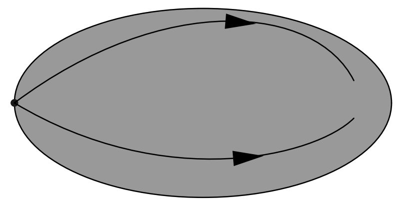
Einstein metrics

$$\mathbf{Ric}(x,y) = (n-1)\sigma(x)F^2(x,y).$$

Open Problem: Is $\sigma(x) = \textit{constant}$ when $n \geq 3$?

Ricci Curvature

Theorem (Auslander-Bonnet-Myers) If (M, F) is forwarded complete with $\mathbf{Ric} \geq (n-1)F^2$, then $\mathrm{Diam}(M) \leq \pi$. In particular, $\pi_1(M)$ is finite.



Every Finsler metric F on a manifold M induces a Riemannian metric $\hat{g}$ on $TM \setminus \{0\}$ of Sasaki type

$$\hat{g} = g_{ij}(x,y)dx^i \otimes dx^j + g_{ij}\delta y^i \otimes \delta y^j,$$

where $\delta y^i := dy^i + \frac{\partial G^i}{\partial y^j}dx^j$. Then it induces the Riemannian metric $\dot{g}$ on the unit tangent sphere bundle SM.

Theorem: (Green-Dazord) Let (M, F) be a closed Finsler space of dimension n. Suppose that the conjugate radius $\mathbf{c}_M \geq \pi$. Then

$$\frac{\int_{\mathrm{S}M} \mathbf{Ric}(y) dV_{\dot{g}}}{\mathrm{Vol}_{\dot{g}}(\mathrm{S}M)} \leq (n-1). \tag{1}$$

The equality holds if and only if F has constant curvature $\mathbf{K} = 1$.

Non-Riemannian Quantities

(1) **Cartan Torsion**: $\mathbf{C} = C_{ijk}(x,y)dx^i \otimes dx^j \otimes dx^k$,

$$C_{ijk}(x,y) = \frac{1}{4}\frac{\partial^3[F^2]}{\partial y^i \partial y^j \partial y^k}(x,y) = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}.$$

Geometric Meaning: F is Riemannian iff $\mathbf{C} = 0$.

(2) **Distortion**: $\mu = \mu(x,y)$.

$$\mu(x,y) := \ln \frac{\sqrt{\det(g_{ij}(x,y))}}{\sigma_F(x)},$$

where $dV = \sigma_F(x)dx^1 \cdots dx^n$ is the Busemann-Hausdorff volume form.

Geometric Meaning: $I_i := g^{jk}C_{ijk}$ is also given by

$$I_i := \frac{\partial \mu}{\partial y^i}.$$

F is Riemannian iff $\mu = 0$.

Non-Riemannian Quantities

The **S-curvature** $\mathbf{S} = \mathbf{S}(x, y)$ is defined by

$$\mathbf{S} = \frac{d}{dt}[\mu(c(t), \dot{c}(t))]|_{t=0},$$

$c = c(t)$ is a geodesic with $c(0) = x$ and $\dot{c}(0) = y$.

Define $\Xi = \Xi_i dx^i$ and $H = H_{ij} dx^i \otimes dx^j$ by

$$\begin{aligned} \Xi_i : &= \mathbf{S}_{\cdot i|m} y^m - \mathbf{S}_{|i}, \\ H_{ij} : &= \frac{1}{2}\mathbf{S}_{\cdot i \cdot j|m} y^m = \frac{1}{4}\{\Xi_{i \cdot j} + \Xi_{j \cdot i}\}, \end{aligned}$$

where "$\cdot$" and "$|$" denote the vertical and horizontal covariant derivatives, respectively, with respect to the Chern connection.

Some Important Identities

$$\Xi_i = -\frac{1}{3}\{2R^m{}_{i\cdot m} + R^m{}_{m\cdot i}\} = I_{i|p|q}y^p y^q + I_m R^m{}_i.$$

$$H_{ij} = -\frac{1}{6}\{R^m{}_{i\cdot m\cdot j} + R^m{}_{j\cdot m\cdot i} + R^m{}_{m\cdot i\cdot j}\}.$$

Volume Comparison

Theorem (Z.S.): Let (M, F) be an n-dimensional forwarded complete Finsler space. Assume that

$$\mathbf{Ric} \geq (n-1)\lambda, \qquad \mathbf{S} \geq -(n-1)\delta.$$

Then the quotient

$$\frac{\mathrm{Vol}(B(p,t))}{\int_0^t \left[e^{\delta s}\mathrm{sn}_\lambda(s)ds\right]^{n-1}}$$

is non-increasing, where sn_λ is the function satisfying

$$\mathrm{sn}_\lambda''(t) + \lambda \mathrm{sn}_\lambda(t) = 0,$$

$\mathrm{sn}_\lambda(0) = 0$ and $\mathrm{sn}_\lambda'(0) = 1$.

Theorem (Mo-Z.S.) Let (M, F) be an n-dimensional closed Finsler manifold $(n \geq 3)$. Suppose that

$$\mathbf{K}(P, y) = \sigma(x, y) \leq -1.$$

Then F is a Randers metric.

Theorem (Z.S.) Let (M, F) be an n-dimensional closed Finsler manifold $(n \geq 3)$ with $\mathbf{S} = (n+1)cF$ $(c = \mathit{constant})$. Suppose that

$$\mathbf{K}(P, y) < 0.$$

Then F is a Riemannian metric.

Finsler Metrics of Constant Flag Curvature

Theorem (Arkbar-Zadeh) : Let (M, F) be a compact Finsler manifold of constant flag curvature $\mathbf{K} = k$. Then

(a) If $k < 0$, then F is Riemannian

(b) If $k = 0$, then F is locally Minkowskian.

Proof: Bianchi identities imply that

$$I_{i|p|q} y^p y^q + kF^2 I_i = -\frac{1}{3}\{2R^m{}_{i\cdot m} + R^m{}_{m\cdot i}\} = \Xi_i = 0.$$

If $k = -1$, then $\mathbf{I} = 0$, hence F is Riemannian.

The Positive Constant Flag Curvature Case

(a) There are non-Riemannian Randers metrics on S^n with $\mathbf{K} = 1$ and $\mathbf{S} = 0$.

(b) (Bryant) Determine the structure of projectively flat metrics on S^n with $\mathbf{K} = 1$. They are not reversible.

(c) (Kim) Every reversible Finsler metric on S^n $(n \geq 3)$ with $\mathbf{K} = 1$ must be Riemannian.

Navigation Representation of Randers Metrics

$F = \alpha + \beta$ can be expressed as

$$\alpha = \frac{\sqrt{[1-(h_{ij}V^iV^j)^2](h_{ij}y^iy^j)^2 + (h_{ij}y^iV^j)^2}}{1-(h_{ij}V^iV^j)^2}$$

$$\beta = -\frac{h_{ij}y^iV^j}{1-(h_{ij}V^iV^j)^2},$$

where $h = \sqrt{h_{ij}(x)y^iy^j}$ is a Riemannian metric and $V = V^i(x)\frac{\partial}{\partial x^i}$ is a vector field.

We call (h, V) the *navigation data* of $F = \alpha + \beta$.

Randers Metrics of Constant Flag Curvature

Theorem (Bao, Robles, Z.S.) Let $F = \alpha + \beta$ be expressed by a navigation data (h, V). F is of constant flag curvature $\mathbf{K} = \sigma$ if and only if

(a) $\mathbf{K}_h = \mu$

(b) $V_{i;j} + V_{j;i} = -4ch_{ij}$.

In this case, $\sigma = \mu - c^2$ and $c\mu = 0$.

Einstein Metrics of Randers Type

Theorem (Bao, Robles) Let $F = \alpha + \beta$ be expressed by a navigation data (h, V). F is Einstein,

$$\mathbf{Ric} = (n-1)\sigma(x)F^2,$$

if and only if

(a) $\mathbf{Ric}_h = (n-1)\mu(x)h^2$

(b) $V_{i;j} + V_{j;i} = -4ch_{ij}$.

In this case, $\sigma = \mu - c^2$ and $c\mu = 0$.

(α, β)-metrics

- A Randers metric $F = \alpha + \beta$ can be written as

$$F = \alpha(1+s), \qquad s = \frac{\beta}{\alpha}.$$

- A square-metric $F = (\alpha+\beta)^2/\alpha$ can be written as

$$F = \alpha(1+s)^2, \qquad s = \frac{\beta}{\alpha}.$$

- (Cheng-Z.S.-Tian) For any polynomial metric of non-Randers type

$$F = \alpha \sum_{i=0}^{n} k_i s^i, \qquad s = \beta/\alpha,$$

  it is Einstein if and only if it is Ricci flat.

Ricci-flat (α,β)-metrics of Douglas Type

(E. Sevim, Z.S. and L. Zhao, Y. Cheng and Y. Tian) Let $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ and $\beta=b_i(x)y^i$. For an (α,β)-metric of Douglas type on an n-manifold ($n\geq 3$) with $\phi(0)=1$:

$$F=\alpha\phi(s), \qquad s=\frac{\beta}{\alpha},$$

$\mathbf{Ric}=0$ if and only if

$$\begin{aligned}
{}^{\alpha}\mathbf{Ric} &= -\frac{4}{25}\tau^2\{[25k_2(b^2k_1+1) \\
&\quad +3(k_1+4k_3)(k_1-k_3)][(n-2)\beta^2+b^2\alpha^2] \\
&\quad +5(n-1)[5k_1(b^2k_1+1)+3(k_1-k_3)]\alpha^2\}, \\
b_{i|j} &= 2\tau(x)\{(1+k_1b^2)a_{ij}+(k_2b^2+k_3)b_ib_j\}, \\
\phi(s) &= \frac{1+(4k_1+k_3)s^2/5}{\sqrt{1+(3k_1+2k_3)s^2/5}}+\epsilon\beta,
\end{aligned}$$

where $\epsilon^2=\frac{4}{5}(k_1-k_3)$, $25k_2=(2k_1+3k_3)(3k_1+2k_3)$, and $\tau_{x^i}(x)=-\frac{4}{5}\tau(x)^2(k_1-k_3)b_i$.

The Finsler metric can be written as

$$F = \frac{\alpha^2 + (4k_1 + k_3)\beta^2/5}{\sqrt{\alpha^2 + (3k_1 + 2k_3)\beta^2/5}} + \epsilon\beta.$$

where $\epsilon := \pm 2\sqrt{\frac{k_1 - k_3}{5}}$.

Note that for the function ϕ, $k_1 \geq k_3$. If $k_1 = k_3$, then F is Riemannian. If $k_1 > k_3$, let

$$\tilde{\alpha} := \sqrt{\alpha^2 + \frac{3k_1 + 2k_3}{5}\beta^2}, \qquad \tilde{\beta} := \pm\sqrt{\frac{k_1 - k_3}{5}}\beta.$$

Then

$$F = \frac{(\tilde{\alpha} + \tilde{\beta})^2}{\tilde{\alpha}}.$$

$\mathbf{Ric} = 0$ if and only if

$$\begin{aligned} {}^{\tilde{\alpha}}\mathbf{Ric} &= -4\tau^2\{-6[(n-2)\tilde{\beta}^2 + \tilde{b}^2\tilde{\alpha}^2] \\ &\quad +(n-1)[(1+2\tilde{b}^2)+3]\tilde{\alpha}^2\}, \\ \tilde{b}_{i|j} &= 2\tau\{(1+2\tilde{b}^2)\tilde{a}_{ij} - 3\tilde{b}_i\tilde{b}_j\}, \end{aligned}$$

where $\tau_{x^i} = -4\tau^2\tilde{b}_i$.

Ricci-flat Square-metrics of Douglas Type

(B. Chen, Z.S. and L. Zhao) For non-Berwaldian square-metric of Douglas type on M:

$$F = \frac{(\alpha+\beta)^2}{\alpha}$$

Ric $= 0$ if and only if locally, $M = R \times \breve{M}$ and

$$\begin{aligned} \alpha^2 &= dt \otimes dt + (\varphi'(t))^2 \breve{\alpha}^2, \\ \beta &= \frac{1}{10}\varphi(t)^{-3/5}\varphi'(t)dt \end{aligned}$$

where

$$\varphi'' = 20\varphi^{1/5} + \frac{2}{5}\varphi^{-1}(\varphi')^2$$

$$\breve{Ric} = (n-2)\lambda\breve{\alpha}^2$$

where $\lambda = 400(1-b^2)\varphi^{2/5}$ is a constant.