Euler Characteristics of Tautological Sheaves on Hilbert Schemes of Points

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1. Introduction

Professor Chern recruited geometers, physicists and combinatorists to work at Nankai Institute of Mathematics (now Chern Institute of Mathematics).

This fact might give some indications on his deep insight on the connections among geometry, physics, and combinatorics.

A proper subtitle of my talk might be

Chern classes, Chern characters, moduli spaces, symmetric functions, and integrable hierarchies.

It reflects on Chern's influence on my researches on enumerative algebraic geometry of moduli spaces and some related problems in string theory.

I hope to provide a humble testimony to the statement that Chern's work has penetrated all fields of mathematics.

For a complex vector $\pi : E \to X$,

$$c_k(E) = e_k(x_1, \ldots, x_r),$$
$$\mathrm{ch}_k(E) = \frac{1}{k!} p_k(x_1, \ldots, x_r) = \frac{1}{k!}(x_1^k + \cdots + x_r^k),$$

where $x_1, \ldots, x_r$ are formal Chern roots of E, and e_k, p_k are the elementary symmetric polynomial and the Newton power polynomial respectively.

These identities already demonstrate the connection of Chern classes and Chern characters to the theory of symmetric functions.

We will show that one can go very far from this connection.

Five themes have emerged in the studies of moduli spaces in algebraic geometry:

Theme 1. Mumford's principle: The cohomology ring of a moduli space is often generated by the Chern classes of tautologically defined vector bundles.

Theme 2. Relationship with the theory of symmetric functions: The topology of moduli spaces are often related to theory of symmetric functions.

Theme 3. Relationship with the theory of integrable hierarchies: Generating functions of intersection numbers on moduli spaces are often tau-functions of integrable hierarchies.

Theme 4. Localization techniques: Consider the fixed point contributions from natural actions on moduli spaces.

This method has its root deep in Chern's work on Gauss-Bonnet theorem and characteristic classes.

Theme 5. Relationship with the theory of modular forms.

We will focus on the first four themes today.

Because the theory of symmetric functions is closely related to the representation theory and to the theory of integerable hierarchies, we have roughly the following picture:

$$
\begin{tikzcd}
 & \text{Moduli spaces} \arrow[r] \arrow[d] & \text{Modular forms} \arrow[dd] \\
 & \text{Symmetric functions} \arrow[dl] \arrow[dr] & \\
\text{Integrable hierarchies} & & \text{Representation theory}
\end{tikzcd}
$$

We will present three types of examples:

1. Grassmannians and flag manifolds

2. Moduli spaces of algebraic curves

3. Hilbert schemes of points in algebraic surfaces

Other theories on moduli problems like Donaldson theory or Gromov-Witten theory share similar features.

2. A classical enumerative problem and its solution

Question. How many lines in $\mathbb{C}^3$ intersect 4 given lines in general position?

Answer. 2 .

To find this answer, we consider the space of all lines in $\mathbb{C}^3$.

This space is not compact, so we projectivize the picture.

Grassmannian as moduli space

First $\mathbb{C}^3$ is compactify to become the projective space $\mathbb{P}^3$, the space of all complex lines through the origin in $\mathbb{C}^4$.

In this projective picture, a line in $\mathbb{C}^3$ corresponds to

(a) a rational curve of degree 1 in $\mathbb{P}^3$; or

(b) a plane in $\mathbb{C}^4$ through the origin.

Denote by $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3; 1)$ the moduli space of all rational curves of degree 1 in $\mathbb{P}^3$.

By the equivalence between (a) and (b), $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3; 1) \cong Gr_2(\mathbb{C}^4)$, the Grassmannian manifold of complex 2-subspaces of $\mathbb{C}^4$.

From intersections to integrals

Now $Gr_2(\mathbb{C}^4)$ is a 4-dimensional compact complex manifold.

On $Gr_2(\mathbb{C}^4)$ there is a tautological rank 2 holomorphic vector bundle $\pi : \xi \to Gr_2(\mathbb{C}^4)$. Take its Chern classes $c_1 := c_1(\xi)$ and $c_2 := c_2(\xi)$.

Fixing a line L in $\mathbb{P}^3$, the space of all lines passing through it form a 3-dimensional submanifold $C(L)$ of $Gr_2(\mathbb{C}^4)$.

It turns out that $C(L)$ is an example of a Schubert cell and is the Poincaré dual of $-c_1(\xi)$.

Therefore

$$C(L_1) \cap C(L_2) \cap C(L_3) \cap C(L_4) = \int_{Gr_2(\mathbb{C}^4)} c_1(\xi)^4.$$

Torus actions and equivariant cohomological setting

To evaluate the integral to get the number 2, we notice that there is a natural $T := T^4$-action on $\mathbb{C}^4$:

$$(e^{i\theta_1}, \ldots, e^{i\theta_4}) \cdot (z_1, \ldots, z_4) = (e^{i\theta_1} z_1, \ldots, e^{i\theta_4} z_4).$$

It induces natural T-actions on $Gr_2(\mathbb{C}^4)$ and the vector bundle ξ.

So one can consider the equivariant cohomology $H_T^*(Gr_2(\mathbb{C}^4))$ and the equivariant Chern classes $c_i(\xi)_T$ of ξ.

For dimension reasons,

$$\int_{Gr_2(\mathbb{C}^4)} c_1(\xi)^4 = \int_{Gr_2(\mathbb{C}^4)} c_1(\xi)_T^4.$$

The latter can be computed by the Atiyah-Bott localization formula.

Localization formula

Let M be a compact T-manifold, $\alpha \in H_T^*(M)$, then one has (Atiyah-Bott):

$$\int_M \alpha = \sum_F \int_F \frac{\alpha|_F}{e_T(\nu(F/M))},$$

where F runs through fixed point components of the T-action,

$\nu(F/M)$ is the normal bundle of F in M,

e_T: equivariant Euler class.

In our case, we get 6 isolated fixed points on $Gr_2(\mathbb{C}^4)$ and

$$
\begin{aligned}
&\int_{Gr_2(\mathbb{C}^4)} c_1(\xi)_T^4 \\
= \;& \frac{1}{2} \sum_{\sigma\text{: permutation of } \{a,b,c,d\}} \sigma\left(\frac{(a+b)^4}{(a-c)(a-d)(b-c)(b-d)}\right) \\
= \;& 2\,.
\end{aligned}
$$

At a fixed point:

$$
\begin{aligned}
&c_1(\xi)_T = a+b, \\
&e_T(N(F/Gr_2(\mathbb{C}^4))) = (a-c)(a-d)(b-c)(b-d).
\end{aligned}
$$

3. Intersections on general Grassmannians and flag manifolds

The Grassmannian $Gr_k(\mathbb{C}^n)$ is the moduli space of k-dimensional subspaces of $\mathbb{C}^n$.

Let $\pi : \xi \to GR_k(\mathbb{C}^n)$ be the tautological vector bundle and $\eta := \underline{\mathbb{C}^n}/\xi$.

Let $c_j := c_j(\xi^*)$ and $d_j = c_j(\eta^*)$. They are related by

$$d_j = \begin{vmatrix} c_1 & c_2 & c_3 & \cdots & c_j \\ 1 & c_1 & c_2 & \cdots & c_{j-1} \\ 0 & 1 & c_1 & \cdots & c_{j-2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & c_1 \end{vmatrix}$$

The cohomology ring of Grassmannian has the following presentation:

$$H^*(Gr_k(\mathbb{C}^n)) = \mathbb{Z}[c_1, \ldots, c_k]/(d_{k+1}, \ldots, d_n).$$

Hence one can obtain an additive basis of $H^*(Gr_k(\mathbb{C}^n))$ by polynomials $c_\mu = e_{\mu_1} \cdots e_{\mu_l}$ in $e_1, \ldots, e_k$, where $k \geq \mu_1 \geq \mu_2 \cdots \geq \mu_l \geq 0$.

It has another additive basis given by Schubert cycles.

Fix a complete flag:

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = \mathbb{C}^n$$

Given a partition $\mu = (\mu^1, \ldots, \mu_l)$, such that

$$n - k \geq \mu_1 \geq \cdots \geq \mu_l \geq 0, \;\; l \leq k,$$

define Σ_μ to be the set of k-planes W such that for $i = 1, \ldots, l$,

$$\dim(W \cap V_{n-k+i-\mu_i}) = i, \;\; \dim(W \cap V_{n-k+i-\mu_i-1}) = i - 1.$$

Then $PD(\overline{\Sigma}_\mu$'s provide an additive basis of $H^*(Gr_k(\mathbb{C}^n))$.

One has

$$c_j = PD(\overline{\Sigma}_{(1,\ldots,1)}), \;\; d_j = PD(\overline{\Sigma}_{(j)})$$

and the Giambelli formula:

$$PD(\overline{\Sigma}_\mu) = \begin{vmatrix} c_{\mu_1^t} & c_{\mu_1^t+1} & c_{\mu_1^t+2} & \cdots & c_{\mu_1^t+n-k-1} \\ c_{\mu_2^t-1} & c_{\mu_2^t} & c_{\mu_2^t+1} & \cdots & c_{\mu_2^t+n-k-2} \\ c_{\mu_3^t-21} & c_{\mu_2^t-1} & c_{\mu_2^t} & \cdots & c_{\mu_2^t+n-k-3} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{\mu_{n-k}^t-n+k+1} & c_{\mu_{n-k}^t-n+k+2} & c_{\mu_{n-k}^t-n+k+3} & \cdots & c_{\mu_{n-k}^t} \end{vmatrix}$$

This establishes relationship between $\{c_\mu\}$ and $\{s_\mu\}$ of $H^*(Gr_k(\mathbb{C}^n))$.

The relationship to the symmetric polynomials is as follows:

$$
\begin{aligned}
c_j &\mapsto e_j(x_1, \ldots, x_k), \\
PD(\overline{\Sigma}_\mu) &\mapsto s_\mu(x_1, \ldots, x_k).
\end{aligned}
$$

Some observations on intersection numbers on Grassmannians (Z.):

$$\int_{Gr_2(\mathbb{C}^n)} c_1^{2m} c_2^{n-2-m} = \frac{(2m)!}{m!(m+1)!} \quad \text{(Catalan numbers!)}$$
$$\int_{Gr_3(\mathbb{C}^n)} c_1^{3m} c_3^{n-3-m} = \text{Sequence A151334},$$

Sequence A151334 from The On-Line Encyclopedia of Integer Sequences: Number of walks within the first quadrant of $\mathbb{Z}^2$ starting and ending at $(0,0)$ and consisting of $3m$ steps taken from $\{(-1,0),(0,1),(1,-1)\}$.

For intersection numbers of Chern characters,

$$\int_{Gr_2(\mathbb{C}^4)} (2!\,\mathrm{ch}_2)^2 + tc_1^2 \cdot 2!\,\mathrm{ch}_2 + t^2c_1^4 = 2 + 2t^2,$$

$$\int_{Gr_2(\mathbb{C}^5)} (2!\,\mathrm{ch}_2)^3 + tc_1^2(2!\,\mathrm{ch}_2)^2 + t^2c_1^4 \cdot 2!\,\mathrm{ch}_2 + t^3c_1^6$$
$$= -3 + t + t^2 + 5t^3,$$

$$\int_{Gr_2(\mathbb{C}^6)} (2!\,\mathrm{ch}_2)^4 + tc_1^2(2!\,\mathrm{ch}_2)^3 + t^2c_1^4(2!\,\mathrm{ch}_2)^2 + t^3c_1^6 \cdot 2!\,\mathrm{ch}_2 + t^4c_1^8$$
$$= 6 + 2t^2 + 4t^3 + 14t^4,$$

$$\int_{Gr_2(\mathbb{C}^7)} (2!\,\mathrm{ch}_2)^5 + tc_1^2(2!\,\mathrm{ch}_2)^4 + t^2c_1^4(2!\,\mathrm{ch}_2)^3 + t^3c_1^6(2!\,\mathrm{ch}_2)^2$$
$$+ t^4c_1^8 \cdot 2!\,\mathrm{ch}_2 + t^5c_1^{10} = -10 + 2t + 2t^2 + 6t^3 + 14t^4 + 42t^5.$$

No nice stability as in the case of Chern classes.

The relationship to integrable hierarchy was discovered much later by Sato: One takes $n = 2k$ and $k \to \infty$, and the Plücker relations correspond to the Hirota bilinear relations for KP hierarchy.

One can also consider K-theoretical intersection numbers on Grassmannian and flag varieties.

This establishes a connection with the representation theory of unitary groups via Borel-Weil-Bott theory.

Let $H \subset G$ be compact connected Lie groups such that $rankG = rankH$.

By a theorem of Wang, there is a structure of homogeneous complex manfold on G/H.

For a representation of $\rho : H \to GL(V)$ of H, let $E_\rho := G \times_\rho V$.

One can define two G-equivariant elliptic operators associated to E_ρ:

$$\bar{\partial} \otimes E_\rho : \Omega^{0,even}(G/H, E_\rho) \to \Omega^{0,odd}(G/H, E_\rho),$$
$$(d + d^*) \otimes E_\rho : \Omega^{even}(G/G, E_\rho) \to \Omega^{odd}(G/H, E_\rho).$$

Define virtual characters:

$$\mathrm{Ind}^{\bar{\partial}}(\rho) := \ker \bar{\partial} \otimes E_\rho - \mathrm{coker}\, \bar{\partial} \otimes E_\rho,$$
$$\mathrm{Ind}^{d+d^*}(\rho) := \ker(d + d^*) \otimes E_\rho - \mathrm{coker}(d + d^*) \otimes E_\rho.$$

This defines maps:

$$\mathrm{Ind}^{\bar{\partial}} : R(H) \to R(G),$$
$$\mathrm{Ind}^{d+d^*} : R(H) \to R(G).$$

One can apply Lefschetz formula to these operators.

For $H = T$, ρ given by a dominant weight, one gets the irreducible representation associated with ρ (Borel-Weil) and its character is given by applying the Lefschetz formula to $\bar{\partial} \otimes E_\rho$ (Atiyah-Bott).

For $G = U(n)$, $H = U(k) \times U(n-k)$, applying Lefschetz formula to $(d + d^*) \otimes E_\rho$, one can show that $\tilde{R}(\bullet) = \oplus_{n \geq 0} \tilde{R}(U(n))$ is a Hopf algebra (Z.), where $\tilde{R}(U(n))$ is the representation ring of polynomial representations, with multiplication map defined by $m := \mathrm{Ind}^{d+d^*} : \tilde{R}(U(k)) \otimes \tilde{R}(U(n-k)) \to \tilde{R}(U(n))$, and co-multiplication given by the restriction $\Delta = \mathrm{Res} : \tilde{R}(U(n)) \to \tilde{R}(U(k)) \otimes \tilde{R}(U(n-k))$. (Z.)

Conjecture 1 *(Z.) Let $\pi : \xi \to Gr_k(\mathbb{C}^n)$ be the tautological bundle on the Grassmannian. For a partition μ of length $\leq k$, denote by ξ_μ the vector bundle constructed from ξ by the Young symmetrizer associated to μ. Then one has*

$$\chi(Gr_k(\mathbb{C}^n), \xi_\mu^*)(t_1, \ldots, t_n) = s_\mu(t_1, \ldots, t_n).$$

In particular,

$$\chi(Gr_k(\mathbb{C}^n), \Lambda_u \xi^*)(t_1, \ldots, t_n) = \prod_{j=1}^{n} (u + t_j),$$
$$\chi(Gr_k(\mathbb{C}^n), S_u \xi^*)(t_1, \ldots, t_n) = \prod_{j=1}^{n} \frac{1}{u - t_j}.$$

These results can serve as paradigm for studies of other moduli spaces.

4. Moduli spaces of curves

Let $\overline{\mathcal{M}}_{g,n}$ be the Deligne-Mumford moduli space of stable curves of arithmetic genus g, with n marked points.

They are smooth orbifolds of dimension $3g-3+n$.

There are naturally defined holomorphic vector bundles on $\overline{\mathcal{M}}_{g,n}$: cotangent line bundles $L_1, \ldots, L_n$ and the Hodge bundle $\mathbb{E}$.

$$L_i|_{[(C;x_1,\ldots,x_n)]} \cong T^*_{x_i}C,$$
$$\mathbb{E}|_{[(C;x_1,\ldots,x_n)]} \cong H^0(C,\omega_C).$$

Define

$$\psi_i = c_1(L_i), \;\; \lambda_j = c_j(\mathbb{E}).$$

Intersection numbers of such classes are called Hodge integrals:

$$\langle \tau_{a_1} \cdots \tau_{a_n} \rangle_{g,n} := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n}.$$

Witten-Kontsevich Theorem. The generating series of such correlators is a tau-function of the KdV hierarchy.

Denote by $H_{g,\mu}$ the Hurwitz number of almost simple ramified cover of $\mathbb{P}^1$ with ramification type μ and genus g. By Burnside formula, $H_{g,\mu}$ can be expressed in terms of representations of S_n.

ELSV formula:

$$H_{g,\mu} = \frac{1}{|\operatorname{Aut}(\mu)|} \prod_{i=1}^{l(\mu)} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\Lambda_g^\vee(1)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)},$$

where $\Lambda_g^\vee(1) = \sum_{i=0}^{g}(-1)^i\lambda_i$.

There are many ways to derive Witten-Kontsevich Theorem from this formula.

A more general formula is the Mariño-Vafa formula (proved by Liu-Liu-Z. and Okounkov-Pandharipande).

It was conjectured by physicists based on duality between topological string theory with Chern-Simons theory (colored HOMFLY polynomials).

Mariño-Vafa formula ——Chern-Simons link invariants of the unknot

2-partition Hodge integral formula (conjectuted by Z. and proved by Liu-Liu-Z.) ——Chern-Simons link invariants of the Hopf link

Such formulas can be used to establish some relationship between Hodge integrals and KP hierarchy and Toda hierarchy (Z.).

There is a natural S_n-action on $\overline{\mathcal{M}}_{g,n}$ by permuting the marked points.

So one expects a relationship with the theory of representation theory of S_n and hence to the theory of symmetric functions and integrable hierarchy by exploiting this action.

This has not been extensively studied.

I make a conjecture on the equivariant K-theory intersection numbers on $\overline{\mathcal{M}}_{0,n}$.

Conjecture (Z.)

Let $\mu = (k_1 k_2 \cdots k_l 1^m)$ be a partition of weight $n := k_1 + \cdots + k_l + m$. Suppose that $k_1 > k_2 > \cdots > k_l > 1$ are relatively prime to each other, then one has

$$\chi_{\sigma_\mu}(\overline{\mathcal{M}}_{0,n+N}, \prod_{i=1}^{n} \frac{1}{1-qL_i} \otimes \bigotimes_{i=1}^{N} \frac{1}{1-q_i L_{n+i}})$$

$$= \frac{(1+\frac{mq}{1-q}+\sum_{i=1}^{N}\frac{q_i}{1-q_i})^{m+N-3}}{(1-q)^m \prod_{i=1}^{N}(1-q_i)} \cdot \prod_{j=1}^{l} \frac{1+\frac{mq^{k_j}}{1-q^{k_j}}+\sum_{i=1}^{N}\frac{q_i^{k_j}}{1-q_i^{k_j}}}{1-q^{k_j}},$$

where σ_μ is an element of S_n of cycle type μ.

The $l = 1$ and $l = 2$ cases have been proved by Z. by holomorphic Lefschetz formula.

In the above we have used the following notations:

$$\prod_{i=1}^{n} \frac{1}{1 - qL_i} := \sum_{m=0}^{\infty} q^m S^m(L_1 \oplus \cdots \oplus L_n).$$

$$\bigotimes_{i=1}^{n} \frac{1}{1 - q_i L_i} := \sum_{d_1,\ldots,d_n \geq 0} q_1^{d_1} \cdots q_n^{d_n} \bigotimes_{i=1}^{n} L_i^{d_i}.$$

The above formula generalize the following formula due to Y.-P. Lee:

$$\chi(\overline{\mathcal{M}}_{0,n}, \bigotimes_{i=1}^{n} \frac{1}{1 - q_i L_i}) = (1 + \sum_{i=1}^{n} \frac{q_i}{1 - q_i})^{n-3} \prod_{i=1}^{n} \frac{1}{1 - q_i}.$$

5. Hilbert schemes of points

Consider the Hilbert schemes $X^{[n]}$ of points in an algebraic surface X.

For $X = \mathbb{C}^2$,

$$(\mathbb{C}^2)^{[n]} = \{ideal\ I \subset \mathbb{C}[z_1, z_2] : \ \dim(\mathbb{C}[z_1, z_2]/I) = n\}$$

By a theorem of Forgarty, $X^{[n]}$ are nonsingular projective varieties of dimension $2n$, and each Hilbert-Chow morphism $\pi_n : x^{[n]} \to X^{(n)}$ to the n-symmetric product $X^{(n)}$ is a resolution of singularities.

$X^{(n)} = X^n/S_n$, S_n is the symmetric group formed by permutations of n objects.

A general phenomenon is that many invariants of $S^{[n]}$ are identical to the corresponding *orbifold* invariants of $S^{(n)}$. This leads to nice expressions for the generating series of these invariants.

For example, for the Betti numbers one has Göttsche's formula:

$$\sum_{n=0}^{\infty} Q^n \sum_{i=0}^{4n} b_i(S^{[n]})(-t)^i$$
$$= \prod_{m=0}^{\infty} \frac{(1-t^{2m+1}Q^{m+1})^{b_1(S)}(1-t^{2m+3}Q^{m+1})^{b_3(S)}}{(1-t^{2m}Q^{m+1})^{b_0(S)}(1-t^{2m+2}Q^{m+1})^{b_2(S)}(1-t^{2m+4}Q^{m+1})^{b_4(S)}}$$
$$= \exp \sum_{n=1}^{\infty} \frac{Q^n}{n(1-t^{2n}Q^n)} \sum_{i=0}^{4} (-t^n)^i b_i(S).$$

By taking $t=1$ one gets

$$\sum_{n=0}^{\infty} Q^n \chi(S^{[n]}) = \frac{1}{\prod_{n=1}^{\infty}(1-Q^n)^{\chi(S)}} = \exp \sum_{n=1}^{\infty} \frac{1}{n} \frac{Q^n}{1-Q^n} \chi(S).$$

For the Hodge numbers one has (G ottshce-Soergel):

$$\sum_{n=0}^{\infty} Q^n \sum_{i=0}^{2n} h^{i,j}(S^{[n]})(-x)^i(-y)^j$$
$$= \exp \sum_{n=1}^{\infty} \frac{Q^n}{n(1-x^n y^n Q^n)} \sum_{0\le i,j\le 2} (-x^n)^i(-y^n)^j h^{i,j}(S).$$

By taking $x=0$ and $y=1$ one gets

$$\sum_{n=0}^{\infty} Q^n \chi(S^{[n]}, \mathcal{O}_{S^{[n]}}) = \exp \sum_{n=1}^{\infty} \frac{Q^n}{n} \chi(S, \mathcal{O}_S).$$

By taking $y=1$ and change x to y one gets

$$\sum_{n=0}^{\infty} Q^n \chi(S^{[n]}, \Lambda_{-y} T^* S^{[n]}) = \exp \sum_{n=1}^{\infty} \frac{Q^n}{n(1-y^n Q^n)} \chi(S, \Lambda_{-y^n} T^* S).$$

As noted by Vafa and Witten, these formulas can be understood as the character formula for some Heisenberg algebra action on $\bigoplus_{n\geq 0} H^*(S^{[n]})$.

Such actions were constructed geometrically by Nakajima and Grojnowski.

These results provide a construction of additive basis of $H^*(S^{[n]})$.

There are several results that relate Hilbert schemes to integrable hierarchies (Okounkov-Pandhripande, Li-Qin-Wang).

Intersection numbers of cohomology classes on Hilbert schemes?

Not many results.

For Euler characteristics of tautological sheaves on Hilbert schemes:
Some nice formulas.

Let X be a smooth projective or projective k-variety.

Let $\mathcal{Z}_n \subset X \times X^{[n]}$ be the universal family of subschemes parameterized by $X^{[n]}$.

Denote by $p_1 : \mathcal{Z}_n \to S$ and $\pi : \mathcal{Z}_n \to X^{[n]}$ the projection onto the X and $X^{[n]}$ respectively.

For any locally free sheaf F on X let $F^{[n]} = \pi_*(\mathcal{O}_{\mathcal{Z}_n} \otimes p_1^* F)$.

With this notation we write $\xi_n = \xi_n^X = \mathcal{O}_X^{[n]}$.

Conjecture 2 *(Z.) For an arbitrary smooth k-dimensional projective variety X and arbitrary holomorphic line bundle $L/$ on X, one has*

$$\begin{aligned}
&\sum_{n\geq 0} Q^n \chi(X^{[n]}, \Lambda_{-u} L^{[n]} \otimes \Lambda_{-v} L^{[n]*}) \\
= \ &\exp \sum_{n=1}^{\infty} \frac{Q^n}{n} \chi(X, \Lambda_{-u^n} L \otimes \Lambda_{-v^n} L^*).
\end{aligned}$$

We prove this conjecture for projective surfaces.

Strategy of proof:

Step 1. Reduction to the case of $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$.

Step 2. Use localization to reduce to the equivariant case on $\mathbb{C}^2$.

Step 3. On $\mathbb{C}^2$ establish a connection to Macdonald polynomials.

Step 1.

For a complex n-manifold X, let $\Psi : K(X) \to H^{\times}[u,v]$ be a group homomorphism from the additive group $K(X)$ to the multiplicative group $H^{\times}$ of units of $H(X;\mathbb{Q})$.

We require Ψ is functorial with respect to pull-backs and is polynomial in Chern classes of its argument.

Also let $\phi(x) \in \mathbb{Q}[[x]]$ be a formal power series and put $\Phi(X) := \phi(x_1)\cdots\phi(x_n) \in H^*(X;\mathbb{Q})$ with $x_1, \cdots, x_n$ the Chern roots of T_X.

For $x \in K(X)$, define a power series in $\mathbb{Q}[[z,u,v]]$ as follows:

$$H_{\Psi,\Phi}(S,x) := \sum_{n=0}^{\infty} \int_{S^{[n]}} \Psi(x^{[n]})\Phi(S^{[n]})z^n.$$

Theorem (Ellingsrud-Gottšche-Lehn) For each integer r there are universal power series $A_i \in \mathbb{Q}[[z,u,v]]$, $i = 1, \cdots, 5$, depending only on Ψ, Φ and r, such that for each $x \in K(S)$ of rank r we have

$$H_{\Psi,\Phi}(S,x) = \mathrm{exp}(\int_S (c_1^2(x)A_1 + c_2(x)A_2 + c_1(x)c_1(S)A_3 \\ + c_1^2(S)A_4 + c_2(S)A_5)).$$

Hence one can reduce to the case of $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$.

Step 2.

This is an easy step.

Consider the natural 2-torus actions on $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, and the induced actions on their Hilbert schemes.

The fixed points on $(\mathbb{P}^2)^{[n]}$ and $(\mathbb{P}^1 \times \mathbb{P}^1)^{[n]}$ can be analyzed.

They "come from" fixed points on $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$, where locally is just $\mathbb{C}^2$, with suitable T^2-actions.

Step 3.

Let $T^k = (\mathbb{C}^*)^k$ act on $\mathbb{C}^k$ whose actions on the linear coordinates $z_1, \ldots, z_k$ are given by

$$(t_1, \ldots, t_k) \cdot z_j = t_j z_j.$$

This action induces actions on $(\mathbb{C}^k)^{[n]}$ and ξ_n.

Since $(\mathbb{C}^k)^{[n]}$ are quasiprojective, one can consider the equivariant indices of equivariant coherent sheaves on them.

For a vector $A = (a_1, \ldots, a_k) \in \mathbb{Z}^k$, denote by $\mathcal{O}^A_{\mathbb{C}^k}$ the T^k-equivariant line bundle on $\mathbb{C}^k$ with weight A. Recall the universal family $\mathcal{Z}_n$ lies in $\mathbb{C}^k \times (\mathbb{C}^k)^{[n]}$, denote by $p_1 : \mathcal{Z}_n \to \mathbb{C}^k$ the projection onto the first factor. Let $\xi^A_n = \pi_*(\mathcal{O}_{\mathcal{Z}_n} \otimes p_1^* \mathcal{O}^A_{\mathbb{C}^k})$.

Conjecture 3 *(Z.) The following identity holds for $k \geq 2$:*

$$\sum_{n\geq 0} Q^n \chi((\mathbb{C}^k)^{[n]}, \Lambda_{-u}\xi_n^A \otimes \Lambda_{-v}(\xi_n^A)^*)(t_1, \ldots, t_k)$$

$$= \exp(\sum_{n=1}^{\infty} \frac{(1-u^n t^{nA})(1-v^n t^{-nA})Q^n}{n \prod_{i=1}^k (1-t_i^n)}),$$

where

$$t^{nA} = t_1^{na_1} \cdots t_k^{na_k}.$$

By holomorphic Lefschetz formula,

$$\sum_{n\geq 0} Q^n \chi((\mathbb{C}^2)^{[n]}, \Lambda_{-u}\xi_n^A \otimes \Lambda_{-v}(\xi_n^A)^*)(t_1,t_2)$$

$$= \sum_{\mu} Q^{|\mu|} \prod_{(i,j)\in\mu} \frac{(1-ut^A t_1^{i-1} t_2^{j-1}) \cdot (1-vt^{-A} t_1^{-(i-1)} t_2^{-(j-1)})}{(1-t_1^{-(\mu_j^t - i)} t_2^{\mu_i - j+1})(1-t_1^{\mu_j^t - i+1} t_2^{-(\mu_i - j)})}$$

$$= \sum_{\mu} Q^{|\mu|} \prod_{s\in\mu} \frac{(1-ut^A t_1^{l'(s)} t_2^{a'(s)}) \cdot (1-vt^{-A} t_1^{-l'(s)} t_2^{-a'(s)})}{(1-t_1^{-l(s)} t_2^{a(s)+1})(1-t_1^{l(s)+1} t_2^{-a(s)})}$$

Summations over all partitions of nonnegative integers

Combinatorics results in the theory of symmetric function.

Key observation (with Zhilan Wang):

$$
\begin{aligned}
& \sum_{n \geq 0} Q^n \chi((\mathbb{C}^2)^{[n]}, \Lambda_{-u}\xi_n^A \otimes \Lambda_{-v}(\xi_n^A)^*)(t_1, t_2) \\
= \ & \sum_{\mu} (\frac{-Qvt^{-A}}{t_1 t_2)})^{|\mu|} \epsilon^x_{ut^A, t_1^{-1}} P_\mu(x; t_2, t_1^{-1}) \cdot \epsilon^y_{v^{-1}t^A, t_2^{-1}} P_{\mu^t}(y; t_1, t_2^{-1}).
\end{aligned}
$$

Macdonald polynomials: $P_\mu(x_1, x_2, \ldots; q, t)$ Complicated definitions not to be recalled here.

Summation formula for macdonald polynomials:

$$
\sum_\mu v^{|\mu|} P_\mu(x; q, t) P_{\mu^t}(y; t, q) = \prod_{j,k} (1 + v x_j y_k).
$$

Specialization: $\epsilon^x_{u,t} : \Lambda_F \to F$ the specialization homomorphism defined by

$$\epsilon^x_{u,t} p_n(x) = \frac{1-u^n}{1-t^n}$$

for each integer $n \geq 1$. Then we have:

$$\epsilon^x_{u,t} P_\mu(x;q,t) = \prod_{s\in\mu} \frac{t^{l'(s)} - q^{a'(s)}u}{1-q^{a(s)}t^{l(s)+1}}.$$

Changing μ to μ^t, we also have

$$\epsilon^x_{u,t} P_{\mu^t}(x;q,t) = \prod_{s\in\mu} \frac{t^{a'(s)} - q^{l'(s)}u}{1-q^{l(s)}t^{a(s)+1}}.$$

Mark Haiman established a different relationship between Hilbert schemes and Macdonald polynomials.

Thank you very much for your attentions!