MAX-PLANCK-INSTITUT FÜR WISSENSCHAFTSGESCHICHTE
Max Planck Institute for the History of Science

PREPRINT 117 (1999)

Peter Damerow

The Material Culture of Calculation

A Conceptual Framework for an Historical Epistemology of the Concept of Number

ISSN 0948-9444

The Material Culture of Calculation

A Conceptual Framework for an Historical Epistemology of the Concept of Number

Peter Damerow
Max Planck Institute for the History of Science
(translated by Klaudia Englund)

Contents

INTRODUCTION

The starting-point for the following comments is the observation that contradictions appear to exist between the results of the theory of cognition concerning the concept of number on the one hand, and the history of arithmetical techniques on the other. Reflections on numbers and their properties led already in antiquity to the belief that propositions concerning numbers possess a special status, since their truth is dependent neither on empirical experience nor on historical circumstances. In an historical tradition extending from the Pythagoreans through the Platonic tradition of Antiquity, Late Antiquity and the Middle Ages, further through the rationalism and the critical idealism of Kantian and neo-Kantian philosophy to the logical positivism and the constructivism of the present, this fact has been considered proof that there are objects of which we can gain knowledge *a priori*. Like a recurring leitmotif, the conviction that numbers are by nature ahistorical and universal threads through the history of philosophy, for which a variety of reasons have been proposed.[1] The historian, on the other hand, is confronted with the fact that numerical techniques and arithmetic insights have a history that is, at least on its surface, in no way different from other achievements of our culture.[2] In view of the variety of historically documented arithmetical techniques, it is scarcely possible to dismiss the assumption that the concept of number – in the same way as most structures of human cognition – is subject to historical development, which in the course of history exposes it to substantial change.

The problem raised by these conflicting views is not just philosophical; it concerns also concrete, empirical research in psychology, in anthropology and in the history of science. In particular, the question of the relationship between the individual development of cognition studied by psychology and the historical changes of cognitive structures, deserves much interest. If the concept of number expresses a cognitive universal, then the ontogenetic development of the cognitive structure it is based on, which constitutes the object of research in developmental psychology, is basically a process independent of culture-specific conditions of socialization and historically variable circumstances. The historical development of numerical techniques and arithmetical insights is consequently an epi-phenomenon, the fundamental conditions of which are unlikely to be adequately addressed by the historian. If, on the other hand, the concept

---

1 Compare the diverse arguments for this »Platonic« view, for example, in transcendental idealism, particularly in Kant, *Prolegomena*, p. 36, and in his *Kritik der reinen Vernunft*, p. 201; in Neo-Kantianism, particularly in Cassirer, *Philosophie der symbolischen Formen*, vol. 1, p. 198, and vol. 3, p. 400ff.; in logical positivism, for example, in Frege, *Die Grundlagen der Arithmetik*, and in Carnap, *Grundlagen der Logik und Mathematik*; further, in constructivism in Lorenzen, *Einführung in die operative Logik und Mathematik*.

2 Compare the historical accounts: Tropfke, *Geschichte der Elementar-Mathematik*; Menninger, *Zahlwort und Ziffer*; Gericke, *Geschichte des Zahlbegriffs*; Ifrah, *Universalgeschichte der Zahlen*.

of number is affected fundamentally by historical development, that is, if it constitutes a genuine historical phenomenon, then the ontogenetic development of the cognitive structures it is based on cannot be sufficiently perceived with psychological means, but can only be comprehended through the conditions of the socialization of the individual subject in a particular culture and in a particular historical situation.

If we ask which of the two alternatives is correct or if, possibly, the alternative itself has to be called into question, we must remember that this is a matter not regarded as solved in either of the disciplines mentioned, but rather, on the contrary, one which has for some time been the object of fundamental and ongoing controversy. This is particularly true in the case of psychology.

In the initial phase of modern psychology under the influence of neo-kantianism, numbers and similar mathematically determined objects were primarily regarded as results of thought processes found in all humans alike. Only in connection with the reaction of gestalt psychology to the challenge by the relativization in cultural anthropology of universal concepts of number[3] did the question of the nature of numbers find entrance into psychology. In particular the empirical evidence, provided by Piaget, that the concept of number is not already imprinted in a child at birth, but is rather formed during the development of the child in a number of developmental stages,[4] contributed to undermining the belief in the non-empirical nature of the concept of number. Piaget himself, however, still interpreted his results entirely in the spirit of neo-kantianism. In his theory the development of the concept of number in ontogenesis rests on experience; however, the result of the development is, according to this theory, determined epigenetically – similar to the biologically determined characteristics of humans – and is thus cross-culturally an *a priori* universal that only appears at the end of this development.[5] Ethnographic research, conducted in the tradition of the developmental psychology that reflects Piaget’s work and methods, has predominantly used this premise as its point of departure, and has accordingly arrived for the most part at the conclusion that the speed of the development of logico-mathematical thought varies markedly under diverse sets of cultural circumstances, not so however the structure of the logico-mathematically structured concepts themselves.[6]

Unlike most psychologists, who as a rule avoid discussions of the historical implications of their theories that go beyond their own discipline,[7] Piaget expressly also drew culture-historical and science-historical conclusions from his theory of the psychogenesis of basic categories of

---

[3] Compare Wertheimer, *Über das Denken der Naturvölker*; Lévy-Bruhl, *Das Denken der Naturvölker*.

[4] Piaget/Szeminska, *Die Entwicklung des Zahlbegriffs beim Kinde*.

[5] Piaget, *Biologie und Erkenntnis*; id., *Die Entwicklung des Erkennens*.

[6] Bruner/Olver/Greenfield, *Studies in Cognitive Growth*; Dasen/Heron, *Cross-cultural Tests of Piaget’s Theory*; Dasen/Ribaupierre, *Neo-Piagetian Theories*; Hallpike, *Foundations of Primitive Thought*.

logico-mathematical thought.[8] Central to his considerations was above all the question of how logico-mathematically structured concepts, for instance the concept of number, can be founded on cognitive universals on the one hand, and on the other be subject to fundamental historical changes. He offered a number of arguments suggesting that cognition in those primitive cultures that have no developed arithmetical activities is comparable to the pre-logical stage of ontogenetic development of the child, in which cognition cannot yet revert to mental operations that are characteristic of the subsequent concrete operational stage of ontogenesis.[9] In following the consequences of such considerations, he distinguished two fundamentally different phases of development for each logico-mathematical concept: an initial phase in which the historical development passes through universal stages that are ontogenetically identifiable, and a second one in which the development is no longer subject to universal laws, but rather to an historical logic of development constituted by reflective abstractions. Such an implication of his psychogenetic theory can, however, only with difficulty be forced to agree with historical findings.[10]

Contrary to such universalistic interpretations of the concept of number, the examination of the particular mental processes going on in arithmetic processes led to theoretical approaches in which the emergence of numbers appears as the result of manifold learning processes.[11] Modern cognitive science has increasingly supported this view, recently by providing evidence that many arithmetic accomplishments can be attributed to the construction of relatively simple “mental models”.[12] Further alternatives came into the discussion through the work of psycholinguists and their interpretation of the concept of number as a linguistic phenomenon, without, however, bringing the question closer to resolution. Under the influence of Chomsky’s theory, numbers have been ascribed to a biologically determined syntax of language.[13] Psycholinguis-

7 Brainerd, *The Origins of the Number Concept*, pp. 3-22, at the beginning of his study on the origin of the concept of number, discusses its historical development. His psychological remarks concerning the various historical periods, for example the assertion that an abstract concept of number is already apparent in Egyptian numerical notations from the time around 3500 B.C., remain, however, unsubstantiated and reflect no discernible connection with the remaining content of the study.

8 Compare Piaget/Garcia, *Psychogenesis and the History of Science*.

9 Piaget, *Die Entwicklung des Erkennens*, vol. 2, pp. 73-77. For the parallelism of ontogenesis and historiogenesis compare also Bachelard, *Die Philosophie des Nein*; id., *Die Bildung des wissenschaftlichen Geistes*; Arcà, *Strategies for Categorizing Change*; Strauss, *Ontogeny, Phylogeny, and Historical Development*; Dux, *Die Zeit in der Geschichte*, in particular pp. 23-35.

10 Compare Damerow, *Ontogenese und Historiogenese des Zahlbegriffs*.

11 Compare for example Gelman/Gallistel, The Child's Understanding of Number; Brainerd, The Origins of the Number Concept; Fuson/Hall, The acquisition of early number word meanings; Smith/Greeno/Vitolo, A model of competence for counting; Gallistel/Rochel, *Preverbal and Verbal Counting and Computation*.

12 For the theoretical bases of such explanations compare Minsky, *A Framework for Representing Knowledge*; id., *The Society of Mind*; Davis, *Learning Mathematics*; for the current state of research: Ashcraft, *Cognitive Arithmetic*; Campbell, *The Nature and Origins of Mathematical Skills*.

13 Compare, for example, Hurford, *The Linguistic Theory of Numerals*; id., *Language and Number*.

tic investigations of the representation of logico-mathematical structures in language on the other hand suggest that we might understand such structures and the objects constituted by them better in a culture-relativistic context.[14]

Such contradictions between different conceptions of number obviously can not be solved within the limited point of view of a single discipline, since neither a study of the cognitive functions of the concept of number excluding the question of its historical changes, nor a study of the historical development of arithmetical techniques leaving out of consideration the cognitive functions of those techniques, do justice to the unsolved problems that are revealed in these controversies. In what sense does the concept of number represent a universal? In what respect is it subject to historical changes? What implications result for the relationship of the ontogenetic development of the concept of number to the historical changes of numerical techniques and arithmetical insights? These questions can only be answered by an historical epistemology of arithmetical thought that is compatible with psychological theories as well as with the results of historical research.

This view of the problems determines the theoretical program to be outlined in the following with the draft of a model describing the development of the number concept. The model will be introduced in two steps. In the first, some theoretical principles are explained, and some concepts clarified that are employed in the formulation of the model. To be outlined is in particular the theory of reflection and its relation to the external representation of cognitive structures, which will in turn form the theoretical link between historical developments and those cognitive structures which are individually shaped in ontogenesis. In a second step based on formulated principles, stages of the historical development of the concept of number will be defined, explained, and identified historically.

[14] The classical study on culture-relativistic conceptions of cognition is Whorf, *Sprache, Denken, Wirklichkeit*. Moreover compare Pinxten, *Universalism versus Relativism in Language and Thought*; Levinson, *Relativity in Spatial Conception and Description*.

Principles of an historical epistemology of logico-mathematical thought

On the nature of historical developmental processes of cognition

Since the historical development of cognition is realized through the cognitive activity of individuals, the description of cognitive abilities in the study of their historical development can not, in principle, be different from that in the study of their individual development. The psychological description of cognitive structures is therefore an adequate medium for characterizing the stages of development of cognition in the case of the individual as well as that of historical development. Problematical, however, is the transfer of psychological concepts to historical development in the case of the developmental processes themselves, insofar as individual development of cognitive structures is a process fundamentally different from the historical development of culturally transmitted facts of knowledge and insight.

The *individual* development of cognition is a process in the psyche of the individual person. The identity of this process is founded in the unity of the individual psyche. It starts with the awakening of intelligence in childhood and ends with the death of the person. The *historical* development of cognition, however, is a collective process spanning populations and generations, based on the interaction of various individuals whose psyches are fundamentally independent one from the other. The process of transmitting cognitive structures from one generation to the next takes place in a network of individual paths of tradition, leading from the individuals of one generation to the individuals of the next and realized in symbolic, and in part also in immediate, interactions. There are no obvious reasons to assume that the network of those avenues of tradition might show analogies to the individual development of cognitive structures. The historical development of cognitive structures is by its nature a phenomenon that has to be interpreted socio-historically and not psychologically.

Nonetheless, the historical development of cognitive structures is based on interactive processes, founded on very particular conditions that can be described psychologically. Not every individual process of knowledge influences the historical development of cognition. Results of individual cognitive processes that are not systematically transferable, so as to be acquired in the process of socialization, are obviously largely irrelevant to the historical development. Likewise, the results of universal ontogenetic processes of development naturally cannot exhibit historical changes that might lead to coherent lines of development in the paths of tradition constituted by interactions. The network of those paths of tradition of cognitive abilities can apparently only then be subject to coherent processes of development when, in social interaction, results of individual cognitive processes are systematically reproduced and expanded by consecutive generations.[15] The reproduction of culture-specific forms of cognition in the process of socialization of the individual and the transfer of individual results of cognitive processes to other individuals are therefore the most important psychologically describable conditions for historical processes of cognition.

BASIC ASSUMPTIONS CONCERNING THE DEVELOPMENT OF LOGICO-MATHEMATICAL THOUGHT

The theoretic model of the historical development of logico-mathematical thought which will, in the following, be described with reference to the development of arithmetical thought, is based on the findings of developmental psychology insofar as it assumes that Piaget's theoretical reconstruction of the development of the number concept in ontogenesis correctly reflects this process in all essential points. In order to allow for the specific nature of historical processes in the development of cognition, however, Piaget's application of the ontogenetic stages of development to the historical processes of development is rejected. Instead of accepting Piaget's psychologically defined developmental stages, the stages of historical development are redefined in a specific, historical manner.

The model is essentially based on two assumptions: *Firstly*, it is assumed, following Piaget's genetic epistemology, that logico-mathematical concepts are abstracted not directly from the objects of cognition, but from the coordination of the actions that they are applied to and by which they are somehow transformed. According to this assumption the emergence of mental operations of logico-mathematical thought is based on the internalization of systems of real actions. The internalized actions form the starting-point for meta-cognitive constructions, through which they become elements of systems of reversible mental transformations that, following Piaget's terminology, we will call here 'operations'. The meta-cognitive constructs that are generated by reflective abstractions, that is, the abstract, logico-mathematical concepts, one of which in particular is the concept of number, can thus be understood as internally represented invariables of transformations to which objects are subjected in the course of action. Thus the experience of objects appears to be preformed by logico-mathematical *a priori* forms (such as, for example, the structures of number, space and time); these structures themselves, although they are subject to processes of development that have their origin, at least indirectly, in the experience of objects, can therefore no longer be changed by those experiences.

[15] This aspect has been emphasized in particular by the culture-historical school of psychology; compare for example *Vygotsky, L. S. (1986). Thought and Language*; Leontjew, *Probleme der Entwicklung des Psychischen.*

Secondly, differing from Piaget's theory, it is presumed that the basic structures of logico-mathematical thought are not determined epigenetically, but are developed by the individual growing up in confrontation with culture-specific challenges and constraints under which the systems of action have to be internalized. The challenges are embodied by material means of goal-oriented or symbolic actions that are shared external representations of the logico-mathematical structures. Thus, according to this assumption, the cognitive structures, according to which logico-mathematical competence is defined, are in ontogenesis not construed independently from processes of socialization, but have as their constitutive condition the co-construction of cognitive structures by means of interaction and communication. Such co-constructions on the micro level of social interaction make it possible that cognitive structures are transferred from one individual to the next and so, on the macro level of social development, are transmitted as intersubjectively shared schemata of interpretation. This process gains historical continuity through collective external representations which, as will be shown, embody both cognitive structures and levels of reflection, and thus levels of abstraction.

The first of these two assumptions formulates a developmental-psychological, the second a knowledge-sociological precondition for a theory of the historical development of logico-mathematical thought. By combining both assumptions, a twofold result is achieved. On the one hand, psychological theories and conclusions receive a culture-historical interpretation; on the other, historical stages of the development of thought can be characterized psychologically. As will be shown in the following, particularly in the case of the development of arithmetical thought which is the subject of the present study, the historical stages of this development can be interpreted as subsequent meta-cognitive levels that are connected with each other by reflective abstractions.

On the definition of historical stages of development

The application of psychological concept formation to historical processes raises, however, a number of fundamental problems that need to be addressed first. One of these problems concerns the definition of historical stages of the development of cognition in general. The problem results from the fact that psychological definitions of abilities by their nature do not refer to collective subjects. Psychologically defined abilities can therefore not readily characterize historical stages of development. They can only be attributed to the individual person, to the members of a group, or to all members of the society in a particular historical situation, not however to the society as a whole. A definition of historical stages of development using psychological concepts seems scarcely possible without determining, with a certain arbitrariness, on which of these different distributions of competence the definition of an historical stage as a criterion of its realization is to be based.

The second of the basic assumptions formulated above offers, however, a solution for this problem. If, according to this assumption, the historical development of cognitive structures is essentially based on their intersubjective communication and historical transmission by means of external representations, then the social distribution of the competence is of only secondary importance for this development. It then represents only a framing condition, determining above all the speed of development and the chances of realization for the cognitive potentials embodied in the representations. The historical stages of development, on the other hand, have to be defined primarily on the basis of analyses of such representations, and this in a manner that these definitions express adequately possible functions of the representations which define a given stage for the individual development of cognition.

The theoretical model of the historical development of logico-mathematical thought proposed here is therefore not primarily meant to explain the outstanding achievements of individuals nor the social distribution of abilities, but the historically changing potentials of development of the individual subject to the conditions prevailing at the time. In particular, the level of development of arithmetical thought in the various cultural epochs is not being measured by the actual results of arithmetical thought, but rather by the arithmetical means and external representations of cognitive structures that were, in the historically determined cultures, available for the ontogenetic development of arithmetic abilities, so that these could in principle evolve.

Once stages of the historical development of logico-mathematical thought are defined this way, the resulting theoretical model can be applied both to the analysis of the construction of new representations which result from outstanding individual achievement, as well as to the application of such a representation by a specifically trained group, or even to the general use of such a representation in a society whose educational system generally communicates such use. The results of those naturally diverse analyses illuminate various aspects of the particular historical stage of development that is defined by representations with certain cognitive characteristics, so that at this stage certain individual accomplishments become possible, certain professional qualifications become reasonable and certain goals of education become generally understandable.

The external representations by which the stages of development are defined cannot be deduced abstractly. They are concrete, historically and culture-specifically determined achievements of human history. Whether the culture-historical stages, definable for the individual abilities, for instance the abilities of arithmetical thought, did indeed exist, is therefore a question that cannot be predetermined theoretically, but one that has to be answered subject to an analysis of actual history, in this case the history of arithmetic.

THE TRANSITION FROM ONE STAGE TO THE NEXT

In the historical development of cognition, the transitions from one stage of development to the next higher one can occur in two fundamentally different forms, namely either by cultural exchange or by culture-immanent processes of construction.

The diversity of cultures coexisting and interacting with each other today results in such transitions in most cases taking place in the form of the adoption of representations that have shown themselves to be effective tools of cognition in another culture. For a global reconstruction of the history of cognition, those processes of transmission have to be studied carefully, in particular because this is the only way to judge which of the cognitive structures found in many, or even in all cultures are biologically inherent in human nature and which, on the other hand, are a result of a transfer of representations of these structures to many or all existing cultures and only give the impression of constituting a cognitive universal of the human race.[16]

Determinative for defining historical stages of development in the processes of the culture-historical genesis of cognitive structures is, however, not this first form of development by cultural exchange, but rather the second form, the development of cognitive structures by culture-immanent constructions. This form of development is based first on individual cognitive achievements that lead to the modification of existing representations and to the construction of new ones. These representations become part of a culture by being embedded in existing paths of tradition, so that they can be integrated into the process of reproduction in this culture.

In both cases, a complex composition of conditions of interactive co-constructions and transmissions of cognitive structures from one individual to the next constitutes a necessary precondition for the emergence of a new, higher stage of development. In the light of these considerations, however, the individual, creative achievement, which spontaneous inclination tends to credit with a crucial role in the rise of new forms of thought, turns out to be only a pe-

[16] A typical problem of this kind is, for example, the debate about whether universal structures of language originate from the spreading of a “proto-language” or whether in those structures universal, biologically founded cognitive structures express themselves; see Bickerton, *A Two-Stage Model of the Human Language Faculty*; Renfrew, *Archaeology and Language*; Bateman et al., *Speaking of Forked Tongues*. Similar questions arise in the case of the transition to literacy. Can writing systems originate from completely independent roots? Are there, in particular, completely independent inventions of systems of arithmetic symbols?

ripheral condition of this development, pre-determined by the existing representations of cognitive structures and by contingent historical circumstances. Without submitting to historical determinism, we might note that any historical situation defines, employing an ensemble of historically transmitted representations of cognitive structures, a space of potential cognitive achievements which at once initiates the individual creative achievements and, at the same time, imposes narrow limits on them. But if these assumptions are correct and can, in particular, claim validity for the structures of logico-mathematical thought, then the question of how the meaning of historically pre-determined representations of a cognitive structure can be reconstructed by an individual in the ontogenetic process becomes a theoretical key question for the understanding of culture-historical development, one to which logico-mathematical thought is obviously also subject.

Therefore, in the following an attempt will first be made to satisfactorily address the question concerning the possibilities of adequate individual reconstructions of the cognitive structures embodied in collective representations of logico-mathematical thought. For this purpose, two kinds of representations will be distinguished that are fundamentally different with regard to the level of reflection crucial for their meaning. The former will be called first-order external representations, the latter second, or more generally, higher-order external representations. The difference consists, briefly stated, in the fact that first-order external representations stand for real objects and actions, higher-order external representations, however, for ideas and mental activities. How can such a differentiation be theoretically specified?

First-order external representations

Definition: First-order external representations (or briefly: first-order representations) are material representations of real objects by symbols or by models composed of symbols and rules of transformation, with which essentially the same actions can be performed as with the real objects themselves.[17]

[17] This and the definitions given in the following do not correspond with the terms introduced by Bruner for the characterization of external representations; compare Bruner, *On Cognitive Growth*. In particular, his classification into enactive, iconic and symbolic representations is not applied here, since, as will be discussed in the following, it appears unsuited to adequately conceptualize the reflective structure of representations and the reflective dynamic of the relationship between symbols and the objects they represent.

Some simple examples may make this definition plausible:

1. The most elementary form of representation is the identification of a concrete object (or an attribute, a perception, etc.) with a name, a word or a sign. These symbols are first-order representations of the quality of objects to be constant, identifiable objects. Concrete objects can be imagined, recognized and in a given manner put together or grouped with other objects, and these actions can be performed in the same way with the symbols representing them.

2. Counters and similar symbolic counting aids that can be simultaneously configured in space are first-order representations of the cardinal structure of sets of objects. When they are linked with real objects in one-to-one correspondences, for example with the animals of a herd of cattle, the same cardinal transformations (increase, decrease, joining, distribution) can be performed with them on a symbolic level as can occur directly with the represented objects.

3. Names of numbers and similar symbols that can be arranged in temporal and spatial succession are first-order representations of the ordinal structure of intensive or extensive quantities. When they are put in homogeneous correspondence to such quantities, for example to the shades of a color range, the same ordinal operations (comparison, determination of maxima and minima, etc.) can be performed with them on a symbolic level as they can be performed directly with the represented quantities.

4. Constructions with compass, ruler and similar graphic instruments are first-order representations of metrical structures of the Euclidean plane. When compass and ruler are used to design the accurate drawing of the position of objects in a plane, for instance the drawing of the foundation walls of a building, then the same spatial transformations (moving of objects, measurements of distances etc.) can be performed in the drawing as they can be performed directly in the empirical plane that is represented.

5. Cutting and pasting of areas performed in a fitting medium of geometrical representation is a first-order representation of the additive structure of areas. When the geometrical representation of a real area, for example of agriculturally productive land, is assigned, the same additive operations (extending, reducing, dividing, etc.) can be performed with the transformations of the geometrical representation as with any quantity that is linearly dependent on the real area represented (division of fields, distribution of crops, planning of water requirements for irrigation, etc.).

The usefulness of first-order representations is based on the fact that actions can as a rule be performed much more easily with the symbols of the representation than with the real objects they represent, since they are not, to the same degree, subject to accidental restrictions characteristic of real situations. Actions which can be performed with the real objects obviously can not be actually replaced by the symbolic actions. The function of symbolic actions is rather exclusively of a cognitive nature: they are a device to anticipate the results of real actions. The purpose of performing symbolic actions of first-order representations is not to substitute real actions, but to plan and control them.

First-order representations, however, are significant not only for the execution of mental operations of existing cognitive structures, but even to a much higher degree for their construction. Unlike in the case of the application of such structures, the symbolic actions can, in the case of the mental construction of cognitive structures by the internalization of systems of actions, completely substitute the real actions. Symbolic actions in the system of rules of a model which is a first-order representation initiate the construction of the same cognitive structures as actions with the real objects they represent.

This quality of a system of actions is in the following to be indicated by the use of the expression *constructive*. The symbolic actions that are performed with first-order representations are in this sense, concerning the cognitive processing of the objects and actions represented by them, *constructive* operations. They can serve as tools in the construction of cognitive structures, since to perform them adequately is no more than to perform the actions represented by them dependent on the precondition that the acting individual has at his disposal already the cognitive structures, whose construction is initiated by these actions.

First-order representations, therefore, share certain physical qualities with the objects and actions for which they stand. They are, however, more abstract, since the same symbols are always used in diverse contexts. This leads to a differentiation in the meaning of symbols characteristic for this kind of representation. A symbol in a first-order representation embodies an object that is abstract and remains the same in all contexts of application; it is implicitly defined by the rules of transformation of the representation; on the other hand, in each case of application it also represents a concrete object, which, simply because it changes from application to application, cannot be identical with the first.

SECOND-(OR HIGHER-)ORDER EXTERNAL REPRESENTATIONS

Definition: Second-(or higher-)order external representations (or simply second-order representations) are material embodiments of mentally constructed objects by symbols or by models composed of symbols and rules of transformation which correspond to the operations of the cognitive structure that implicitly define the mental objects embodied by them.

Adequate application of second-(or higher-)order representations requires that they are placed in relation to real objects and actions. This happens by assimilating these objects and actions to the cognitive structure that gives the representation its meaning. Its use therefore requires the adequate interpretation of its meaning and does not, as is the case with first-order representations, result already from the assignment of symbols and symbolic actions to underlying real objects and actions.

Here again some examples to illustrate the definition:

1. Conventionally determined names of numbers (one, two, three, …) and non-constructive numerals (1, 2, 3, …) are, from a certain developmental stage of the number concept on, second-order representations of *abstract numbers*. Their application to real objects corresponding to this stage requires the understanding of the concept of number.

2. The use of the word “number” and of a general terminology related to the attributes of numbers,[18] the use of variables as *universal numbers* and even *abstract calculations* with numerals independent of concrete applications, are examples of the use of higher-order representations of the number concept. Their applicability is based on the reflective manipulation of numbers and their representations, for example on the correct use of predicate-logical rules of substitution for variables.[19]

3. Theorems of Euclid’s *Elements*, for instance of the Pythagorean theorem, are second-order representations of the *metric structure of the Euclidean plane*. Albeit propositions on geometric figures, they are independent of these figures insofar as the objects they relate to are

---

[18] That the use of the term *numbe*r can be seen as an indication of a higher level of meta-cognition compared to the simple use of designations of numbers is apparent from the fact that non-literate cultures, even those with developed systems of counting, do not as a rule possess terms of this kind. This is in keeping with the fact that in those cultures even abstract counting without identification of concrete objects of counting is often regarded as meaningless. Even the early high civilizations with developed mathematics, for instance Egypt and Babylonia, do not have a term corresponding to our word *numbe*r.

[19] The use of variables instead of specific designations of numbers offers the possibility to determine precisely the degree of generalization of statements and to verifiably change it by substitution of variables. It thus opens a potential means of representation for a higher level of meta-cognitive insights.

no longer concrete figures but rather "virtual" mathematical objects that are implicitly defined by axioms and definitions within a framework of deductive representation.

Real objects and actions are only indirectly represented by second-order representations. Therefore, such representations are not constructive in the sense indicated above with regard to the cognitive processing of the objects and actions indirectly represented by them. The adequate application of those representations requires that the cognitive structure which implicitly defines the mental objects is already constructed in some way, since the symbolic actions of such a representation correspond to the elements and operations of this structure and not, as is the case with first-order representations, to the indirectly represented real objects and actions.

Since symbols are meaningless without the cognitive structure for which they embodyrepresentthe elements and operations, this structure can also not be reconstructed from the symbolic transformation rules of the representation. Contrary to first-order representations, second-order representations are, therefore, not constructive with regard to their meaning. They are, however, constructive in another respect, that is, with regard to the meta-cognitive objects that are constructible by reflective abstraction. To clarify this, some basic considerations concerning the nature of collective external representations are necessary.

All collective external representations have a material base that serves to produce the symbols and to realize the symbol transformations. Second-order representations are therefore not just indirectly related to the real objects and actions, but also directly. They are indirectly related to those objects and actions to which the cognitive structure represented by them is applied. They are directly related to those real objects and actions (signs and sign transformations) that the symbols and symbol transformations are realized with.

This dual relation to real objects and actions is present in all external representations, but in second-(or higher-)order representations it results in a different form of meaning differentiation of the symbols. As stated above, in the case of first-order representations this dual relation results in a differentiation of meaning into the abstract object that the symbol stands for, and into the concrete object to which it is applied. In the case of second-order representations, the concrete object is replaced by the abstract object and the abstract object is replaced by an object which is implicitly defined by the symbols and symbol transformations and which, insofar as it relates to the cognitive structure represented, is of meta-cognitive nature.

A close relationship between second-(and higher-)order representations and the process of reflective abstraction exposes itself here. This is the process which, according to the first assumption formulated above, creates logico-mathematical concepts. The material symbols and symbolic actions of second-(and higher-)order representations can themselves become objects of cognition. They then initiate the construction of precisely that kind of concept formation and cognitive structures for which Piaget coined the term ‘reflective abstraction’.

The above arguments concerning the constructive nature of first-order representations can thus be analogously applied to second-(and higher-)order representations. The former are constructive with regard to the cognitive processing of the real objects and actions represented by them, the latter with regard to the meta-cognitive processing of the cognitive structures initiated by the former. They can serve as tools in the construction of meta-cognitive structures, because their adequate application and the execution of symbolic actions require knowledge of their meaning, but not of the meta-cognitive structures initiated by their application. second-(and higher-)order representations can therefore be considered as first-order representations of symbols and symbol transformations. While they are not constructive with regard to the cognition of the indirectly represented real objects and actions, they are certainly constructive with regard to the meta-cognitive level of cognition that has as its object these cognitive structures and their first-order representations themselves.[20]

second-(and higher-)order representations are thus constructive tools of meta-cognition. Just as first-order representations are, because of their constructive nature, suitable to represent both collectively and externally the fundamental cognitive structures of logico-mathematical concept formation, second-(and higher-)order representations are suitable to represent both collectively and externally the reflection processes that constitute the meta-cognitive structures.

THE HISTORICAL CHANGE OF COGNITIVE FUNCTIONS OF REPRESENTATIONS

As a rule, representations change their function in the process of historical development as well as in individual cognitive development. In particular, higher-order representations develop from first-order representations.

All systems of counting, for example, were originally first-order representations. They mainly represented ordinal structures, as a rule by the temporal succession of a conventionally determined counting sequence. Primarily, they represented cardinal structures only insofar as, with

[20] An excellent example of such a transfer of the constructive character of the representation to a meta-cognitive level is offered by the emergence of the deductive method in Greek mathematics, which will be discussed below. At first, arithmetic insights were constructed by figured numbers, that are patterns of geometrically arranged counters, geometrical insights by constructions with compass and ruler. The definitions and logic rules of deductive systems (e.g. those of Euclid’s *Elements*) are no longer constructive in this sense, but they are with regard to the structuring of the mathematical knowledge gained in the process.

a one-to-one correspondence of real objects with names of numbers in the process of counting, they also served in the identification of cardinal numbers. With the development of the number concept, abstract numerical qualities were attributed to their meaning, so that they became second-order representations of numbers with all their arithmetic rules; that is to say, they now also represented structures like multiplication which have no parallels in the symbolic action of counting. When the abstract concept of number had finally developed, the names of numbers became the abstract infinite counting sequence, to which were ascribed, step by step, all deduced abstract qualities of numbers, for instance the infinity of the number of prime numbers. The counting sequence thus becomes a higher-order representation of the abstract number concept.

The change of the function of representations was in similar fashion also characteristic of the development of geometry. The prehistory of deductive geometry was shaped by the use of drawings and later also of true-to-scale constructions as first-order representations of relationships in empirical space. Constructions were still playing an essential, though different role in Euclid's *Elements*. The ancient version of Euclid's geometry comprised not only theorems with the proofs of their truth, but in addition, and to almost the same extent, constructions with the proofs that the constructed figures possess the required qualities. The analysis of the proofs by means of modern theory of proof demonstrates that the constructions were indeed essential to the *Elements*, for information was derived from the figures concerning the respective position of points, straight lines, triangles, etc., which tacitly was entered into the proofs. Judging from a modern perspective, it appears that in this way gaps in the proofs were bridged. With respect to developmental history, however, the duality of constructions and proofs in Euclid's *Elements* has to be seen as an indication that figures still served here as first-order representations complementing the deductive second-order representation in written language. The information derived from these figures was later only taken from the so-called visualization (German "Anschauung"), that is, figures that were only *mental images* and the real figures, including the Euclidean constructions, degenerated in the new editions and revisions of the *Elements* into helpful but basically dispensable illustrations. They now served only as second-order representations of the geometric meanings implicitly defined by the deductive structures, insignificant for deduction. All the more important became their role now at the meta-level of judging the epistemological function of Euclidean deductions. The representability of all results of such deductions in real geometric figures appeared to prove the *a priori* nature of deductive geometry. With the construction of non-Euclidean geometries and the development of modern formalism, however, this view had to be revised. The visualization lost its constitutive importance as it became, in a technical manner, dispensable in a proof. Hilbert's *Grundlagen der Geometrie* might be interpreted as proto-typical of this transition. This rephrasing of Euclid's geometry represents a developed system of meta-cognitively determined knowledge that was purposefully construed to prove the complete independence of geometric deduction from any real geometric figure and any "visualization", that is, also of any mental image of geometric figures. Geometric figures have since been understood as "models" of abstract structures, i.e., as higher-order representations, and this reinterpretation opened completely new possibilities for their use as means of achieving and processing knowledge.[21]

Ontogenetic reconstruction and the historical tradition of cognitive structures of logico-mathematical thought

Having attempted to clarify the diverse roles of first and higher-order representations, we can now return to the key question, mentioned above, of a theory of the historic development of cognitive structures based on the assumption that this development depends on the historical transmission and elaboration of external representations. This is further the question of how the individual can, in the process of ontogenesis, reconstruct the meaning of representations. The preceding considerations demonstrate that to answer this question three different processes and their combined effect have to be examined:

1. the process of the ontogenesis of universal structures which are independent of culture-specific representations, and which constitute the common precondition of all history-specific structures of logico-mathematical thought,
2. the process of the reconstruction of the meaning of first-order representations, and
3. the process of the reconstruction of second-(and higher-)order representations.

The basic assumptions constitutive for the theory presented here: first, that logico-mathematical concepts are abstracted invariants of transformations (transformations which are realized by actions), and second, that those abstractions are historically transmitted by collective external representations, imply a particular relationship between these processes. Culture-dependent logico-mathematical abilities, part of which are doubtless arithmetical skills, cannot originate from processes of the first kind alone. Their formation in ontogenesis also requires the abstrac-

[21] The meta-constructive character of higher-order representations may, for example, be used to prove, through the construction of models, the relative consistency of a deductive system. The so-called "Klein model" is a Euclidean model of the hyperbolic type of non-Euclidean geometry; it demonstrates that a contradiction in hyperbolic geometry would generate a contradiction in the Euclidean geometry. By means of meta-mathematical semiotic reflection, a geometric figure is here constructed which, contrary to Euclid's constructions, serves only meta-cognitive purposes.

tion of cognitive structures from historically transmitted symbolic actions in the context of representations of these structures and, in this connection, possibly also communication with already competent interaction partners.

Since the mere reconstruction of the meaning of first-order representations does not yet require specific cognitive structures of logico-mathematical thought, they play a special role: first-order representations are the starting-point of abstraction and thus to a certain extent determine the structure of the semantics, reconstructible through reflective abstraction, of second-(and higher-)order representations. In the same way that first-order representations can, as representations of real objects and actions, initiate the construction of new cognitive structures, they also make possible the reconstruction of those cognitive structures which already exist historically and which give a specific logico-mathematical meaning to the culture-specific representations.

second-(and higher-)order representations, that is, representations of mental objects, require, however, that these objects are already constituted by the construction of corresponding cognitive structures. They have, in other words, already been constructed or reconstructed by means of first-order representations. The properties of the mental objects that are represented by the symbols can, in general, not be inferred from the transformation rules for these symbols. But since second-(and higher-)order representations represent, at the first order, the mental objects and transformations on a meta-cognitive level, namely on the level of operating with the symbolically represented cognitive operations as these relate to objects, they have, as soon as the cognitive conditions for their application are present, the same function in the reconstruction of reflected meaning as first-order representations have for the meaning of concrete logico-mathematical activities themselves.

But on a meta-cognitive level, namely on the level of operating with the symbolically represented cognitive operations, second-(and higher-)order representations represent the mental objects and transformations in first order. Thus, they have, as soon as the cognitive conditions for their application are present, the same function in the reconstruction of the reflected meaning of mental transformations as first-order representations have for the meaning of concrete logico-mathematical activities themselves.

This model of the process of ontogenetic reconstruction of the meaning of representations, differentiated according to the kinds of representation, provides a powerful theoretical tool for the explanation of historical as well as individual processes in the development of logico-mathematical thought. According to this model, any process of abstraction of logico-mathematical structures starts with first-order representations, for instance with counters, standards of measurement, drawings or with terms for intuitively comprehended quantities, for geometrical objects, for relations, etc. As long as thinking about real objects and actions occurs without reference to such means of representation, they do not gain any meaning beyond their physical perception. Two sheep in this kind of thinking are two individual sheep and not a cognitive construct formed by the general term sheep and the number two as an abstract mental object.

Representations are by their nature, however, more general than the objects represented. Two counters that stand for two sheep may in a different context represent two cows. Yet even counters are at first only perceived as counters and not as representatives of numbers. They are concrete objects, tally objects, which by their nature nonetheless represent potential abstractions. This abstraction is not arbitrary; rather, it reflects a universal pattern in the actions with the represented objects which are in the first order represented by the symbolic actions performable with the tally objects.

By performing symbolic actions with first-order representations instead of performing real actions with the represented objects, the structure of the actions is isolated from its real context and thus the construction of a corresponding cognitive structure by reflective abstraction is initiated. This cognitive structure constitutes a mental object that is more abstract than the objects of the real actions and is no longer based on direct empirical experience. At the same time, the first-order representation develops into a link between the abstract concept constructed in this fashion and its adequate application to the real objects that formed the starting-point of the process of reflective abstraction.

On the other hand, the constructed abstract concept may itself be embodied by a global symbol or by an elaborate symbol system, for example by a *terminus technicus* or a system of axioms. From this kind of embodiment originates a second-order representation. This second-order representation reflects the real objects and actions on the level of cognitive competence constituted by the abstract concept, and at the same time it initiates on a meta-level the abstraction of a new structure. Again, the abstraction is not arbitrary, but rather reflects a universal structure of the mental operations that are characteristic of thought about the real objects and actions on the level of the abstract concepts constructed in first order.

Since such a second-order representation is at the same time a first-order representation of the meaning of the abstract concept represented, it is a useful tool for performing the mental operations that constitute this concept. The execution of symbolic actions with the symbols of the abstract concept, instead of only mentally performing the operations (for example, working with an arithmetic algorithm), objectifies the mental activity and so again initiates the construction of meta-cognitive operations in the same way as the original first-order representation initiated the construction of the abstract concept, which is now embodied by the second-order representation. Any process of development of the logico-mathematical thought may thus be interpreted as an iteration and recombination of such reflective abstractions that are initiated by culturally transmitted representations.

The historic development of logico-mathematical thought is thus based, according to this theory, on two psychologically explicable processes, namely on individual construction and on the ontogenetic reconstruction of the meanings of representations. These processes, however, come about in culture-specific, historically changing symbolic scenarios which cannot themselves be the objects of psychological explanations. The scenario of culturally transmitted representations of cognitive structures in logico-mathematical thought is an external precondition for the psychological explanation of ontogenetic processes of development. The historical development and transmission of these structures is a stochastic consequence of innumerable individual constructions and reconstructions. Systematically influenced by culture-specific scenarios, however, these constructs represent a development whose regularities can be explained historically and sociologically.

Thus the historical development of logico-mathematical thought does not simply occur, as is for example assumed in Piaget's theory, in parallelism with the ontogenetic development. It is true that in ontogenesis the meaning of higher-order representations has to be reconstructed, and this reconstruction of logico-mathematical meaning establishes a connection with the same elementary systems of action that also constituted the historical starting-point for the construction, reflection and representation of the cognitive structures of logico-mathematical thought; but the individual can only reconstruct those meanings that are indeed present in the systems of action and representations of a particular cultural environment.

Three different kinds of systems of action are thus of consequence for a reconstruction of the historic development of logico-mathematical thought.

First, there are systems of action of universal nature, they are quasi part of the biological properties of humans and can consequently be found in all cultures. Among those systems are, for example, such actions as the reaching for objects, the arranging of objects, the movement within a given space, etc. These most universal human activities correspond to the universals of logico-mathematical thought. Such universals are supercultural cognitive structures that are not subject to historical changes.[22]

[22] According to Piaget, the logico-mathematical operations of operative intelligence pre-exist as practical intelligence in the pre-operative phase of ontogenetic development in the form of senso-motorical schemata of action. These "proto-logic" systems of action have been systematically examined, in particular by Langer; compare Langer, *The Origins of Logic*.

Second, there are very common systems of action that are not part of our biological properties, are therefore subject to historical changes, and are not found in all cultures in the same fashion. Once these systems exist, they exhibit in their simplicity little if any fundamental variation and are therefore largely transmitted by the same kind of first-order representations.[23] To these systems of action are to be reckoned, for example, the techniques of counting which are manifest in almost all cultures.

Third and finally, there are countless culture-specific systems of action and higher-order representations that are of importance for the reconstruction of the historical development of logico-mathematical thought since without them the complex cognitive structures of culture-specific forms of abstract thinking that they embody cannot evolve in ontogenesis.[24]

A reconstruction of the historical development of logico-mathematical thought needs to explain in particular the origin of that characteristic peculiarity of the cognitive structures underlying this thought pattern which constitutes the starting-point of the present inquiry: the properties of the implicitly defined objects appear to be logically determined. They apparently are no longer connected to the empirical knowledge that constituted the starting-point for their construction. If the theory of the function of higher-order representations submitted here is correct, there is, however, a simple explanation for this characteristic. The independence of implicitly defined objects may be interpreted as the immediate consequence of the relative independence of higher-order representations from the meaning of lower-order representations that are their object; for the elements of the meta-cognitive structures they represent cannot be related directly to their reflected objects, so that the relations between them cannot be empirically falsified.

This results particularly in the fact that the symbolic transformations of a higher-order representation at the level of immediate identification of real objects and actions in a cognitive structure constituted by a first-order representation no longer need to be any meaningful operations. This consideration provides an explanation for one of the most peculiar phenomena to be found in the individual as well as in the historical development of mathematical thought: in the course

[23] Such systems and their cognitive effects have been made the object of research in developmental psychology, in particular by Piaget; compare Piaget, *Die Entwicklung des Erkennens*. To the extent that such systems (for instance, the technique of constructing one-to-one correspondences to a standard of counting) are indeed nearly universal (compare Pinxten, *Epistemic Universals*; Dasen, *Are Cognitive Processes Universal?*), they can be identified in historical representations with relative ease ; compare for the concept of number, for example, Menninger, *Zahlwort und Ziffer*; Tropfke, *Geschichte der Elementar-Mathematik*; Ifrah, *Universalgeschichte der Zahlen*.

[24] All arithmetical skills that result from processes of learning are probably based on these culture-specific systems of action and representations. Those skills have been described and examined for example in Campbell, *The Nature and Origins of Mathematical Skills*. Even though there are precious few ethnographic studies concerning those skills (compare the summary article Ashcraft, *Cognitive Arithmetic*), there is no doubt that they are largely of culture-specific nature and depend on the arithmetical tools that are known to us from ethnological and historical sources.

of development, formal conclusions contradict more and more interpretations of mathematical concepts at a lower level of abstraction, until finally, at a sufficiently high level of abstraction, those contradictions become irrelevant. In light of the theory on the function of representations presented here, this phenomenon appears as an inevitable consequence of reflective abstractions.

From those abstractions result, for example, seemingly unreal constructs like “negative”, “irrational”, “transcendental” and “imaginary” numbers. Those numbers can no longer be interpreted as “natural” numbers in the sense of the original representation of sets of objects by counters. As long as numbers are applied exclusively to the context of their origin in actions with concrete sets of discrete objects, such numbers therefore appear to be artificial concept formations, absent any real correlate. Once numbers are understood formally, that is, as implicitly defined entities, however, the unreal character of these objects disappears.

A similar example is provided by non-Euclidean geometries. Those geometries appear absurd and unthinkable as long as geometrical concepts are, as Euclid’s postulates indicate, applied in principle to finite figures constructed with compass and ruler. The canonical application of the concepts makes the construction of Euclidean models of non-Euclidean geometries,[25] completely familiar to us today, at first appear as unmotivated reinterpretations of the basic concepts. What is used as a convincing technique of proof at a meta-cognitive level, appears to common sense as a violation of the quasi “natural” meaning of geometrical concepts.

Historical stages of the development of the number concept

Having developed principles of an historical epistemology of logico-mathematical thought in section one, we will attempt to construct in the following section a theoretical model of the historical development based on those principles. After the description of basic arithmetical activities, stages of development that can result from these actions through reflective abstraction will be illustrated with the help of the theory outlined in section one.

[25] For example the »Klein model« of hyperbolic geometry mentioned above.

ARITHMETICAL ACTIVITIES AS A BASIS OF THE CONCEPT OF NUMBER

According to the considerations presented above, assumptions about systems of action that were historically constitutive for the concept of number, that is, assumptions about the historical nature of certain arithmetical activities and their relation to cognitive universals, have to be the starting-point for the construction of the model. Those systems of action from which the concept of number can be abstracted have been described in particular by developmental psychology. However, the possibilities for psychology to identify and empirically test the causal connections between conditions of development and results of development are very limited in the case of actions that are as fundamental and ubiquitous in our culture as those on which the concept of number is based, for such conditions of development can scarcely be varied systematically. Only ethnographic and historical examinations can here provide information more reliable for our purposes.

The arithmetical activities that have to be reviewed as a possible basis for the historical origin and development of the concept of number are all closely related to the most fundamental structures of the number concept, that is, to the structures of the so-called "natural" numbers, the positive integers.[26]

1. Natural numbers are ordered linearly. Their use as ordinal numbers, that is, as a tool to identify orders, is based on this order.

2. Natural numbers can be interpreted as equivalence classes of finite quantities of discrete objects. They can therefore be used as cardinal numbers for the identification of quantities.

3. Two fundamental arithmetical operations are defined on the set of natural numbers, namely, addition and multiplication. It is therefore possible to calculate with the numbers.

These fundamental structures that are, in natural numbers, integrated in a specific way into an overall structure, now indeed presuppose the existence of certain systems of action. To determine in a specific case the relations of order that are to be represented by numbers, comparisons have to be made. To determine quantities, one-to-one correspondences have to be constructed.

[26] The following list is not to be understood as a paraphrase of an axiomatic representation. The structure of natural numbers can be characterized axiomatically in various ways. Through the Peano axioms, numbers are reduced to the iteration of units. Their definition as equivalence classes of sets with equal cardinality characterizes them as a structure of one-to-one correspondences. In a definition as semigroup with specific characteristics numbers are characterized by arithmetical operations. Those reductions of the structure of natural numbers to a few basic conditions are of only limited interest for the reconstruction of the origin and development of numbers, for numbers have historically certainly not been deduced from an (axiomatically describable) original definition, but probably rather originated through the integration of structural elements of their overall structure.

The additive relationship between three numbers corresponds to the combination of two sets of objects to form a third, the multiplicative relationship corresponds to a repetition of the same action or the reproduction of the same configuration of objects.

These actions of comparison, correspondence, combination and repetition, integrated into systems, will here be designated as *arithmetical activities.* They link the cognitive structures of arithmetical thought with the concrete objects and situations of empirical experience, for they constitute the means by which quantitative values and relations are attributed to empirical experience. Their importance in the development of the fundamental structures of the number concept lies in the fact that they can in principle also be reasonably executed independent of each other and, in particular, independent of an idea of number which acts to integrate them. If according to the assumption formulated above, logico-mathematical concepts are therefore constructed from internalized actions by reflective abstraction, it is obvious that these arithmetical activities in particular are the ones from which the fundamental concept of number is abstracted in individual development.

Developmental stages of the number concept as meta-cognitive levels of reflection

Studies in developmental psychology have shown that such arithmetical activities are to a considerable degree based on cognitive universals independent of culture.[27] On the other hand, it is obvious, considering the great cultural differences in the actual formation of even such fundamental arithmetical activities, that in their core they are already of an historical nature. They are not based on cognitive universals alone, but are also constituted by certain culture-historical processes of transmission. If developmental stages of arithmetical thought are therefore defined as reflective abstractions of historically developing arithmetical activities, that is, as meta-cognitive levels reflecting actions of comparison, correspondence, combination and repetition, this definition does not anticipate the answer to the question concerning the degree to which the concept of number is determined by cognitive universals, but rather the definition provides, to the contrary, an analytical tool to study the question in historical sources.

If the historical development is thus interpreted as a sequence of gradually attained levels reflecting the systems of action of arithmetical activities, this results in a describable structure of process. In the first place, basic cognitive constructs of arithmetical thought are abstracted from

[27] Compare Langer, *The Origins of Logic*.

the actions of arithmetical activities themselves, then from symbolic actions with vicariously used tally systems, body-counting, calculi, etc.; further, from manipulating symbols that only indirectly represent counted objects, that is, from the signs of developed systems of numerical signs and calculation aids, and, finally, from the formal use of the written representation of logical conclusions concerning properties of numbers and numerical relations. This way, a model of the historical development of arithmetical thought is provided, in which four distinguishable phases in the history of the number concept can be described.

0. Before the development of the concept of number, there must have been a period characterized by the complete *lack of arithmetical activities* in the above defined sense.

1. In a first stage of the development of the concept of number, the real actions of fundamental *arithmetical activities* were created and became part of culturally transmitted techniques. Their symbolic representation resulted in first-order representations of these arithmetical activities in the form of concrete tools for the control of quantities.

2. A second stage of development was reached when cognitive constructs that originated from the reflection of the real and *symbolic actions* of the first stage were represented by symbols, and culturally transmitted by means of those representations. With regard to the original arithmetical activities, the resulting symbol systems were thus second-order representations. The transformations performed with the symbols no longer represented the fundamental arithmetical activities with real objects directly; rather, they now represented mental operations with the concepts reflectively abstracted from them.

3. A third stage of development was finally achieved by coding, in written language, these concepts and the mental operations that constituted them; i.e., they were submitted to *formal rules for logical transformations* in a form specified in writing.

Since reflection and abstraction are two processes which are at the same time dependent on each other, yet are not identical, and since the specific form of abstraction which produces logico-mathematical concepts can be interpreted as a consequence of the reflection, the last two stages can be further subdivided into two substages according to their degree of abstraction.

The stages of development thus definable as levels of reflection of the systems of action of arithmetical activities will below be designated in the following ways:

0: pre-arithmetical quantification

1: proto-arithmetic

2: symbol-based arithmetic

2a: level of context-dependent symbol systems

2b: level of abstract symbol systems

3: theoretical arithmetic

3a: level of deduction in natural language

3b: level of formal deduction

These theoretically postulated stages have to be validated by historical analyses. This will be done here in a preliminary form by specifying the stages, and by identifying them historically, allowing them to be compared and contrasted with the results of historical research in order to test the assumptions they are based on. For this purpose each stage will *first* be defined theoretically. *Second*, semiotic characteristics will be described that may serve as criteria for assigning arithmetical techniques of a cultural context to this particular stage of development. *Third*, some concrete historical examples will in each case illustrate the respective goals of research that result from the proposed theoretical model for an historical epistemology of the development of the concept of number.

Pre-arithmetical quantification (stage 0)

Definition. The level of pre-arithmetical quantification designates here a stage of development in which no arithmetical activities can be found with the exception of comparisons, so that judgments of quantities, if they are made at all, can only be based on those comparisons. Pre-arithmetical quantifications in the sense of this definition are consequently based only on comparisons, and require neither the construction of correspondences and thus the identification of quantities, nor the composition of quantities by arithmetical operations, for instance the construction of numbers through the repetition of units.

Semiotic characteristics of pre-arithmetical quantification. The most noticeable difference of the level of pre-arithmetical quantification from all levels of genuine arithmetical activities is the absence of socially transmitted standards that might serve as tools for the construction of one-to-one correspondences. On the pre-arithmetical level there are no structured sequences of counting and no tally systems such as finger counting, counting notches, counting knots or calculi. No words, signs or other symbols are used that possess any arithmetical meaning. The language at this stage possesses terms for quantities, yet these are of exclusively qualitative nature. Insofar as there are any rudimentary words for numbers at all, these are not used for counting; rather, they are simple special terms of quality, that is, designations for intuitively and globally understood quantities. The quantitative aspects of an object of cognition are not yet distinguished from its specific physical appearance and from implications of its quantitative aspects.

Historical identification of pre-arithmetical quantification. An historiogenetic theory of arithmetical thought first addresses the question as to which original conditions for the historical development of this thought pattern are already preconditioned by the cognitive universals founded in human nature and therefore not subject to historical change. In particular, the question arises as to whether cultures ever existed that corresponded to the definition of a pre-arithmetical level given here. The answer to this question ensues from the fact that non-literate cultures existed until recently, cultures that had no counting techniques before their contact with European cultural tradition. The definition of a pre-arithmetical level certainly applies to such cultures.

The number of cultures for which the assignment to this level can be clearly demonstrated is, however, small. The best known examples are the Australian aborigines[28] and South American natives who are, following older reports, frequently also referred to as cultures without words for numbers.[29] It is difficult to identify such cultures with certainty today, however, since even the most remote primitive peoples have been in extended contact with modern civilization. Through trade, which often provided the first systematic contacts with Western culture, those cultures very quickly assimilated arithmetical activities and concept formations. Thus the possibility cannot be ruled out that the number of cultures to be assigned to the pre-arithmetical level was much larger in the period before their contact with the later colonial powers. Proof that many of these cultures did not possess developed counting procedures before their contact with Western culture is currently only feasible through often speculative linguistic inferences.

---

[28] Compare Dixon, *The Languages of Australia*, pp. 107f.; Blake, *Australian Aboriginal Languages*, p. 33; further the numerous examples in Dixon/Blake, *Handbook of Australian Languages*.

[29] Compare for example Lévy-Bruhl, *Das Denken der Naturvölker*, pp. 156f.; Gnerre, *Quantification and Numerals in an Amazon Language*, p. 74.

Such proof is made the more difficult by the fact that many peoples not only adapted the arithmetical techniques of colonial settlers and conquerors, but they also changed quantifications in their own language, for example, by inventing new, indigenous words for counting and new arithmetical techniques.[30] Moreover, the linguistic material can often only be collected by interviewing a few elderly informants,[31] whose children and grandchildren attend public schools in order to learn reading, writing and arithmetic.[32] The influence of such contacts in many cases invalidates the information that can be gained from the informants, in particular because it is not the explicit transmission in oral communication, leaving identifiable traces in the language, which results in the transition from the pre-arithmetic level to proto-arithmetic, but rather the adoption of arithmetical activities which begins changing the semantics of existing terms. Identifying with certainty the pre-arithmetic level in the development of arithmetical thought *historically* constitutes an even larger problem, since written sources usually do not reach far enough in to the past even to gain linguistic material for an identification according to semiotic criteria. An important clue is provided by the fact that from periods before the Late Neolithic no objects or signs, for example counting notches or calculi, have been identified that might have served as tally systems, or might have had another kind of arithmetical function. It is true that Paleolithic, Mesolithic and in particular Neolithic finds, especially of bones, often exhibit repeated signs such as regular patterns of notches, and that these have occasionally been interpreted to be representations of numbers,[33] but such an interpretation can hardly be justified factually, since these sign repetitions lack the characteristic subdivision by counting levels that would be expected in signs with numerical meaning, and which is indeed present in all known real counting systems.

30 Compare Dixon, *The Languages of Australia*, p. 108; further in particular: Saxe, *Culture and the Development of Numerical Cognition*.

31 The sequence of counting of the Iqwaye, studied by Mimica, was, for example, reconstructed mainly from the information given by two persons; compare Mimica, *Intimations of Infinity*, p. 27.

32 Compare the detailed description of the cultural environment in Lancy, *Cross-Cultural Studies in Cognition and Mathematics*.

33 Compare for example Heinzelin, *Ishango*; further in particular Marshack, *The Roots of Civilization*; compare also the controversy about Marshack’s interpretations, which is documented by: D'Errico, *Paleolithic Lunar Calendars*; Marshack, *On Wishful Thinking and Lunar “Calendars”*; D'Errico, *On Wishful Thinking and Lunar “Calendars”: Reply*.

PROTO-ARITHMETIC (STAGE 1)

Definition. As the proto-arithmetic level in the historical development of the concept of number a stage of development will be designated here in which first-order representations of quantities are constructed by means of one-to-one correspondences to standard sets of concrete objects or other symbols.

Semiotic characteristics of proto-arithmetic. The earliest genuine arithmetical activities historically attested are without exception based on objects themselves being represented by symbols, their quantity however by the repetition of these symbols. Symbols are the most simple tools for the construction of one-to-one correspondences that can be transmitted from generation to generation. Structured and standardized systems of symbols, from which standard amounts can be formed that are assigned to the quantities to be identified, are therefore the oldest tools for the identification and control of quantities.

Since symbols, in order to represent quantities, may be repeated either temporally or spatially, in principle two different kinds of such simple standards for the representation of quantities can be distinguished. The former will here be called counting sequences, the latter tallies.

A *counting sequence* in this sense is a standardized sequence of words or symbolic actions which is assigned to the elements of a given set in a fixed sequence, realized in time, a process that is generally called counting. Through the process of counting, a linear (temporal) first-order representation of the ordinal structure of such a given quantity is realized.

Tallies, on the other hand, are to be understood here as concrete objects such as signs, knots, notches or calculi, that can be arranged and combined in a simple way for the purpose of constructing correspondences. By assigning such objects to the elements of a given set, a spatial first-order representation of the cardinal structure of a given quantity is realized.

Thus, counting sequences and tallying systems are two different forms of first-order representations of finite sets of objects, namely, representations of different aspects of their quantity bound with specific advantages and disadvantages regarding the practical problems that are to be solved with their assistance. Common to both forms are certain characteristic structures originating from their function. The deliberate construction of correspondences initiates, in the course of historical development, processes acting to extend their range of application. Genuine counting sequences and tallying systems therefore exhibit two typical structural patterns that are a consequence of their progressive improvement in the direction of an infinite counting sequence. First, older counting limits are preserved as inherent steps of the counting procedure. Second, counting limits are systematically overcome by certain techniques.

The predominant procedure for extending the range of application of counting sequences and tallying systems is, when a counting limit is reached, to start over with counting in connection with a counting procedure of higher order which determines how often the primary procedure had been used. The repeated application of such techniques of passing counting limits generates a hierarchically structured symbol system which can, with this technique, be extended to virtually any required order of magnitude.

The mentioned characteristic structures of genuine counting sequences and tallying systems resulting from this procedure are important indicators of arithmetical activities that have to be attributed to the proto-arithmetic level of the development of the concept of number. On the basis of their hierarchically organized structures, tools for the construction of one-to-one correspondences that have served in the control of quantities can be identified even when there are no historical sources that give us definite information concerning their original purpose. Thus there is in principle no problem in distinguishing cultures that have reached the proto-arithmetic level of development from cultures at the pre-arithmetic level. Nevertheless, this distinction may in some cases be difficult. There are, for example, cultures which use rudimentary sequences of less than ten number words without any structure that would indicate their use in the construction of correpondences. According to the here submitted definitions, however, developed symbol systems for ordinal relations without reference to arithmetical activities on which the concept of number is based should occur at most as a transitional phenomenon leading from the pre-arithmetic to the proto-arithmetic level. Linguistic research, however, appears to indicate that there are cultures in which rudimentary counting sequences have been transmitted over an extended period of time without substantial modification. There are two possible explanations for such cases. Those cultures either developed and transmitted techniques of intuitively identifying small quantities without the construction of one-to-one correspondences, or the rudimentary counting sequences indeed exhibit techniques for the construction of one-to-one correspondences which nonetheless remained extremely limited and did not experience any development of their inherent potentials even over extended periods of time, because they were of only small practical significance under the particular conditions of the culture in question. In the first case those cultures would have to be assigned to the pre-arithmetic, in the second case to the proto-arithmetic level.

The proto-arithmetic level is distinguished from higher levels in the development of the concept of number through the absence of arithmetical procedures, that is, of symbolic transformations, that correspond to such arithmetical operations as addition and multiplication. Those arithmetical procedures require that the symbols which are transformed according to formal rules no longer represent the counted objects, but rather their quantity. According to the definition above of the proto-arithmetic level, however, this level is precisely characterized by the fact that symbolic transformations apply to representations of objects by tallies.[34]

Historical identification of proto-arithmetic. Proto-arithmetical tools and techniques are known to us mainly from surviving non-literate cultures. Most of these cultures are probably to be assigned to the proto-arithmetic level insofar as they actually command arithmetical techniques of their own and have not already adopted arithmetical techniques and concepts from European cultural tradition. Such cultures do display a wide variety of counting sequences and tallying systems for the construction of correspondences, realized in all kinds of forms, whereas arithmetical techniques that are based on the symbolic representation of quantities and the numerical relations between them are encountered relatively seldom. Their proto-arithmetical tools are used almost exclusively for the identification of quantities, not, however, for symbolic transformations with the purpose of quantitative prognosis of results from real interaction with sets of objects.

The study of the proto-arithmetic level in surviving non-literate cultures is of great significance for the reconstruction of the development of the concept of number insofar as there is almost no opportunity to study this stage of development in historical sources. Provided that the conditions in surviving non-literate cultures are comparable to those in historically early periods, which display similar cultural techniques and are of similar scale in the disposal of natural resources, the study of such surviving cultures provides hints for the interpretation of pre-literate period achaeological finds with possible arithmetical functions.

The assignment of surviving cultures to historically early non-literate cultures can, however, hardly proceed other than through criteria that only indirectly indicate arithmetical thought. Comparisons of various surviving non-literate cultures with regard to their stage of development in arithmetical thought suggest that the forms of agricultural cultivation, of animal husbandry and housekeeping connected with sedentariness rendered social conditions that made proto-arithmetical techniques useful and their systematic transmission and development possible. Beginning with this assumption, a proto-arithmetic level of development can be identified in the Late Neolithic and the Early Bronze Age.

---

[34] Compare Lévy-Bruhl, *Das Denken der Naturvölker*, in particular pp. 155ff.; Gay/Cole, *The New Mathematics and an Old Culture*; Hallpike, *Foundations of Primitive Thought*, in particular pp. 236ff.

A system of clay tokens possessing apparent proto-arithmetical functions has indeed been identified as one widely used in the Near East during that period (we are speaking here of a time span from the beginning of sedentariness in the areas surrounding the Mesopotamian lowland plain and in the Nile valley around 8000 B.C. until the emergence of cities around 4000 B.C.).[35] These tokens were found by the thousands in excavations, especially in those of the Mesopotamian alluvial plain and the Persian highland. The oldest clay objects ascribed to these symbols are dated to the beginning of the 8th millennium.[36] Their identification as belonging to a tallying system is based, as will be shown below, on finds from the transitional phase to symbol-based arithmetic in the cuneiform texts developed around 3200 B.C.

The transition from proto-arithmetic to symbol-based arithmetic

The first stage in the development of arithmetical thought about which we have detailed information from historical sources is the level of symbol-based arithmetic. Although, as was shown, we have relatively exact knowledge of the proto-arithmetic level indirectly through surviving non-literate cultures, the degree to which conclusions about the past history of symbol-based arithmetic can indeed be drawn from insights into the proto-arithmetic in those cultures cannot be known with certainty. It is an unfortunate fact that in surviving non-literate cultures the transition from proto-arithmetic to arithmetic occurs almost exclusively through the adoption of techniques of the Indian-Arabic tradition from European culture and not by the culture-immanent development of a symbol-based arithmetic from its proto-arithmetic precursors, corresponding to historical development.

Considering this problematic situation, it is of fundamental importance for the understanding of the origin of arithmetic that in recent years the study of archaeological sources from the dawn of literacy in the Near East has lead to the identification of peculiar forms of arithmetical activities which are, in all likelihood, phenomena of the transition from proto-arithmetic to symbol-based arithmetic. Therefore, some remarks are in order here concerning this transition, which may be interpreted as the origin of arithmetic proper.

---

[35] The discovery of this function is of recent date. The first publications in which these objects were, among other things, interpreted as arithmetical objects are the 1966 article by Amiet, *Élamites inventaient l'Écriture*, and the 1977 work by Schmandt-Besserat, *An Archaic Recording System.* Current literature on these clay tokens, however, is extensive .

[36] Schmandt-Besserat, *Before Writing*; for the date in particular pp. 36f.

Definition. The transitional phase from proto-arithmetic to symbol-based arithmetic will be understood here as an historical phase during which, the same as at the proto-arithmetic level, exclusively first-order representations of quantities were used. These are, however, already represented in a developed symbol system and not only by techniques of counting and tallying.

Semiotic characteristics of the transitional phase. The characteristic semiotic feature of this transitional phase are complex symbol systems used as counting units, whose numerical values, however, are not constant, but vary with the context of their application. They do not represent context-independent fixed numerical meanings, but rather units of counting and measurement of products whose numerical relations are determined by the social context in which they are standardized by conventions. They differ in particular from the counting and tallying systems of the proto-arithmetic level in that they are already the subject of genuine symbolic transformations. The basis is formed by transformations that still are first-order representations of real actions; that means they have to be interpreted as representations of economic transactions or real administrative activities. In addition, transformations occur that are not only symbolic representations of real actions, but use the potential of symbol systems for performing transformations with the purpose of purely getting knowledge about the outcome in situations, in which the performance of the respective actions with the real units represented would not be useful or even possible.

Historical identification of the transitional phase. The transition from proto-arithmetic to symbol-based arithmetic is obviously closely related to the invention of writing. With the exception of cuneiform, there is, however, insufficient documentation of the other early writing systems preserved from the time of their origin which would permit the identification of a period of transition from proto-arithmetic to symbol-based arithmetic. Only with cuneiform tablets do we possess rich sources from the period of that writing system's origin around 3200 B.C. These are about 7000 proto-cuneiform and proto-Elamite texts and text fragments, almost exclusively economical texts with records of quantities. The numerical notations in these texts offer unique source material for the study of the transition from proto-arithmetic to symbol-based arithmetic.

Seen from an historical perspective, this transition occurred during a relatively short period of time. The starting-point for the development was constituted by the clay tokens mentioned above, which had obviously been in general use as proto-arithmetical tools in the Neolithic period of the entire Near East. Together with the emergence of Mesopotamian cities and the beginnings of a form of state organization and the centralization in the administration of estates during the second half of the 4th millennium, a change in the system of clay tokens took place. The forms became more varied. Their safekeeping in closed and sealed clay balls demonstrates that they were used for the encoding of important information. Finally, they were complemented and later completely replaced by markings that were impressed onto the surface of such clay balls or onto the surface of small clay tablets which were also sealed.

These so-called numerical tablets were the precursors of proto-cuneiform inscribed tablets. With the invention of this writing system around 3200 B.C., the numerical notations were supplemented with pictograms, and the tablets achieved a more complex structure through the inclusion in texts of several quantitative notations arranged according to their function, as in an administrative form. The number of numerical signs increased to some 60 signs, each representing various units of measurement and counting. But obviously even this differentiation of signs was not sufficient to fulfill the numerous demands for the representation of such units, since most of the signs were used to represent more than just one unit of counting or measurement, regardless of the quantities represented by these units. The unusual consequence of this – at least in our modern view – was that the numerical signs had no fixed numerical values. Rather, their numerical values were determined by their respective metrological context, and changed with their area of application, without any apparent attempt to attain unambiguous numerical values for the signs.[37]

Thus the transition from proto-arithmetic to symbol-based arithmetic occurred in a close relationship with the emergence of writing and the possibilities which expanded through this new form of symbolic representation. At first, the proto-arithmetic system of clay tokens became the central tool for the control of a locally centralized economic administration. The proto-arithmetical tools for the control of quantities with the help of clay symbols as representatives of the products to be controlled were exploited to the limit of their capacity, finally to be transformed into a more powerful form by the innovation of literacy, that is, into the form of representation through signs that could be flexibly created and manipulated. In a transitional phenomenon, a symbol-based arithmetic still closely related to the cognitive tools of proto-arithmetic thus developed in the early Mesopotamian city states.

The resulting peculiar semantic structures that characterized the numerical signs in this period of transition to symbol-based arithmetic existed for only a short period of time. Already the first texts from a phase following the archaic period, that is, the texts of the Early Dynastic period, exhibit the typical semantic structures of the numerical signs of symbol-based arithmetic and scarcely any traces of the semantics of numerical signs from the archaic transitional period.

[37] For the decipherment of proto-literate systems of numerical signs compare Vaiman, *Über die protosumerische Schrift*; Friberg, *A Method for the Decipherment*; id., *Metrological Relations in a Group of Semipictographic Tablets*; furthermore in particular Damerow/Englund, *Die Zahlzeichensysteme der Archaischen Texte aus Uruk*; Nissen/Damerow/Englund, *Archaic Bookkeeping*.

This supports the assumption that the use in the archaic texts of numerical signs as counting units that are semantically determined by their context must be interpreted as a transitional phenomenon and not as an indication of the existence of a separate stage of development.

SYMBOL-BASED ARITHMETIC (STAGE 2)

Definition. The level of symbol-based arithmetic designates here a stage in the historical development of the concept of number in which second-order representations of quantities and arithmetical activities are constructed by the reflection of proto-arithmetical mental constructs and by their representation in a symbol system. This development produces semiotically structured systems of numerical signs. According to the degree of abstraction from the specific context in which these products of reflection originate, the level of symbol-based arithmetic is subdivided into two further levels, level 2a of context-dependent symbol systems and level 2b of abstract symbol systems.

Semiotic characteristics of symbol-based arithmetic. The striking characteristic of the level of symbol-based arithmetic is the emergence of complex systems of numerical symbols and of formal rules for their application. These symbol systems present themselves mainly in two different forms fulfilling different purposes, namely as systems of numerical signs and of calculation aids. The former are used predominantly for the registration of quantitative information, the latter for its processing.

Both forms of symbol systems usually consist first of signs for counting and measuring units that are hierarchically related. The signs are combined in numerical notations and as a rule are repeated according to the number of units they represent.

Such systems are thus still based on first-order representation of quantities, on a representation by sign combinations that are constructed to be in one-to-one correspondences with the units of real objects they represent. But contrary to simple counting sequence and tallying systems, these systems contain strict transformation rules for dealing with numerical signs and notations. Even if the transformations of sign configurations according to such rules correspond to the meaning of the signs, they remain strict insofar as they are no longer bound to contingent conditions of dealing with the symbolized units of real objects.

The rules according to which the symbolic transformations are performed may relate directly to the symbols, and may thus be explicitly stated. They may on the other hand only emerge from the strict application of the symbols according to their meaning, that is, they can remain implicit, transmitted in the context of meaning. The latter is still mainly the case at the level of symbol-based arithmetic. In historical sources those rules may be discerned from their consequences. In the case of arithmetical symbol systems, they can be identified in particular because of the exact numerical relations caused by them among numerical signs and notations.

Although at the level of symbol-based arithmetic the rules for symbolic transformations are still only implicit rules which can, moreover, be of a specific nature for their respective different contexts of application, they represent the first genuine *arithmetical techniques*. They constitute a new level of arithmetical activities in which objects are no longer the original real objects, but the numerical notations and their meanings. Therefore, at this level the direct relationship between the counting unit and the symbol representing it becomes more and more obsolete and the repetition of symbols gradually loses its function of supporting the arithmetical activities. Numerical notations as second-order representations are reduced to standardized signs.

The immediate consequence of the development of complex symbol systems is the emergence of formally determined *technical terms*. Terms that are at first related to specific applications of numerical symbols attain second meanings which are implicitly defined by their function in symbolic transformations. Even this technical meaning is, because of the implicitness of the semiotic rules, at first only associated with the objects and not with the symbols. Thus, in the beginning the technical terms in particular do not yet exhibit a differentiation between general and specific representational aspects, which later leads, for example, to the differentiation of general properties of numbers and specific properties of special representations of numbers.

Context-dependent and abstract symbol systems. Since numerical symbol systems characteristic of the level of symbol-based arithmetic are, due to the implicitness of their semiotic rules, at first still closely related to the meanings of the symbols in specific contexts of application, they are not necessarily from the outset applicable to arbitrary contents. Two sublevels of the level of symbol-based arithmetic can therefore be differentiated, based on the degree of independence of their cognitive constructs, represented by the numerical signs, from particular contexts of application: a level of context-dependent and a level of abstract symbol systems.

It is characteristic of the *level 2a of context-dependent symbol systems* that numerical notations still possess specific areas of application. Although quantities at this level are already symbolically represented by second-order representations, there is not yet a system of notations through which all special forms of notations can be transformed into a standardized form of representation. Consequently, the rules of sign transformation are intermingled with the specific meanings of the signs in their particular context of application and thus are rarely formal, that is, depending only on the form of the signs and sign combinations.

On the *level 2b of abstract symbol systems*, however, there is an arithmetical symbol system that is generally applicable, independent of the context of application. It allows the standardized representation of all quantities, into which all particular application-specific representations can be transformed. There are no external limits to its application. Consequently, with such symbol systems formal operations can be performed, that is, without reference to any specific interpretation in their actual context of application.

Such abstract semiotic constructs without canonical reference to specific contexts of real objects and actions make possible for the first time forms of cognitive processing of abstract ideas that may be interpreted as early forms of genuine mathematical thought. The constitution of meanings reflecting formal symbolic transformations leads to knowledge of entirely implicitly defined artificial objects, with at best metaphorical reference to real contexts of application. This knowledge can be acquired, represented and historically transmitted as a first body of mathematical knowledge. However, the knowledge of abstract objects that are only mentally constructible is at this level not yet integrated into deductive systems with formal rules of inference. Thus the qualities of these abstract objects cannot yet be derived or substantiated with logical derivations. But mental operations with such objects created by reflection are for good reason usually considered an early form of mathematical thought.[38]

Historical identification of symbol-based arithmetic. Most, if not all, advanced civilizations, in particular the Egyptian empire, the Mesopotamian city states, the Mediterranean cultures, the Chinese empire, the Central American cultures and the Inca culture, have, independent from each other, developed or adapted from other cultures systems of numerical symbols that exhibit the semiotic characteristics of the level of symbol-based arithmetic. The basic symbols of these systems were, the same as at the proto-arithmetical level, signs for units and not for numbers, but part of the systems were now complex symbol transformations, that is, arithmetical techniques such as the Egyptian calculation using unit fractions,[39] the sexagesimal arithmetical technique of the Babylonians,[40] the transformations of rod numerals on the Chinese counting board,[41] the calendar calculations in the pre-Columbian culture of the Maya,[42] or the technique of the use of knotted cords (quipu) as administrative tools by the Inca.[43]

---

[38] Although detailed studies of the sources have been presented, there still exists the occasional prejudice that the Greeks were the first to develop abstract mathematics and that all pre-Greek mathematics was oriented exclusively towards practical purposes. This view can no longer be supported considering the original sources that have come down to us.

[39] Compare Neugebauer, *Die Grundlagen der ägyptischen Bruchrechnung*, pp. 137ff.; Chace, *The Rhind Mathematical Papyrus*; Vogel, *Vorgriechische Mathematik*, vol. 1, pp. 31-44; Gillings, *Mathematics in the Times of the Pharaohs*, pp. 20ff.; Damerow, *Abstraction and Representation*, pp. 188-199.

The discussion of the transitional phase from proto-arithmetic to symbol-based arithmetic has already shown clearly that the development of these numerical techniques was historically closely related with the administrative problems that had arisen through the concentration of economic goods and services in the governmental centers of those cultures. There is nowhere an indication that such techniques might have existed in the rural cultures of the Neolithic and the Early Bronze Age periods before the emergence of these centers and might have only been adopted from them. Only the emergence of forms of state organization led to problems that were solved by means of the arithmetical techniques of the early civilizations: a dramatic rise in the quantities of products that had to be controlled, and an immense variety of decision-making implications that had to be executed administratively.

It is thus very probable that administrative bureaucracies have always been the institutions that created those complex techniques for the transformation of numerical symbols characteristic of the level of symbol-based arithmetic. As is obvious from preserved sources, these techniques as a rule served either the control of economic goods or other technical purposes, and this circumstance also determined the structure of symbolic transformations. Perhaps with the exception of addition, which, as a trivial consequence of the representation of quantities by repetition of symbols, developed in all cultures in almost the same form, the arithmetical techniques of the early civilizations reflect for the most part the culture-specific differences of the areas of application for which the respective systems had been developed.

Historical identification of the transition to abstract symbol systems. While the transition from proto-arithmetic to symbol-based arithmetic is relatively easy to identify, the historical identification of the transition from the exclusive use of context-dependent symbol systems to the construction of a unified abstract symbol system presents us with greater difficulties. The sources preserved from early civilizations are often too meager to allow of a sufficiently precise assessment of the area of application of a symbol system that would distinguish between context-dependent and abstract use of such a system.

---

[40] Compare Neugebauer, *Vorgriechische Mathematik*, pp. 4ff.; Vogel, *Vorgriechische Mathematik*, vol. 2, pp. 15-35; Damerow, *Abstraction and Representation*, pp. 204-211.

[41] Compare Needham, *Mathematics and the Sciences of the Heavens and the Earth*; Juschkewitsch, *Geschichte der Mathematik im Mittelalter*, pp. 12ff.; Li/Dù, *Chinese Mathematics*, pp. 3-19.

[42] Compare Thompson, *Maya Artihmetic*; id., *Maya Hieroglyphic Writing*, pp. 51ff.; Closs, *The Mathematical Notation of the Ancient Maya*; Gaida/Tear, *Kalender, Numerologie and lunare Astronomie*.

[43] Compare Locke, *The Ancient Quipu*; Ascher/Ascher, *Numbers and Relations from Ancient Andean Quipus*; Scharlau/Münzel, *Quellqay*, pp. 80-93.

Even at a stage when they are not yet universally applicable, moreover, the numerical symbol systems exhibit such great differences with regard to the scale of their area of application that some, when compared to the others, appear to be much more abstract systems. These are mostly the counting sequence systems, for counting procedures in contrast to measuring procedures are as a rule so object-neutral that fortuitous peculiarities of the counted objects can scarcely influence the counting procedure. The existence of such a neutral counting procedure is, however, a necessary, though not a sufficient criterion to assign a culture that uses such a counting procedure to the level of abstract arithmetic. Only the actual use of the procedure that allows the rendering of different forms of context-dependent numerical notations for qualitatively incongruous quantities in a unified form of representation, either mentally or through a symbolic technique, provides the numerical notation with that semiotic structure which is the criterion for attaining the level of abstract numerical symbol systems according to its definition given here. The identification of this generalized application of a system of numerical notations requires that sufficient sources have been preserved documenting the use of the system. The distinction proposed here between a level of context-specific numerical symbol systems and a level of abstract systems has been made in particular in view of the historical development of arithmetic in Babylonia, which fulfills this condition due to propitious circumstances of source material. Hundreds of thousands of preserved administrative texts offer an almost complete picture of the early use of arithmetical techniques.

These sources demonstrate that from the moment of the emergence of writing around 3200 B.C. until the invention of the sexagesimal place value system about 2000 B.C., exclusively context-specific symbol systems were used.[44] One of these systems, a sexagesimal system[45] that was already strictly structured at the moment of the emergence of writing and which was used in counting discrete objects, probably corresponding to a similar sequence of number words,[46] did possess a very wide range of application. It was, however, not used for any objects whatever.

Only with the invention of the sexagesimal place value system around 2000 B.C. was an abstract system of numerical notations introduced that unified all forms of notations.[47] This invention had two far-reaching consequences for the development of arithmetic.

---

44 Compare Damerow, *Abstraction and Representation*, chap. 7.

45 System S in Damerow/Englund, *Die Zahlzeichensysteme der Archaischen Texte aus Uruk*; not to be confused with the later sexagesimal place value system.

46 The recorded Sumerian sequence of number words was strictly sexagesimally structured; compare Powell, *Sumerian Numeration and Metrology*. Since this sequence of number words has only been preserved in very late sources, however, the possibility cannot be excluded that it was artificially created by the scribes of periods following that of the emergence of writing, based on sexagesimal numerical notations in the cuneiform texts available to them.

47 This date is based on the arguments presented in Damerow, *Abstraction and Representation*, chap. 7. A different opinion is held by Powell, *The Antecedents of Old Babylonian Place Notation*.

First, it revolutionized the *Babylonian arithmetical technique*. Before the invention of the sexagesimal place value system, a multitude of arithmetical procedures obtained for specific problems, without any indication in preserved sources that these procedures might have been applications of universal, non-object-specific, symbolic transactions. Within the framework of the centralized administration of all resources, products and services, for example, were standardized but highly specialized book-keeping procedures that guaranteed a universal, continuous control of goods at their disposal; the bookkeeping rules, however, always remained subject to the administrative units organized according to the units of production.[48] The computation of the area of a field, the computation of the expected yield of grain and the computation of the labor necessary for cultivation were, for instance, performed with different numerical notation systems and with different procedures, although, from a modern perspective, all three cases were simple multiplications. Even in the artificial conversions of products and services into uniform equivalents of value, which are documented for the period of the 3rd Dynasty of Ur (ca. 2100 to 2000 B.C. – the latest phase of the exclusive use of context-specific numerical systems), the standard could change from one administrative unit to the other, so the equivalencies of value could, according to their tasks, be expressed in silver, in fish, in barley or in human labor.[49] Beginning at the time of the invention of the sexagesimal place value system, however, hundreds of arithmetical tablets are preserved that demonstrate on the one hand the conversion of traditional numerical systems into this new system, itself no longer bound to a specific area of application, on the other the use of abstract operations, in particular a uniform multiplication procedure.[50]

Second, the invention of the sexagesimal place value system had as an immediate consequence the development of so-called *Babylonian mathematics*, a system of technical terms, canonical types of problems and abstract mental operations which was only metaphorically related to its areas of application and which made possible the solution of such complex, but at the same time practically irrelevant problems as the computation of the sides of a field from its perimeter and its area, that is of the problem of identifying two unknown quantities from their sum and their product.[51] In contrast to the procedures of economic administration to which it was still meta-

---

[48] Nissen/Damerow/Englund, *Archaic Bookkeeping*.

[49] Compare Englund, *Organisation und Verwaltung der Ur III-Fischerei*, in particular pp. 18ff., 96ff. and 181ff.

[50] Compare Damerow, *Abstraction and Representation*, in particular pp. 246ff.

[51] The so-called mathematical cuneiform texts mainly come from this period. Compare for this the standard literature on the history of mathematics, in particular Neugebauer, *Mathematische Keilschrifttexte*; Neugebauer/Sachs, *Mathematical Cuneiform Texts*; Friberg, *Mathematik*. The interpretation of these texts has, however, been subjected to a sensational revision in recent years through new translations of the mathematical *termini technici* for arithmetical operations presented by Høyrup; compare Høyrup, *Algebra and Naive Geometry.* The view argued here of Babylonian mathematics resulting from the reflection of culture-specific arithmetical activities at a level of a unified abstract symbol system is based on these new philological findings.

phorically related, and although performed by the same persons, namely the so-called scribes, the officials in the administration of state bureaucracies, Babylonian mathematics was an esoteric art not directly related to the practical problems from which the numerical techniques it used had originated.

Thus the development of arithmetic in Babylonia can be subdivided very precisely into two extensive phases corresponding to the definitional distinction presented above of a level of context-dependent and a level of abstract symbol systems. The question, however, whether this kind of subdivision of the level of symbol-based arithmetic is characteristic of all early civilizations where a similar degree of abstraction of numerical notations and techniques was reached has to remain open. Assuming our sources permit this at all, the question can only be decided based on intensive historical study.

CONCEPT-BASED ARITHMETIC (STAGE 3)

Definition. The stage of concept-based arithmetic designates here a stage in the historical development of the concept of number in which second and higher-order representations of symbolic actions with representations of symbol-based arithmetic are constructed by means of the reflection of arithmetical concepts for properties of numbers and numerical relations, and of the representation of the thus gained meta-cognitive insights in the medium of written language. Those representations are in the form of logically structured systems of arithmetical propositions that make possible their deductive derivation and proof. The cognitive constructs that are thus abstracted from the sign systems of symbol-based arithmetic are independent from the characteristics of specific systems for the representation of numbers. According to the degree of abstraction of these cognitive constructs from the specific context of application of written language representation, the stage of concept-based arithmetic is subdivided into two levels, a level 3a of deduction in natural language and a level 3b of formal deduction.

Semiotic characteristics of concept-based arithmetic. The most important semiotic characteristics of the level of concept-based arithmetic are general propositions concerning the properties of abstract numbers. Since such propositions can only be deduced, they are naturally embedded in a system of deductive and interrelated relations. Under certain conditions such deductive webs of relations can be linearized, that is, the propositions can be globally structured in a way that all circular arguments are removed and all propositions appear to be systematically deduced from but a few basic propositions. Accordingly, it is customary following Euclidean tradition to fix in writing such propositions in a deductive order as a *theory*.

Arithmetical propositions at this level of concept-based arithmetic are no longer statements about real arithmetical activities, are moreover no longer statements about numerical relations in a symbolic representation system for such actions; rather, such propositions are statements about objects whose properties can be entirely elaborated by mental operations. They are statements about *abstract numbers*. Thus, at the level of concept-based arithmetic not only some – as at the level of symbol-based arithmetic – but *all* terms are technical terms in the sense of the characterization given above.

Further, theoretically established arithmetic concepts are, through the deductive context that constitutes them, also determined in a fashion other than is the case with simple arithmetical technical terms, determined as they are through the context of a system of symbolic representations of numerical relations. We have seen that, although the meanings of arithmetical technical terms in symbol-based arithmetic are determined by their function in arithmetical techniques, this determination of their meaning occurs only implicitly. In a deductive system, however, the meanings of arithmetical terms are, in general, established not just implicitly, but are reasonably determined by explicit definitions.

Thus, theoretically established concepts do not owe their structure directly to the arithmetical techniques that constitute them, but to the knowledge that can be gained by the application of those techniques. These concepts are consequently embedded in structures of argumentation in natural language, that is, in structures at a meta-level of reflection of simple technical terms. At this level of reflection, the seemingly canonical meanings of arithmetical concepts that had been established at the stage of symbol-based arithmetic, may again be subject to development and can, if necessary, be modified by considerations of suitability of a higher kind. Thus they can, to a certain extent, be freely fashioned.

Theoretical concepts, for example the concept of *prime number* or the concept of *perfect number*[52], do not have the same kind of immediate technical meaning as do technical terms of symbol-based arithmetic such as sum, factor or divisor. Only the reflection of all possible summations, multiplications, divisions, and their results that are logically determinable in a deductive system, leads to the formation of such concepts as a prime number being a number that cannot be further decomposed into factors other than one and the number itself, or a perfect number being a number equal to the sum of its factors.

The relevance of such concepts is no longer based on their practical applicability. We could not justify the formation of the concept of prime number, for example, by any practical significance of the question of whether it might be impossible or not to decompose a given number further

[52] Book VII of Euclid's *Elements*; compare Heath, *The Thirteen Books of Euclid's Elements*.

into factors, although of course such a question may well be of some relevance for certain arithmetical techniques. Rather, the definition of the concept of prime number obtains its relevance from such theorems as the proposition that every natural number can be unambiguously factored into a product of prime numbers, or that the series of prime numbers is infinite. The concept of prime number is not justified by its applicability but by the structuring capability it achieves in the system of arithmetical knowledge.

The establishment of meanings of concepts by reflections of this kind seemingly frees statements about numbers from their real representations. Numerical notations that are used to handle them for practical purposes, for example, appear to have no influence on the truth of such statements. Abstract numbers appear to exist *a priori*, because they have their origin in reflection.

Deduction in natural language and formal deduction. Similar to the level of symbol-based arithmetic, the level of concept-based arithmetic can, according to the degree of independence from their origin of the cognitive constructs deriving from reflective abstraction, be divided into two sublevels, namely, into a level of deduction in natural language and a level of formal deduction.

The *level 3a of* deduction in natural language is characterized by the fact that the deductive systems still consist of statements and proofs that are formulated in natural language. Mathematical terms, for instance the concept of number, are explicitly defined and are abstract insofar as they are determined within the logical structure of a deductive system. They still refer, however, to concrete objects and actions, since a representation in natural language entails connotations of the concepts that are determined by their origin in real actions. Numbers, for example, have an abstract structure on the one hand, on the other a connection to the arithmetical activities that the numerical notations are based on; they not only have provable properties, but these properties also apply to quantities of real objects.

Arithmetical concepts at this level of development thus have at the same time *extrinsic meanings* that result from their origin in arithmetical activities and *intrinsic meanings* that are deduced from seemingly self-evident axioms. This results in particular in a canonical meaning of the concept of number at this level of development, usually expressed by the term *natural number*. Natural numbers are extrinsically determined as reflectively constructed structures of actions of correspondence and comparison. As cardinal numbers of quantities, they have thus a canonical object that determines their quasi natural properties. Intrinsically, their properties are determined by universal laws such as the distributive law or the commutative law of addition and multiplication. Those laws can be arranged deductively in a way that makes them appear to be logical conclusions of a few, seemingly self-evident axioms, for example the Peano axioms.

At *level 3b of formal deduction*, the concepts formulated in natural language are substituted by terms of formal languages, so connotations with their original meanings are systematically prevented. The concepts can be subsumed in generalized, unifying concepts that may be constructed artificially, and be entirely determined by mathematical structures. Their meanings no longer appear to be determined naturally, but through axioms that seem to be presupposed arbitrarily.

Numbers at this level appear as superposition of algebraic structures, structures of order and topological structures. These structure can be precisely defined and distinguished from each other by axioms. These axioms can be modified and in various ways combined with each other into structures for which artificial concepts have been created, for example, the concept of semigroup, the concept of group, the concept of topological group, the concept of ring and the concept of body. The only condition to be fulfilled is the condition that there may be no contradiction deducible from the axioms that define a structure. A mathematical object is considered to exist when the defining system of axioms is consistent. Numbers are no longer in any way distinguished among these mathematical objects; their apparent “naturalness”, characteristic for the level of deduction in natural language, appears to be an fortuitous historical relict of the history of mathematical thought.

Historical identification of concept-based arithmetic. The definition of the level of concept-based arithmetic shows that the transition to this level is not a process specific to the concept of number. The criterion for the transition to concept-based arithmetic is the emergence of representations of arithmetical facts in written language and of deductions based on this form of representation. The historical identification of this transition thus appears to be straightforward, since a recurring topos of mathematical historiography states that pre-Classical mathematics did not know proofs.[53] If this observation were correct, the transition from the level of symbol-based to concept-based arithmetic in the sense of the definitions given here would not have been accomplished in any of the early literate civilizations.

Historians who study pre-Classical mathematics have with good reason repeatedly raised the objection that here a particular form of representing deductive conclusions is being confused with the mental operations themselves and thus that European mathematical tradition, of which

[53] Compare for example Becker/Hofman, *Geschichte der Mathematik*, p. 41; Vogel, *Vorgriechische Mathematik*, vol. 2, pp. 84f.; Wussing, *Mathematik in der Antike*, pp. 56f.; Gericke, *Geschichte des Zahlbegriffs*, p. 20; Gericke, *Mathematik in Antike und Orient*, pp. 71ff.

this form of representation is characteristic, is being eurocentrically overestimated.[54] Besides, the logical structures of pre-Classical mathematical activities have unfortunately been only insufficiently studied. For a long time historians of science have, in the assumption – influenced by the rationalistic-idealistic tradition – that logical and mathematical thought is by nature universal, failed to see the problems connected with the translation of documents of pre-Classical mathematical thought into modern mathematical terminology. Pre-Classical mathematical sources have often been translated in modern transcription, with the consequence that information on the *Eigenbegrifflichkeit* of the sources, in particular on the structures of technical terms and of immanent deductive relations, can only be gained by reverting to the original texts.[55]

Thus, for the moment it can only be stressed that the first examples for the explicit representation of chains of inferences, which described abstract mathematical objects and led to universal propositions about such mental objects, come from the Classical ancient world and that the question whether comparable structures of abstract concept formations and deductive thought as implicit structures of cognition already existed in the early civilizations has to remain unanswered.

The oldest known example of a deductively ordered system of universal propositions is the so-called doctrine of even and odd numbers, a theory that has not come down to us in its original form through preserved sources, but which can be reconstructed in its outlines from definitions and theorems of Euclid's *Elements.*[56] The doctrine goes back to the Pythagorean tradition of the 5th century B.C. The universally formulated arithmetical insights of the doctrine are mainly concerned with how the property of a number created by a calculation to be it either even or odd is dependent on the respective properties of the original numbers. The 14 relevant theorems that have come down to us have, in all probability, been found with the help of geometrical configurations of Greek counters, the so-called figured numbers, in connection with the attempt by the Pythagoreans, to associate all objects with numbers.[57] Later, the theorems of the doctrine were deductively arranged according to the logical dependencies among them. In the modified form in which the doctrine has been transmitted, the theorems appear to be arranged in a deductive schema, which obviously does not correspond to the original considerations that led to the

---

54 Compare Joseph, *The Crest of the Peacock*; Gerdes, *Ethnogeometrie*, in particular pp. 24ff.; for Chinese mathematics: Chemla, *Theoretical Aspects of the Chinese Algorithmic Tradition*; for Babylonian mathematics: Friberg, *Mathematics*, in particular pp. 582ff.

55 Compare for this the results of more recent philological studies, for example for Greek mathematics: Szabó, *Anfänge des Euklidischen Axiomensystems*; id., *Die Entfaltung der griechischen Mathematik*; for Babylonian mathematics: Høyrup, *Algebra and Naive Geometry*.

56 The definitions VII 1 through VII 2, VII 6 through VII 10 and VII 12, as well as the theorems IX 21 through IX 34.

57 Heath, *The Thirteen Books of Euclid's Elements*, p. 67; Becker, *Die Lehre vom Geraden und Ungeraden*; van der Waerden, *Die Arithmetik der Pythagoreer*.

formulation of the theorems.[58] Euclid's version, for example, uses as evidence the distributive law, which is of much more general nature and does not belong to the special doctrine of even and odd numbers.

The development of the doctrine of even and odd numbers therefore begins with the reflection of the sophisticated use of a symbolic representation of numbers. The results were on the one hand *technical terms* such as the concept of odd and even, on the other hand general insights about logical relations between those abstract concepts, for example the realization that the sum of two even numbers is again even. The preliminary conclusion of the development was constituted by the proof of such propositions based on explicit definitions of concepts as well as on more universal theorems about abstract numbers, for example the distributive law, a proof that has become completely independent of the origin of the insights. This development of the doctrine from the symbolic representation of numbers by counting units to a representation in written language and deduction of their properties from universal basic assumptions marks precisely the transition from symbol-based to concept-based arithmetic in the sense of the definition given here.

Historical identification of the transition to formal deduction. Like the transition to concept-based arithmetic in general, the transition from the first sublevel of deduction in natural language to the second sublevel of formal deduction is not a process specific to the concept of number, but a process that is characteristic of the development of mathematics in general and which includes all mathematical concepts.

In the case of the concept of number, the starting point is constituted by deductions in natural language of properties of numbers, whereby the concrete relation of the arithmetical concepts to the arithmetical activities, conveyed by the language, firmly connects these with a specific real meaning and thus gives the impression that they are not of merely formal nature. Through a reflected dealing with a more and more refined technique of logical proofs and its heuristics, however, more and more differentiated formal structures of higher-order develop as products of reflection, structures which are increasingly distant from the basis of arithmetic activities, from which they have been abstracted.

Historically, this process presents itself as a recurring, crisis-prone analysis of the bases of the concept of number which through the processes of reflection is becoming increasingly problematic. Conditioned by the origin of the concept of number, deductive arithmetic was understood as the theory of natural numbers, although the limitations of such a notion soon became apparent. The expansion of the number concept from natural numbers to numbers that no longer rep-

[58] Lefèvre, *Rechenstein und Sprache*.

resent cardinal numbers of finite sets, the expansion to fractions, negative, irrational and imaginary numbers, appears at the level of deduction conditioned by natural language as an unnatural, artificial construction. It is incompatible with the connotation that the concept number can as quantity be attributable to concrete sets of objects.

Thus the problem already existed in antiquity that fractions could no longer be related to the origin of the number concept in counting sequences and tallying systems. They did not come under Euclid's definition "number is a quantity composed of units", from which the properties of numbers in Euclid's *Elements* were deduced.[59] In order not to endanger this whole deductive edifice, Platonism dogmatically excluded from theoretical arithmetic all numerical structures that did not come under this definition.[60] The theory of proportions functioned as a substitute for the expanded concept of number. The problem was aggravated with the renaissance of ancient mathematics in the Early Modern Era, since numerical structures constructed by reflective abstractions which did not fit the ancient definition of number became ever more extensive. As long as the ancient concept of number determined thinking, they were perceived as "absurd numbers"[61], "irrational numbers" and "imaginary numbers".

The precondition for overcoming this understanding of number and thus for the transition to the level of formal deduction was the development of the technique of using in deduction, instead of words, variables without connotations regarding content. This went so far as to cause the concept of number to lose its apparent association with specific meanings of words. Only with this transition to the level of formal deduction, at which consistency is the only criterion for the existence of a mathematical object, do constructions become acceptable that are connected with the expansion of the number concept and which relativize the canonical meaning of the concept of natural numbers.

Leibniz, who systematically freed himself of traditional deductive systems and made use of the potentialities of symbolic representations of mathematical facts with at times entirely new invented symbol systems in constructing new deductive systems, is rightly seen as an early advo-

---

[59] Def. VII 2, compare Heath, *The Thirteen Books of Euclid's Elements*, p. 277.

[60] In Plato's *Republic* Socrates instructs his interlocutor Glaukon: 'I mean, I'm sure you're aware that the experts in the field pour scorn on any attempt to divide the actual number one and refuse to allow it. If you chop it up, they multiply it; they take steps to preserve one's oneness and to prevent it ever appearing to contain a multiplicity of factors (…) What do you think they'd say, then, Glaucon, if someone were to ask them, in surprise, "What are these numbers you're talking about? What numbers involve a oneness which fulfills your requirements, where every single unit is equal to every other unit, without even the smallest variation, and without being divisible in the slightest?" 'I think they'd reply that the numbers they're talking about are only accessible to thought, and cannot be grasped in any other way.' Plato, *Republi*c, book VII., §§. 525d-526a (translated by R. Waterfield, Oxford University Press 1994).

[61] Michael Stifel called the negative numbers "numeri absurdi", compare Tropfke, *Geschichte der Elementar-Mathematik*, vol. 2, p. 98.

cate of such a formal concept of number. Through the analytical tradition co-founded by him, mathematical concepts became obsolete that were concrete and mostly oriented to meanings based on geometry.

This tradition, which originated at the beginning of the modern era, but which was insufficiently developed to generally assert itself in mathematics before the end of the 19th century, finally brought about the transition to formal arithmetic. This transition coincided with the emergence of modern theory of proof. Hilbert's characterization of mathematical concepts – as in principle implicitly defined by the axioms – made the formal perception of mathematical objects which constitutes the condition of the level of formal deduction programmatic for mathematical thought.

Final remarks

Historical epistemology does not have as its object the real history of acts of thought and cognition, but it is supposed to answer the question whether in this history theoretically interpretable, universal stages and processes of the development of thought can be identified, stages that mark, for a real history of processes of cognition, the horizon of the insights possible in particular historical situations. Accordingly, the historical epistemology of the development of the concept of number does not comprise the development of the external form of arithmetical techniques that can be directly documented through sources, but the development of arithmetical thought which finds its expression in those techniques. The purpose of the theoretical reflections presented here is to create the conditions for a reconstruction of this history of arithmetical thought.

The starting point for our considerations was constituted by a basic problem one encounters when attempting, in line with our purpose, to draw conclusions, by means of cognitive psychology, on the development of arithmetical thought from historical sources that document arithmetic operations or their results. The problem becomes particularly evident from the fact that there is no satisfactory answer to the question: which of the cognitive structures and processes of arithmetical thought are in some way variable and can thus in principle be explained historically? It results from the contradiction that appears to exist between the non-empirical, logically necessary character of the properties of numbers and the historical changes of arithmetical techniques. Psychologically, it appears reasonable to infer that propositions which are non-empirically valid because they are logically deducible must be universal. Historical and cross-cultural studies, however, encounter the continuous development and change of arithmetical techniques. The uncertainty resulting from this problem as to which conclusions about arithmetical thought are in principle admissible from our historical documents can obviously not be eliminated through the perspective and with the means of a single discipline. It rather requires theoretical considerations that bridge psychology and history, and which can be reconciled with findings by both scientific disciplines.

Which structures and processes of arithmetical thought have their origin in ontogenesis and which are, on the other hand, the result of historiogenesis? Very different general answers have been given to the question of the relation between ontogenesis and historiogenesis.

In the quasi "Platonic" tradition, which extends from the Pythagoreans to logical positivism, an attempt has been made to find a sharp boundary between mathematics and empirical knowledge, that is, between logical thinking and sensory experience, and to ascribe the logico-mathematical side to an historically immutable human intellect, the empirical side, however, to an historically changing experience of things. Following the assumptions of this theoretical tradition, the development of cognition in ontogenesis and in historiogenesis would be concerned with fundamentally different objects of cognition.

Against these attempts to draw an absolute boundary, attempts which implicate that ontogenesis and historiogenesis exhibit only marginal points of contact, developmental theory has formulated the genetic principle. This principle postulates parallelism of ontogenesis and historiogenesis in two mutually exclusive ways. According to the genetic principle of the theory of evolution, the individual, in order to attain the phylogenetic and historiogenetic climax of his development, must in ontogenesis recapitulate the developmental process of the human species, both as a biological, and as a social being. According to genetic epistemology as it was brought forth by developmental psychology, however, the historical development of fundamental structures of logico-mathematical thought only makes explicit ontogenetic stages of development that are epigenetically determined, just like stages in the biological development of the individual. Thus, in the former case the individual recapitulates in ontogenesis the evolution of the human species, in the latter the the human species follows a developmental sequence which is predisposed in ontogenesis.

The solution of the problem which has been outlined in considerations presented here, cannot be subsumed in any of these alternatives. It is, rather, based on the assumption that ontogenesis and historiogenesis represent processes which are, albeit concerned with the same objects of developing cognition, fundamentally different. This solution is based on a particular interpretation of a concept of cognitive psychology that is central for the understanding of communication and transmission of cognitive structures, that is, the concept of the external representation of cognitive structures. External representations of cognitve structures are interpreted here from the perspective of the history and sociology of knowledge. If the assumption of cognitive psychology is correct that the cognitive structures of logico-mathematical thought in ontogenesis are constructed by reflective abstractions from coordinations of actions, and if the external representations can represent stages of reflection in a way that their meaning might be reconstructed in symbolic actions by the individual, then there is no longer an insurmountable basic contradiction between the alterability of logico-mathematical constructs in history and their independence of experience. With the interpretation of the historically transmittable tools of arithmetical techniques as embodiments of the cognitive structures constructed by reflective abstraction, ontogenesis gains a genuine historical dimension – embodied in the culture-specific symbolic scenarios as conditions of development.

What is achieved with this answer to the question about the historical or ahistorical nature of logico-mathematical and in particular arithmetical thought?

The result of such considerations certainly can not consist of replacing historical arguments with theoretical arguments. Indeed, the theoretical considerations in the first part of the work have by no means made historical research redundant. The answer to the question, to what degree the structures and processes of arithmetical thought represent culture-independent and historically unalterable universals of the nature of *homo sapiens*, and what part of arithmetical thought on the other hand derives ultimately from cultural achievements and what structures of this cognition have developed in which historical periods, is by no means theoretically preempted by the considerations presented here. These considerations, to the contrary, create one decisive precondition to study and answer such psychologically oriented questions by historical and cross-cultural studies. For the theory presented here concerning the transmission of cognitive structures of logico-mathematical thought provides an opportunity of identifying such structures not only along the direct way of experimental psychology, but also indirectly by reconstructing them through an analysis of their external representations.

In the second section of the work conclusions were indeed drawn which concern our understanding of the historical development of arithmetic. This occurred in relating developmental stages of reflective abstraction which can be psychologically identified with historically identifiable stages of the levels of reflection represented by external representations of arithmetical thought. The result is a psychological interpretation of the global historical development which leads from cultures without arithmetical techniques to the modern arithmetical thought of the industrial age, in the following stages of development:

Stage 0 *Pre-arithmetical quantification*: approximately until the end of the Mesolithic period (in the Near East until ca. 10,000 B.C.).

No arithmetical activities. All judgments about quantities are based on direct comparisons of amounts and sizes. Communication and transmission only by transmittable techniques of comparison and by comparative expressions of language.

Stage 1 *Proto-arithmetic*: Neolithic period and Early Bronze Age (in the Near East until ca. 3000 B.C.).

Quantities are precisely identified by one-to-one correspondences. Communication and transmission with the aid of conventionalized counting sequences and tallying systems.

Stage 2a *Symbol-based arithmetic with context-dependent symbol systems*: Period of the early city cultures (in the Near East until the invention of the sexagesimal place value system around 2000 B.C.). Quantities are structured by metrological systems. Communication and transmission of these systems and of the corresponding mental constructs through complex symbol systems and developed techniques for the transformation of symbol configurations.

Stage 2b *Symbol-based arithmetic with context-independent symbol systems*: Period of developed city cultures (in the Near East until the beginning of Classical Antiquity around 500 B.C.).

Quantities are structured by abstract numerical systems with object-independent arithmetical operations. Communication and transmission of these systems by unified, context-independent, but culture-specific symbol systems for the representation of arbitrary quantities, including abstract “rules of calculation”. Emergence of first forms of “pre-Classical mathematics” that are abstract but dependent on culture-specific symbol systems.

Stage 3a *Concept-based arithmetic with deductions in natural language*: Classical Antiquity, Late Antiquity, Middle Ages and Early Modern Era (until the emergence of analytical mathematics in the 18th century A.D.).

Abstract number concept with “a priori” provable properties. Communication and transmission with the aid of a written representation of “propositions” about abstract numbers and their mathematical properties. Propositions are logically ordered and systematically arranged by deductive theories according to the modelof Euclid’s *Elements*.

Stage 3b *Concept-based arithmetic with formal deductions*: The modern mathematical tradition until the present.

Formal understanding of arithmetical structures and expansion of the number concept by construction of new arithmetical structures. Communication and transmission with the aid of formal language systems.

The theoretical model of the history of arithmetical thought, gained through the assignment of stages of reflective abstraction to external representations, establishes a connection between theoretical assumptions of psychology and historical facts. This connection, however, concerns only general characteristics of the development of arithmetic such as the emergence, the change and the increasingly more complex representation of numbers which are so elementary that any theory of the historical development of arithmetical thought has to consider them and to explain them satisfactorily in one way or another. Thus the ultimate historical validity of the model is not a given. In this sense, the considerations offered are to be regarded only as a first step towards an historical epistemology of the development of the concept of number.

Such preliminary considerations are, however, indispensable. So long as we are unsuccessful in finding a satisfactory explanation for the extraordinary differences between logico-mathematical thought on the one hand in surviving non-literate cultures, cognition which at its face seems identifiable with first stages of the historical development of arithmetical thought, on the other the abstract and highly complex arithmetical thought of the present; so long as we are unsuccessful in theoretically illustrating how the forms of the latter have developed from the forms of the former, the studies of historical details of this development have no chance of even proposing plausible hypotheses about the psychological determinants of arithmetical thought in the various periods of the development of arithmetical techniques. In this sense the considerations offered here are not just preliminary discussions, but might be taken seriously as an attempt to create the theoretical framework for an historical epistemology of the number concept as the outcome of the developing material culture of calculation.

REFERENCES

Amiet, P. (1966). Il y a 5000 ans les Élamites inventaient l'Écriture. *Archaeologia*, 12, 16-23.

Arcà, M. (1984). Strategies for Categorizing Change in Scientific Research and in Children's Thought. *Human Development*, 27, 335-341.

Ascher, M., & Ascher, R. (1971/72). Numbers and Relations from Ancient Andean Quipus. *Archive for History of Exact Sciences*, 8, 288-320.

Ashcraft, M. H. (1992). Cognitive Arithmetic: A Review of Data and Theory. *Cognition*, 44, 75-106.

Bachelard, G. (1978). *Die Bildung des wissenschaftlichen Geistes*. Frankfurt: Suhrkamp.

Bachelard, G. (1978). *Die Philosophie des Nein*. Wiesbaden: Heymann.

Bateman, R., Goddard, I., O'Grady, R., Funk, V. A., Mooi, R., Kress, W. J., & Cannell, P. (1990). Speaking of Forked Tongues: The Feasibility of Reconciling Human Phylogeny and the History of Language. *Current Anthropology*, 31(1), 1-24.

Becker, O. (1965). Die Lehre vom Geraden und Ungeraden. In O. Becker (ed.), *Zur Geschichte der griechischen Mathematik* (pp. 125-145). Darmstadt: Wissenschaftliche Buchgesellschaft.

Becker, O., & Hofman, J. E. (1951). *Geschichte der Mathematik*. Bonn: Athenäum.

Bickerton, D. (1988). A Two-Stage Model of the Human Language Faculty. In S. Strauss (ed.), *Ontogeny, Phylogeny, and Historical Development* (pp. 86-105). Norwood: Ablex.

Blake, B. J. (1981). *Australian Aboriginal Languages*. London: Angus & Robertson.

Brainerd, C. J. (1979). *The Origins of the Number Concept*. New York: Praeger.

Bruner, J. S. (1966). On Cognitive Growth. In Bruner, J. S.,Olver, R. R., & P. M. Greenfield (1966). *Studies in Cognitive Growth: A Collaboration at the Center for Cognitive Studies* (pp. 1-67). New York: John Wiley & Sons.

Bruner, J. S.,Olver, R. R., & P. M. Greenfield (1966). *Studies in Cognitive Growth: A Collaboration at the Center for Cognitive Studies*. New York: John Wiley & Sons.

Campbell, J. I. D. (ed.) (1992). *The Nature and Origins of Mathematical Skills*. Advances in Psychology, Vol. 91. Amsterdam: North-Holland.

Carnap, R. (1975^2). *Grundlagen der Logik und Mathematik*. Darmstadt: Wissenschaftliche Buchgesellschaft.

Cassirer, E. ($1953/54^2$). *Philosophie der symbolischen Formen*. Darmstadt: Wissenschaftliche Buchgesellschaft.

Chace, A. B. (1927). *The Rhind Mathematical Papyrus*. Oberlin: Mathematical Association of America.

Chemla, K. (1991). Theoretical Aspects of the Chinese Algorithmic Tradition (First to Third Century). *Historia Mathematica*, 42, 75-98.

Closs, M. P. (1986). The Mathematical Notation of the Ancient Maya. In M. P. Closs (ed.), *Native American Mathematics* (pp. 291-369). Austin: University of Texas Press.

Damerow, P. (1993). Zum Verhältnis von Ontogenese und Historiogenese des Zahlbegriffs. In W. Edelstein & S. Hoppe-Graff (eds.), *Die Konstruktion kognitiver Strukturen: Perspektiven einer konstruktivistischen Entwicklungstheorie* (pp. 195-259). Bern: Huber.

Damerow, P. (1995). *Abstraction and Representation: Essays on the Cultural Evolution of Thinking*. Boston Studies in the Philosophy of Science, Vol. 175. Dordrecht: Kluwer Academic Publishers.

Damerow, P., & Englund, R. K. (1987). Die Zahlzeichensysteme der Archaischen Texte aus Uruk. In M. W. Green & H. J. Nissen (eds.), *Zeichenliste der Archaischen Texte aus Uruk (ATU 2)* (pp. 117-166). Berlin: Gebr. Mann.

Dasen, P. R. (1977). Are Cognitive Processes Universal? A Contribution to Cross-cultural Piagetian Psychology. In N. Warren (ed.), *Studies in Cross-cultural Psychology* (pp. 155-201). London: Academic Press.

Dasen, P. R., & de Ribaupierre, A. (1988). Neo-Piagetian Theories: Cross-cultural and Differential Perspectives. In A. Demetriou (ed.), *The Neo-Piagetian Theories of Cognitive Development: Toward an Integration* (pp. 287-326). Amsterdam: North-Holland.

Dasen, P. R., & Heron, A. (1981). Cross-Cultural Tests of Piaget's Theory. In H. C. Triandis & A. Heron (eds.), *Handbook of Cross-Cultural Psychology* , Vol. 4 (pp. 295-341). Boston: Allyn and Bacon.

Davis, R. B. (1984). *Learning Mathematics: the Cognitive Science Approach to Mathematics Education*. London: Croom Helm.

D'Errico, F. (1989). On Wishful Thinking and Lunar "Calendars": Reply. *Current Anthropology*, 30(4), 494-500.

D'Errico, F. (1989). Palaeolithic Lunar Calendars: A Case of Wishful Thinking? *Current Anthropology*, 30(1), 117-118.

Dixon, R. M. W. (1980). *The Languages of Australia*. Cambridge: Cambridge University Press.

Dixon, R. M. W., & Blake, B. J. (eds.). (1979 ff.). *Handbook of Australian Languages*. Amsterdam: Benjamins.

Dux, G. (1992). *Die Zeit in der Geschichte: Ihre Entwicklungslogik vom Mythos zur Weltzeit*. Frankfurt a. M.: Suhrkamp.

Englund, R. K. (1990). *Organisation der Verwaltung der Ur III-Fischerei*. Berliner Beiträge zum Vorderen Orient, Vol. 10. Berlin: Reimer.

Frege, G. (1988). *Die Grundlagen der Arithmetik: Eine logisch mathematische Untersuchung über den Begriff der Zahl.* Hamburg: Meiner.

Friberg, J. (1978). *A Method for the Decipherment, through Mathematical and Metrological Analysis, of Proto-Sumerian and Proto-Elamite Semi-pictographic Inscriptions*. Göteborg: Chalmers Technische Hochschule.

Friberg, J. (1979). *Metrological Relations in a Group of Semi-pictographic Tablets of the Jemdet Nasr Type, Probably from Uruk-Warka*. Göteborg: Chalmers Technische Hochschule.

Friberg, J. (1990). Mathematik. In D. O. Edzard (ed.), *Reallexikon der Assyriologie und Vorderasiatischen Archäologie,* Vol. 7 (pp. 531-585). Berlin: de Gruyter.

Fuson, K. C., & Hall, J. W. (1983). The Acquisition of Early Number Word Meanings: A Conceptual Analysis and Review. In H. P. Ginsburg (ed.), *The Development of Mathematical Thinking* (pp. 49-107). New York: Academic Press.

Gaida, M., & Tear, D. (1984). Kalender, Numerologie und lunare Astronomie auf Copán-Monumenten. *Beiträge zur allgemeinen und vergleichenden Archäologie*, 6, 310-353.

Gallistel, C. R., & Rochel, G. (1992). Preverbal and verbal counting and computation. *Cognition*, 44, 43-74.

Gay, J., & Cole, M. (1967). *The New Mathematics and an Old Culture*. New York: Holt, Rinehart & Winston.

Gelman, R., & Gallistel, C. R. (1978). *The Child's Understanding of Number*. Cambridge Mass.: Harvard University Press.

Gerdes, P. (1990). *Ethnogeometrie: Kulturanthropologische Beiträge zur Genese und Didaktik der Geometrie*. Bad Salzdetfurth: Franzbecker.

Gericke, H. (1970). *Geschichte des Zahlbegriffs*. Mannheim: Bibliographisches Institut.

Gericke, H. (1984). *Mathematik in Antike und Orient*. Berlin: Springer.

Gillings, R. J. (1972). *Mathematics in the Times of the Pharaohs*. Cambridge Mass.: MIT Press.

Gnerre, M. C. (1986). Some Notes on Quantification and Numerals in an Amazon Language. In M. P. Closs (ed.), *Native American Mathematics* (pp. 71-91). Austin: University of Texas Press.

Hallpike, C. R. (1979). *The Foundations of Primitive Thought*. Oxford: Clarendon.

Heath, S. T. L. (ed.) (1956). *The Thirteen Books of Euclid's Elements*. New York: Dover.

Heinzelin, J. de (1962). Ishango. *Scientific American*, 206 (Juni), 105-116.

Høyrup, J. (1990). Algebra and Naive Geometry. *Altorientalische Forschungen*, 17, 27-69, 262-354.

Hurford, J. R. (1975). *The Linguistic Theory of Numerals*. Cambridge: Cambridge University Press.

Hurford, J. R. (1987). *Language and Number*. Oxford: Basil Blackwell.

Ifrah, G. (1986). *Universalgeschichte der Zahlen*. Frankfurt a.M.: Campus.

Joseph, G. G. (1992). *The Crest of the Peacock: The Non-European Roots of Mathematics*. London: Penguin.

Juschkewitsch, A. P. (1966). *Geschichte der Mathematik im Mittelalter*. Basel: Pfalz.

Kant, I. (1911). *Kritik der reinen Vernunft*. Kant's Werke, Vol. 3. Berlin: Reimer.

Kant, I. (1911). Prolegomena zu einer jeden künftigen Metaphysik, die als Wissenschaft wird auftreten können. In *Kant's Werke*, Vol. 4 (pp. 253-383). Berlin: Reimer.

Lancy, D. F. (1983). *Cross-Cultural Studies in Cognition and Mathematics*. Developmental Psychology Series. New York: Academic Press.

Langer, J. (1980). *The Origins of Logic: Six to Twelve Months*. Developmental psychology series. New York: Academic Press.

Langer, J. (1986). *The Origins of Logic: One to Two Years*. Developmental psychology series. New York: Academic Press.

Lefèvre, W. (1981). Rechensteine und Sprache. In P. Damerow & W. Lefèvre, *Rechenstein, Experiment, Sprache: Historische Fallstudien zur Entstehung der exakten Wissenschaften* (pp. 115-168). Stuttgart: Klett-Cotta.

Leontjew, A. N. (1973). *Probleme der Entwicklung des Psychischen*. Frankfurt a.M.: Athenäum Fischer.

Levinson, S. C. (1991). *Relativity in Spatial Conception and Description*. Working Paper No. 1. Cognitive Anthropology Research Group at the Max Planck Institute for Psycholinguistics.

Lèvy-Bruhl, L. (1921). *Das Denken der Naturvölker*. Wien: Braumüller.

Li, Y., & Dù, S. (1987). *Chinese Mathematics: a Concise History*. Oxford: Clarendon.

Locke, L. L. (1923). *The Ancient Quipu or Peruvian Knot Record*. New York: The American Museum of Natural History.

Lorenzen, P. (1955). *Einführung in die operative Logik und Mathematik*. Berlin: Springer.

Marshack, A. (1972). *The Roots of Civilization: The Cognitive Beginnings of Man's First Art, Symbol and Notation*. London: Weidenfeld and Nicolson.

Marshack, A. (1989). On Wishful Thinking and Lunar "Calendars". *Current Anthropology,* 30(4), 491-494.

Menninger, K. (1979). *Zahlwort und Ziffer*. Göttingen: Vandenhoeck & Ruprecht.

Mimica, J. (1988). *Intimations of Infinity: The Mythopoeia of the Iqwaye Counting System and Number*. Oxford: Berg.

Minsky, M. (1975). A Framework for Representing Knowledge. In P. H. Winston (ed.), *The Psychology of Computer Vision* (pp. 211-277). New York: McGraw-Hill.

Minsky, M. L. (1985). *The Society of Mind*. New York: Simon & Schuster.

Needham, J. (1959). *Mathematics and the Sciences of the Heavens and the Earth.* Science and Civilisation in China, Vol. 3. Cambridge: Cambridge University Press.

Neugebauer, O. (1926). *Die Grundlagen der ägyptischen Bruchrechnung*. Berlin: Springer.

Neugebauer, O. (1934). *Vorgriechische Mathematik.* Vorlesungen über Geschichte der Antiken Mathematischen Wissenschaften. Berlin: Springer.

Neugebauer, O. (1935). *Mathematische Keilschrifttexte*. Berlin: Springer.

Neugebauer, O., & Sachs, A. (1945). *Mathematical Cuneiform Texts*. New Haven: American Oriental Society.

Nissen, H. J., Damerow, P., & Englund, R. K. (1993). *Archaic Bookkeeping: Early Writing and Techniques of Economic Administration in the Ancient Near East*. Chicago: Chicago University Press.

Piaget, J. (1974). *Biologie und Erkenntnis*. Frankfurt a. M.: Fischer.

Piaget, J. (1975). *Die Entwicklung des Erkennens*. Stuttgart: Klett.

Piaget, J., & Garcia, R. (1989). *Psychogenesis and the History of Science*. New York: Columbia University Press.

Piaget, J., & Szeminska, A. (1965). *Die Entwicklung des Zahlbegriffs beim Kinde*. Stuttgart: Klett.

Pinxten, R. (1976). Epistemic Universals: A Contribution to Cognitive Anthropology. In R. Pinxten (ed.), *Universalism versus Relativism in Language and Thought* . Den Haag: Mouton.

Pinxten, R. (ed.). (1976). *Universalism versus Relativism in Language and Thought*. Den Haag: Mouton.

Platon (1988). *Der Staat*. Sämtliche Dialoge, ed. O. Apelt, Vol. 5. Hamburg: Meiner.

Powell, M. A. (1973). *Sumerian Numeration and Metrology*. Ann Arbor: University Microfilms.

Powell, M. A. (1976). The Antecedents of Old Babylonian Place Notation and the Early History of Babylonian Mathematics. *Historia Mathematica*, 3, 417-439.

Renfrew, C. (1988). Archaeology and Language. *Current Anthropology*, 29(3), 437-441.

Saxe, G. B. (1982). Culture and the Development of Numerical Cognition: Studies among the Oksapmin of Papua New Guinea. In C. J. Brainerd (ed.), *The Development of Logical and Mathematical Cognition* (pp. 157-176). New York: Springer.

Scharlau, B., & Münzel, M. (1986). *Quellqay: Mündliche Kultur und Schrifttradition bei Indianern Lateinamerikas*. Frankfurt a. M.: Campus.

Schmandt-Besserat, D. (1977). An Archaic Recording System and the Origin of Writing. *Syro-Mesopotamian Studies*, 1, 31-70.

Schmandt-Besserat, D. (1992). *Before Writing*. Austin: University of Texas Press.

Smith, D. A.,Greeno, J. G., & Vitolo, T. M. (1989). A Model of Competence for Counting. *Cognitive Science*, 13, 183-211.

Strauss, S. (Ed.). (1988). *Ontogeny, Phylogeny, and Historical Development*. Norwood: Ablex.

Szabó, Á. (1965). Anfänge des Euklidischen Axiomensystems. In O. Becker (ed.), *Zur Geschichte der griechischen Mathematik* (pp. 355-461). Darmstadt: Wissenschaftliche Buchgesellschaft.

Szabó, Á. (1994). *Die Entfaltung der Griechischen Mathematik*. Mannheim: Bibliografisches Institut.

Thompson, J. E. S. (1941). *Maya Arithmetic*. Contributions to American Anthropology and History, Vol. 36. Washington: Carnegie Institute.

Thompson, J. E. S. (1960). *Maya Hieroglyphic Writing*. Norman: University of Oklahoma Press.

Tropfke, J. (1930-1933). *Geschichte der Elementar-Mathematik*. Berlin: de Gruyter.

Tropfke, J. (1980). *Geschichte der Elementar-Mathematik - Vollständig neu bearbeitet von Kurt Vogel, Karin Reich und Helmuth Gericke*, Vol. 1. Berlin: de Gruyter.

Vaiman, A. (1974). Über die Protosumerische Schrift. *Acta Antiqua Hung.*, 27, 15-27.

Vygotsky, L. S. (1986). *Thought and Language*. Cambridge Mass.: MIT Press.

Vogel, K. (1958/59). *Vorgriechische Mathematik*. Hannover/Paderborn: Schroedel/Schöningh.

Waerden, B. L. van der (1965). Die Arithmetik der Pythagoreer. In O. Becker (ed.), *Zur Geschichte der griechischen Mathematik* (pp. 203-254). Darmstadt: Wissenschaftliche Buchgesellschaft.

Wertheimer, M. (1925). Über das Denken der Naturvölker: Zahlen und Zahlgebilde. In *Drei Abhandlungen zur Gestalttheorie* (pp. 106-163). Erlangen: Palm & Enke.

Whorf, B. L. (1963). *Sprache, Denken, Wirklichkeit*. Reinbek: Rowohlt.

Wussing, H. (1965). *Mathematik in der Antike*. Leipzig: Teubner.