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非完整力学系统的 Lie 对称性直接导致的一种守恒量

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摘要：非完整约束力学系统对称性与守恒量的研究不仅具有重要的理论意义，而且具有重要的实际价值。利用无限小群变换分析方法研究了非完整力学系统的 Lie 对称性直接导致的一种守恒量，给出了系统 Lie 对称性的判据，得到了系统 Lie 对称性直接导致的一种守恒量存在的条件和形式，举例说明了结果的应用。

关键词：一般力学；非完整力学系统；Lie 对称性；守恒量

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A Kind of Conserved Quantity Induced by Lie Symmetry of Nonholonomic Mechanical System

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Abstract: It represents important not only theoretical but also practical significance to study the symmetry and conserved quantity of nonholonomic mechanical system. A kind of conserved quantity directly induced by Lie symmetry is studied by using infinitesimal group transformation analysis. Firstly, the criterion of Lie symmetry for nonholonomic mechanical system is given. Secondly, the existential condition of the conserved quantity and its forms are obtained. Finally, an example is given to illustrate the application of the results.

Key words: general mechanics; nonholonomic mechanical system; Lie symmetry; conserved quantity

0　引言

对称性是力学和物理学中更高层次的法则。对称性理论有许多用途，其中一个重要的用途就是用来寻求守恒量。守恒量不仅具有重要的数学意义，而且反映物理系统深刻的本质，在约束力学系统运动微分方程不可积分的情况下，如果能找到系统的一些守恒量，对了解系统的物理状态，认识其内在特性及规律性具有非常重要的意义。分析力学中寻求约束力学系统守恒量的近代对称性方法主要有：Noether 对称性[1-2]，Lie 对称性[2-3]和 Mei 对称性[4-6]。Noether 对称性是系统的 Hamilton 作用量在无限小变换下的一种不变性，Lie 对称性是系统的微分方程在无限小变换下的一种不变性，Mei 对称性是指系统运动微分方程中的动力学函数经历无限小变换后仍满足原来方程的一种不变性。3 种对称性导致的守恒量主要有：Noether 型守恒量，Hojman 型守恒量和 Mei 型守恒量[7-8]。近年来，约束力学系统的 3 种主要对称性与守恒量的研究取得了一系列重要成果[4-21]。约束力学系统的 Lie 对称性能直接导致 Hojman 型守恒量，间接导致 Noether 型守恒量和 Mei 型守恒量[7]。本文研究非完整力学系统的 Lie 对称性直接导致的另一种守恒量，给出这种守恒量存在的条件和形式，举例说明其应用。

1　系统 Lie 对称性的判据

设力学系统的位形由 n 个广义坐标 $q_s(s=1,2,\cdots,n)$ 来确定，系统的运动受到 g 个双面理想 Chetaev

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型非完整约束

$$f_\beta(t,\boldsymbol{q},\dot{\boldsymbol{q}})=0,\ \beta=1,2,\cdots,g, \tag{1}$$

约束(1)加在虚位移 δq_s 上的 Chetaev 条件为

$$\frac{\partial f_\beta}{\partial \dot{q}_s}\delta q_s=0, \tag{2}$$

则系统的运动微分方程[2,7]可表示为

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}-\frac{\partial L}{\partial q_s}=Q_s+\Lambda_s, \tag{3}$$

式中:

$$\Lambda_s=\Lambda_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})=\lambda_\beta\frac{\partial f_\beta}{\partial \dot{q}_s} \tag{4}$$

为广义非完整约束力,$\lambda_\beta=\lambda_\beta(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ 为约束乘子;$L=L(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ 为系统的 Lagrange 函数;$Q_s=Q_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ 为非势广义力。引入 Euler 算子

$$E_s=\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_s}-\frac{\partial}{\partial q_s}. \tag{5}$$

(4)式可简写为

$$E_s(L)=Q_s+\Lambda_s. \tag{6}$$

Lie 对称性是运动微分方程在无限小群变换下的一种不变性。引进无限小群变换

$$\begin{cases}t^*=t+\varepsilon\xi_0(t,\boldsymbol{q},\dot{\boldsymbol{q}}),\\ q_s^*(t^*)=q_s(t)+\varepsilon\xi_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}),\end{cases} \tag{7}$$

式中:ε 为无限小参数;ξ_0 和 ξ_s 为无限小单参数群变换的生成元。根据力学系统的 Lie 对称性理论,(1)式和(6)式可得到如下判据:

判据　对(1)式和(6)式,如果(7)式的生成元 ξ_0、ξ_s 满足确定方程

$$X^{(2)}[E_s(L)]=X^{(1)}(Q_s+\Lambda_s), \tag{8}$$

则系统具有 Lie 对称性。其中

$$X^{(1)}[f_\beta(t,\boldsymbol{q},\dot{\boldsymbol{q}})]=0, \tag{9}$$

$$X^{(2)}=X^{(1)}+[(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0)-\ddot{q}_s\dot{\xi}_0]\frac{\partial}{\partial \ddot{q}_s}, \tag{10}$$

$$X^{(1)}=X^{(0)}+(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0)\frac{\partial}{\partial \dot{q}_s}, \tag{11}$$

$$X^{(0)}=\xi_0\frac{\partial}{\partial t}+\xi_s\frac{\partial}{\partial q_s}. \tag{12}$$

2　系统 Lie 对称性直接导致的一种守恒量

非完整力学系统的 Lie 对称性能直接导致 Hojman 型守恒量,间接导致 Noether 型守恒量和 Mei 型守恒量[7]。下述定理给出了非完整力学系统的 Lie 对称性直接导致的另一种守恒量。

定理　对(1)式和(6)式,如果存在规范函数 $G=G(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ 使系统 Lie 对称性的生成元 ξ_0,ξ_s 满足条件

$$X^{(1)}\left(\frac{\partial L}{\partial t}\right)+\dot{\xi}_0\frac{\partial L}{\partial t}-\dot{q}_sX^{(1)}(Q_s+\Lambda_s)-\dot{\xi}_s(Q_s+\Lambda_s)-\dot{G}=0, \tag{13}$$

则系统的 Lie 对称性直接导致如下形式的一种守恒量

$$I=\dot{q}_sX^{(1)}\left(\frac{\partial L}{\partial \dot{q}_s}\right)-X^{(0)}(L)+G=\mathrm{const}. \tag{14}$$

证明　将 I 对 t 求导得

$$\frac{\mathrm{d}I}{\mathrm{d}t}=\dot{q}_s\frac{d}{\mathrm{d}t}\left[X^{(1)}\left(\frac{\partial L}{\partial \dot{q}_s}\right)\right]+\ddot{q}_sX^{(1)}\left(\frac{\partial L}{\partial \dot{q}_s}\right)-\frac{d}{\mathrm{d}t}[X^{(0)}(L)]+\dot{G}. \tag{15}$$

由于

$$X^{(1)}\left(\frac{\partial L}{\partial \dot{q}_s}\right)=\frac{\partial X^{(1)}(L)}{\partial \dot{q}_s}-\left[\frac{\partial\xi_0}{\partial \dot{q}_s}\frac{\partial L}{\partial t}+\frac{\partial\xi_k}{\partial \dot{q}_s}\frac{\partial L}{\partial q_k}+\frac{\partial(\dot{\xi}_k-\dot{q}_k\dot{\xi}_0)}{\partial \dot{q}_s}\frac{\partial L}{\partial \dot{q}_k}\right], \tag{16}$$

$$\frac{\partial X^{(1)}(L)}{\partial q_s}=X^{(1)}\left(\frac{\partial L}{\partial q_s}\right)+\left[\frac{\partial\xi_0}{\partial q_s}\frac{\partial L}{\partial t}+\frac{\partial\xi_k}{\partial q_s}\frac{\partial L}{\partial q_k}+\frac{\partial(\dot{\xi}_k-\dot{q}_k\dot{\xi}_0)}{\partial q_s}\frac{\partial L}{\partial \dot{q}_k}\right], \tag{17}$$

因此,有

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[X^{(1)}\left(\frac{\partial L}{\partial \dot{q}_s}\right)\right]=&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial X^{(1)}(L)}{\partial \dot{q}_s}\right)-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\xi_0}{\partial \dot{q}_s}\right)\frac{\partial L}{\partial t}-\\&\frac{\partial\xi_0}{\partial \dot{q}_s}\left(\frac{\partial L}{\partial t}\right)-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\xi_k}{\partial \dot{q}_s}\right)\frac{\partial L}{\partial q_k}-\frac{\partial\xi_k}{\partial \dot{q}_s}\left(\frac{\partial L}{\partial q_k}\right)-\\&\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial(\dot{\xi}_k-\dot{q}_k\dot{\xi}_0)}{\partial \dot{q}_s}\right]\frac{\partial L}{\partial \dot{q}_k}-\frac{\partial(\dot{\xi}_k-\dot{q}_k\dot{\xi}_0)}{\partial \dot{q}_s}\left(\frac{\partial L}{\partial \dot{q}_k}\right)=\\&E_s[X^{(1)}(L)]-E_s(\xi_0)\frac{\partial L}{\partial t}-E_s(\xi_k)\frac{\partial L}{\partial q_k}-\\&E_s(\dot{\xi}_k-\dot{q}_k\dot{\xi}_0)\frac{\partial L}{\partial \dot{q}_k}-\frac{\partial\xi_0}{\partial \dot{q}_s}\left(\frac{\partial L}{\partial t}\right)-\frac{\partial\xi_k}{\partial \dot{q}_s}\left(\frac{\partial L}{\partial q_k}\right)-\\&\left(\frac{\partial\dot{\xi}_k}{\partial \dot{q}_s}-\dot{q}_k\frac{\partial\dot{\xi}_0}{\partial \dot{q}_s}\right)\left(\frac{\partial L}{\partial \dot{q}_k}\right)+\dot{\xi}_0\left(\frac{\partial L}{\partial \dot{q}_s}\right)+X^{(1)}\left(\frac{\partial L}{\partial q_s}\right),\end{aligned} \tag{18}$$

因为

$$\begin{aligned}E_s[X^{(1)}(L)]=&E_s(\xi_0)\frac{\partial L}{\partial t}+\xi_0E_s\left(\frac{\partial L}{\partial t}\right)+\dot{\xi}_0\frac{\partial^2L}{\partial t\,\partial \dot{q}_s}+\\&\frac{\partial\xi_0}{\partial \dot{q}_s}\left(\frac{\partial L}{\partial t}\right)+E_s(\xi_k)\frac{\partial L}{\partial q_k}+\xi_kE_s\left(\frac{\partial L}{\partial q_k}\right)+\dot{\xi}_k\frac{\partial^2L}{\partial q_k\,\partial \dot{q}_s}+\\&\frac{\partial\xi_k}{\partial \dot{q}_s}\left(\frac{\partial L}{\partial q_k}\right)+E_s(\dot{\xi}_k-\dot{q}_k\dot{\xi}_0)\frac{\partial L}{\partial \dot{q}_k}+\end{aligned}$$

$$(\dot{\xi}_k - \dot{q}_k\dot{\xi}_0)E_s\left(\frac{\partial L}{\partial \dot{q}_k}\right) + (\dot{\xi}_k - \dot{q}_k\dot{\xi}_0)\frac{\partial^2 L}{\partial \dot{q}_k\,\partial \dot{q}_s} + \left(\frac{\partial \dot{\xi}_k}{\partial \dot{q}_s} - \dot{q}_k\frac{\partial \dot{\xi}_0}{\partial \dot{q}_s}\right)\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \dot{\xi}_0\left(\frac{\partial L}{\partial \dot{q}_s}\right), \quad (19)$$

$$X^{(2)}[E_s(L)] = \xi_0 E_s\left(\frac{\partial L}{\partial t}\right) + \xi_k E_s\left(\frac{\partial L}{\partial q_k}\right) + (\dot{\xi}_k - \dot{q}_k\dot{\xi}_0)E_s\left(\frac{\partial L}{\partial \dot{q}_k}\right) + (\dot{\xi}_k - \dot{q}_k\dot{\xi}_0)\frac{\partial^2 L}{\partial \dot{q}_s\,\partial q_k} + [(\dot{\xi}_k - \dot{q}_k\dot{\xi}_0)^{\cdot} - \ddot{q}_k\dot{\xi}_0]\frac{\partial^2 L}{\partial \dot{q}_k\,\partial \dot{q}_s}, \quad (20)$$

所以

$$X^{(2)}[E_s(L)] = E_s[X^{(1)}(L)] - E_s(\xi_0)\frac{\partial L}{\partial t} - E_s(\xi_k)\frac{\partial L}{\partial q_k} - E_s(\dot{\xi}_k - \dot{q}_k\dot{\xi}_0)\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial \xi_0}{\partial \dot{q}_s}\left(\frac{\partial L}{\partial t}\right) - \frac{\partial \xi_k}{\partial \dot{q}_s}\left(\frac{\partial L}{\partial q_k}\right) - \left(\frac{\partial \dot{\xi}_k}{\partial \dot{q}_s} - \dot{q}_k\frac{\partial \dot{\xi}_0}{\partial \dot{q}_s}\right)\left(\frac{\partial L}{\partial \dot{q}_k}\right). \quad (21)$$

由（18）式和(21）式,可知

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[X^{(1)}\left(\frac{\partial L}{\partial \dot{q}_s}\right)\right] = X^{(2)}[E_s(L)] + \dot{\xi}_0\left(\frac{\partial L}{\partial \dot{q}_s}\right) + X^{(1)}\left(\frac{\partial L}{\partial q_s}\right). \quad (22)$$

将(22)式代入(15)式,得

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \dot{q}_s X^{(2)}[E_s(L)] + \dot{\xi}_0\dot{q}_s\left(\frac{\partial L}{\partial \dot{q}_s}\right) + \dot{q}_s X^{(1)}\left(\frac{\partial L}{\partial q_s}\right) + \ddot{q}_s X^{(1)}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - (X^{(0)}(L)) + \dot{G}, \quad (23)$$

进一步整理可得

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \dot{q}_s X^{(2)}[E_s(L)] + \dot{\xi}_0\dot{q}_s\frac{\partial^2 L}{\partial t\,\partial \dot{q}_s} + \dot{\xi}_k\dot{q}_s\frac{\partial^2 L}{\partial \dot{q}_k\,\partial q_s} + \dot{\xi}_k\ddot{q}_s\frac{\partial^2 L}{\partial \dot{q}_k\,\partial \dot{q}_s} - \dot{\xi}_0\frac{\partial L}{\partial t} - \dot{\xi}_k\frac{\partial L}{\partial q_k} - \xi_0\frac{\partial^2 L}{\partial t^2} - \xi_k\frac{\partial^2 L}{\partial t\,\partial q_k} + \dot{G} = \dot{q}_s X^{(2)}[E_s(L)] + \dot{\xi}_s E_s(L) - X^{(1)}\left(\frac{\partial L}{\partial t}\right) - \dot{\xi}_0\frac{\partial L}{\partial t} + \dot{G}. \quad (24)$$

将(8)式和(13)式代入(24)式,可得

$$\frac{\mathrm{d}I}{\mathrm{d}t} = 0. \quad (25)$$

3 算例

系统的 Lagrange 函数为

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2), \quad (26)$$

非势广义力为

$$Q_1 = -\frac{t}{1+t^2},\ Q_2 = \frac{1}{1+t^2}, \quad (27)$$

受到的非完整约束为

$$f = \dot{q}_2 - t\,\dot{q}_1 = 0, \quad (28)$$

试研究系统 Lie 对称性直接导致的守恒量(14)式。

(6)式给出

$$\ddot{q}_1 = -\frac{t}{1+t^2} - \lambda t,\ \ddot{q}_2 = \frac{1}{1+t^2} + \lambda. \quad (29)$$

利用(28)式和(29)式可求得

$$\lambda = \frac{\dot{q}_1 - 1}{1+t^2}. \quad (30)$$

系统运动方程为

$$\ddot{q}_1 = -\frac{t\,\dot{q}_1}{1+t^2},\ \ddot{q}_2 = \frac{\dot{q}_1}{1+t^2}. \quad (31)$$

系统 Lie 对称性的确定方程(8)式给出

$$(\dot{\xi}_1 - \dot{q}_1\dot{\xi}_0)^{\cdot} - \ddot{q}_1\dot{\xi}_0 = -\frac{t}{1+t^2}(\dot{\xi}_1 - \dot{q}_1\dot{\xi}_0) - \xi_0\frac{1-t^2}{(1+t^2)^2}\dot{q}_1, \quad (32)$$

$$(\dot{\xi}_2 - \dot{q}_2\dot{\xi}_0)^{\cdot} - \ddot{q}_2\dot{\xi}_0 = \frac{1}{1+t^2}(\dot{\xi}_1 - \dot{q}_1\dot{\xi}_0) - \xi_0\frac{2t}{(1+t^2)^2}\dot{q}_1. \quad (33)$$

限制方程(9)式给出

$$X^{(1)}(f) = (\dot{\xi}_2 - \dot{q}_2\dot{\xi}_0) - t(\dot{\xi}_1 - \dot{q}_1\dot{\xi}_0) - \dot{q}_1\xi_0 = 0. \quad (34)$$

取无限小群变换的生成元为

$$\xi_0 = 0,\ \xi_1 = \ln(t + \sqrt{1+t^2}),\ \xi_2 = \sqrt{1+t^2}. \quad (35)$$

容易证明,对生成元(35)式,(32)式～(34)式满足。

由（13）式得

$$\dot{q}_1\frac{t\,\dot{\xi}_1}{1+t^2} - \dot{q}_2\frac{\dot{\xi}_1}{1+t^2} - \dot{\xi}_1\frac{t\,\dot{q}_1}{1+t^2} + \dot{\xi}_2\frac{\dot{q}_1}{1+t^2} + \dot{G} = 0, \quad (36)$$

于是有

$$G = 0. \quad (37)$$

根据（14）式可得系统 Lie 对称性导致的守恒量

$$I = \frac{1}{\sqrt{1+t^2}}(\dot{q}_1 + t\,\dot{q}_2) = \mathrm{const}. \quad (38)$$

4 结论

本文研究了非完整力学系统 Lie 对称性直接导致的一种守恒量,给出了守恒量的存在条件和形式。这种守恒量不同于 Noether 型守恒量、Hojman 型守

恒量和 Mei 型守恒量。非完整力学系统的 Lie 对称性能直接导致 Hojman 型守恒量,间接导致 Noether 型守恒量和 Mei 型守恒量,本文的研究表明,非完整力学系统的 Lie 对称性还能直接导致形如(14)式的守恒量。本文得到的结果具有更一般的意义,对完整力学系统也适用。本文的结果拓展了非完整力学系统 Lie 对称性导致的守恒量,丰富和发展了约束力学系统的对称性与守恒量理论。

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