Local Coordinates on Formal Path

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I was attracted to pursue my graduate study at Sto

the following two books:

Cheeger and Ebin, Comparison Theorems in Rieman

etry

Lawson, The Theory of Gauge Fields in Four Dime

I was partly supported by the Simons Graduate Fe

some time during my graduate study there.

Going to Stony Brook was clearly one of the mos

steps I took that determines my future life.

I named my daughter by the Chinese characters for S

- Motivations: 1. Integrations on loop spaces or p

  2. Cohomology theory of loop spaces or path sp

  3. Relationship with elliptic genus

- Example: Index as integrations over loop spac

  Bismut, ...)

- Example: Motivatic integrations over the forma

  (Kontsevich, Denef, Loeser, ...)

- Objective: Study path space by local coordinate

- Tool: Taylor expansions.

- Subjects: Smooth functions, vector fields, differe
  etc.

- Outcome: Infinite dimensional Lie algebras
  generated by differential operators.

Formal path space

- $c : (-a, a) \to M$: a path in a smooth manifold

- $\{x^i\}$ local coordinates on M

- Taylor expansion: $x^i(t) = \sum_{k=0}^{\infty} x^{i,k} t^k$.

- Key idea: Use $\{x^{i,k}\}$ as local coordinates to defin

  path space $\mathcal{P}M$.

Coordinate changes

- Let $\{y^j\}$ be another local coordinate system, give

  functions:

$$y^j = y^j(x^1, \ldots, x^n).$$

- The relationship between $\{y^{i,k}\}$ and $\{x^{j,l}\}$ is g

  power series expansion:

$$y^i(x^1(t), \ldots, x^n(t)).$$

- For example,

$$y^{i,1} = \frac{\partial y^i}{\partial x^j} x^{j,1}$$

$$y^{i,2} = \frac{1}{2} \frac{\partial^2 y^i}{\partial x^{j_1} \partial x^{j_2}} x^{j_1,1} x^{j_2,1} + \frac{\partial y^i}{\partial x^j} x^{j,2},$$

$$y^{i,3} = \frac{1}{6} \frac{\partial^3 y^i}{\partial x^{j_1} \partial x^{j_2} \partial x^{j_3}} x^{j_1,1} x^{j_2,1} x^{j_3,1} + \frac{\partial^2 y^i}{\partial x^{j_1} \partial x^{j_2}} x$$

- In general, $y^{i,k}$ is a weighted homogeneous polyn

  $(l \geq 1)$ of degree k.

Notations for partitions

- A *partition of* k is a sequence of integers $\mu =$
  such that $\mu_1 + \cdots + \mu_l = k, \quad \mu_1 \geq \cdots \geq \mu_l \geq 1.$

- $|\mu| := k$ is called the *weight* of μ, $l(\mu) := l$: the
  and is denoted by $l(\mu)$.

- Also write $\mu = (1^{m_1} 2^{m_2} \cdots n^{m_n})$. E.g. $\mu = (3, 3,$
  $l(\mu) = 3$, $|\mu| = 3 + 3 + 1 = 7$.

Some general facts about Taylor series expansi

- Let f be a smooth function in $x^1, \ldots, x^n$.

- Let the Taylor series of $f(x^1(t), \ldots, x^n(t))$ be

$$\mathcal{P}f = \sum_{k=0}^{\infty} f_{(k)} t^k.$$

- Then we have

$$f_{(k)} = \sum_{|\mu|=k} C_\mu \sum_{1\le j_1,\ldots,j_{l(\mu)}\le n} \frac{\partial^{l(\mu)} f}{\partial x^{j_1}\cdots\partial x^{j_{l(\mu)}}} \cdot \frac{1}{\binom{k}{\mu}}$$

where

$$C_{(1^{m_1}2^{m_2}\cdots)} = \frac{(\sum_{i\ge1} i m_i)!}{\prod_{i\ge1}((i!)^{m_1} m_i!)}, \qquad \binom{k}{\mu} =$$

- In particular, we have

$$y^{i,k} = \sum_{|\mu|=k} C_\mu \sum_{1\le j_1,\ldots,j_{l(\mu)}\le n} \frac{\partial^{l(\mu)} y^i}{\partial x^{j_1}\cdots\partial x^{j_{l(\mu)}}} \cdot \frac{1}{\binom{k}{\mu}}$$

We will prove the following two identities:

$$\sum_{j=1}^{n}\sum_{l=1}^{\infty} l x^{j,l} \frac{\partial f_{(k)}}{\partial x^{j,l+a}} = (k-a) f_{(k-a)}.$$

$$\frac{\partial f_{(k)}}{\partial x^{j,l}} = \frac{\partial f_{(k-l)}}{\partial x^{j}}.$$

In particular,

$$\frac{\partial y^{i,k}}{\partial x^{j,l}} = \frac{\partial y^{i,k-l}}{\partial x^{i}}.$$

For example,

$$y^{i,1} = \frac{\partial y^i}{\partial x^j} x^{j,1}$$

$$y^{i,2} = \frac{1}{2} \frac{\partial^2 y^i}{\partial x^{j_1} \partial x^{j_2}} x^{j_1,1} x^{j_2,1} + \frac{\partial y^i}{\partial x^j} x^{j,2},$$

$$y^{i,3} = \frac{1}{6} \frac{\partial^3 y^i}{\partial x^{j_1} \partial x^{j_2} \partial x^{j_3}} x^{j_1,1} x^{j_2,1} x^{j_3,1} + \frac{\partial^2 y^i}{\partial x^{j_1} \partial x^{j_2}} x^{j_1,2}$$

$$\frac{\partial y^{i,1}}{\partial x^{j,1}} = \frac{\partial y^i}{\partial x^j},$$

$$\frac{\partial y^{i,3}}{\partial x^{j,2}} = \frac{\partial^2 y^i}{\partial x^j \partial x^{j_2}} x^{j_2,1} = \frac{\partial}{\partial x^j} \left(\frac{\partial y^i}{\partial x^{j_2}} x^{j_2,1} \right) = \frac{\partial y^{i,}}{\partial x^j}$$

$$\frac{\partial y^{i,3}}{\partial x^{j,1}} = \frac{1}{2} \frac{\partial^3 y^i}{\partial x^{j_1} \partial x^{j_2} \partial x^j} x^{j_1,1} x^{j_2,1} + \frac{\partial^2 y^i}{\partial x^{j_1} \partial x^j} x^{j_1,2}$$

- The Jacobian matrix for the coordinate change

  to $\{y^{i,k}\}$ has a special upper triangular shape:

$$A = \begin{pmatrix} A_0 & 0 & 0 & 0 & 0 & \cdots \\ A_1 & A_0 & 0 & 0 & 0 & \cdots \\ A_2 & A_1 & A_0 & 0 & 0 & \cdots \\ A_3 & A_2 & A_1 & A_0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

- We will study formal path space based on this f

Recursion relations among C_μ

- The coefficients C_μ are some constants which
  on the partition μ.

- Define the following operator on $\mathbb{Z}[p_1, p_2, \ldots]$:
$$A = p_1 + \sum_{i=1}^{\infty} p_{i+1} \frac{\partial}{\partial p_i}.$$

- $A^k 1 = \sum_{|\mu|=k} C_\mu p_\mu$, where $p_\mu = p_{\mu_1} \cdots p_{\mu_{l(\mu)}}$.

- For $m > 0$, define

$$A_{-m} = \frac{1}{\Gamma(m)} p_m + \sum_{l=1}^{\infty} \frac{\Gamma(l+1)}{\Gamma(l+m)} p_{l+m} \partial_{p_l}.$$

  In particular, $A_{-1} = A$.

- For $m \geq 0$, define

$$A_m = \sum_{l=1}^{\infty} \frac{\Gamma(l+m+1)}{\Gamma(l)} p_l \partial_{p_{l+m}}.$$

- Clearly $A_m 1 = 0$ for $m \geq 0$.

Lemma 1 *The operators* $\{A_m\}_{m \geq -1}$ *span half of algebra with central charge* 0*:*

$$[A_m, A_{m'}] = (m - m')A_{m+m'},$$

for $m, m' \geq -1$*, or* $m, m' < 0$*.*

Lemma 2 *For* $k, m \geq 1$ *we have*

$$A_m A_{-1}^k 1 = k(k-1)\cdots(k-m)A_{-1}^{k-m}1.$$

- The above two Lemmas yield the following rec

  tions:

$$k(k-1)\cdots(k-a)C_{(1^{m_1}2^{m_2}\cdots)}$$
$$=\sum_{l\geq 1}\frac{\Gamma(l+a+1)}{\Gamma(l)}(m_{l+a}+1)C_{(1^{m_1}\cdots l^{m_l-1}\cdots(l+$$

Combined with

$$f_{(k)} = \sum_{|\mu|=k} C_\mu \sum_{1\le j_1,\ldots,j_{l(\mu)}\le n} \frac{\partial^{l(\mu)} f}{\partial x^{j_1} \cdots \partial x^{j_{l(\mu)}}} \cdot \frac{1}{\binom{k}{\mu}} x^{j_1,\mu}$$

one gets:

Corollary 1 *We have*

$$\sum_{j=1}^{n} \sum_{l=1}^{\infty} l x^{j,l} \frac{\partial f_{(k)}}{\partial x^{j,l+a}} = (k-a) f_{(k-a)}.$$

The method can be generalized to get simpler relat

We use

$$[\partial_{p_l}, A_{-1}] = \begin{cases} 1, & l = 1, \\ \partial_{p_{l-1}}, & l > 1 \end{cases}$$

and induction to get the following

Lemma 3 *For* $k \geq l \geq 1$ *we have*

$$\partial_{p_l} A_{-1}^k 1 = \binom{k}{l} A_{-1}^{k-l} 1.$$

- The above Lemma yields:

$$\binom{k}{l} C_{(1^{m_1}2^{m_2}\cdots)} = (m_l + 1) C_{(1^{m_1}2^{m_2}\cdots l^{m_l+1}$$

where $\sum_i im_i = k - l$.

- Combined with

$$f_{(k)} = \sum_{|\mu|=k} C_\mu \sum_{1\le j_1,\ldots,j_{l(\mu)}\le n} \frac{\partial^{l(\mu)} f}{\partial x^{j_1}\cdots\partial x^{j_{l(\mu)}}} \cdot \frac{1}{\binom{k}{\mu}}$$

one gets:

$$\frac{\partial f_{(k)}}{\partial x^{j,l}} = \frac{\partial f_{(k-l)}}{\partial x^j}.$$

- In particular, $\frac{\partial y^{i,k}}{\partial x^{j,l}} = \frac{\partial y^{i,k-l}}{\partial x^{i}}$.

- This means the Jacobian matrix for the coordi

  from $\{x^{j,l}\}$ to $\{y^{i,k}\}$ has a special upper triangul

$$A = \begin{pmatrix} A_0 & 0 & 0 & 0 & 0 & \cdots \\ A_1 & A_0 & 0 & 0 & 0 & \cdots \\ A_2 & A_1 & A_0 & 0 & 0 & \cdots \\ A_3 & A_2 & A_1 & A_0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

- This means $T\mathcal{P}M$ and $T\mathcal{P}^*M$ are *filtered vector*

Filtered vector spaces

- Let V be a vector space over a field k.

- A *forward filtration* of V we mean a sequence o

  spaces

$$0 = V^{-1} \subset V^{0} \subset V^{1} \subset \cdots$$

such that for any $v \in V$, there exists some $n \geq 0$ s

V^{n} and for each $n \geq 0$, $W^{n} := V^{n}/V^{n-1}$ is finite-

- The graded vector space

$$Gr^*V := \bigoplus_{n=0}^{\infty} V^n/V^{n-1}$$

will be called the *graded vector space associate*

- Example, $T^*\mathcal{P}M$ is a forward filtered vector b

$Gr^nT^*\mathcal{P}M \cong T^*M$ for each $n \geq 0$.

- Dually, a *backward filtration* of V is a sequen
subspaces

$$V = V_0 \supset V_1 \supset \cdots$$

such that for any $v \in V$, there exists some $n \geq 0$ s
V_n and for each $n \geq 0$, $W_n := V_n/V_{n+1}$ is finite-c

- The graded vector space

$$Gr_*V := \bigoplus_{n=0}^{\infty} V_n/V_{n+1}$$

will be called the *graded vector space associate*

- Example, $T\mathcal{P}M$ is a backward filtered vector b

$Gr_nT\mathcal{P}M \cong TM$ for each $n \geq 0$.

Ring of functions on $\mathcal{P}M$

- Denote by $\mathcal{A}(\mathcal{P}M)$ the space of functions on $\mathcal{P}I$

  be locally written as

$$f = f_{i_1,k_1;\ldots,i_l,k_l}(x^1, \ldots, x^n) \cdot x^{i_1,k_1} \ldots x^{i_l,}$$

  where the coefficients are smooth functions i

  $k_1 \ldots, k_l \geq 1$.

- Define the *conformal weight* of f by

$$deg f := k_1 + \cdots + k_l.$$

- This is independent of the choices of local coor

- Denote by $\mathcal{A}^k(\mathcal{P}M)$ the subspace of $\mathcal{A}(\mathcal{P}M)$ of

  conformal weight k.

An order on local generators of $\mathcal{A}^k(\mathcal{P}M)$

- Locally, $\mathcal{A}^k(\mathcal{P}U)$ is generated over $C^\infty(U)$ by

$$x^{J,\mu} := x^{j_1,\mu_1} \dots x^{j_l,\mu_l},$$

where $\mu = (\mu_1, \dots, \mu_l)$ are partitions of k,

$J = (j_1, \dots, j_l)$ is a multiple index.

- Monomials with the same partition μ have the

  The order of $x^{J,\mu}$ is small than $x^{K,\nu}$ if $\mu < \nu$.

- We use the reverse lexicographic order for parti

  if the first nonzero $\mu_i - \nu_i$ is negative.

Filtered vector bundles associated with $\mathcal{A}(\mathcal{P}M)$

Proposition 1 *The space $\mathcal{A}^k(\mathcal{P}M)$ is isomorphic to*

sections to a forward filtered vector bundle V_k on M,

cal forward filtered frame consisting of monomials

listed in the reverse lexicographic order of partitions

thermore,

$$Gr^{\mu} V_k \cong \bigotimes_{i \geq 1} S^{m_i(\mu)} T^* M.$$

- As a consequence, we get:

$$\sum_{k=0}^{\infty} q^k V_k = \bigotimes_{i=1}^{\infty} S_{q^i} T^* M.$$

- The index of the Dirac operator twisted by $\bigotimes_{i=}^{\infty}$

  peared in the study of elliptic genus.

Examples

For $k = 2$,

$$y^{i_1,1}y^{i_2,1} = \frac{\partial y^{i_1}}{\partial x^{j_1}}\frac{\partial y^{i_2}}{\partial x^{j_2}} \cdot x^{j_1,1}x^{j_2,1},$$
$$y^{i,2} = \frac{1}{2}\frac{\partial^2 y^i}{\partial x^{j_1}\partial x^{j_2}} \cdot x^{j_1,1}x^{j_2,1} + \frac{\partial y^i}{\partial x^j}x^{j,2},$$

$Gr^{(1,1)}V_2 \cong S^2T^*M$, $Gr^{(2)}V_2 \cong S^1T^*M$.

Examples

For $k = 3$,

$$y^{i_1,1}y^{i_2,1}y^{i_3,1} = \frac{\partial y^{i_1}}{\partial x^{j_1}}\frac{\partial y^{i_2}}{\partial x^{j_2}}\frac{\partial y^{i_3}}{\partial x^{j_3}} \cdot x^{j_1,1}x^{j_2,1}x^{j_3,1},$$

$$y^{i_1,2}y^{i_3,1} = \frac{1}{2}\frac{\partial^2 y^{i_1}}{\partial x^{j_1}\partial x^{j_2}}\frac{\partial y^{i_3}}{\partial x^{j_3}} \cdot x^{j_1,1}x^{j_2,1}x^{j_3,1} + \frac{\partial y^{i_1}}{\partial x^{j}} \cdots$$

$$y^{i,3} = \frac{1}{6}\frac{\partial^2 y^{i}}{\partial x^{j_1}\partial x^{j_2}\partial x^{j_3}} x^{j_1,1}x^{j_2,1}x^{j_3,1} + \frac{\partial^2 y^{i}}{\partial x^{j_1}\partial x^{j_2}} x^{j_1,2} \cdots$$

$Gr^{(1,1,1)}V_3 \cong S^3T^*M$, $Gr^{(2)}V_3 \cong S^1T^*M \otimes S^1T^*M$ …

S^1T^*M.

Generalized Euler vector fields

- Define the *generalized Euller vector fields* on $\mathcal{P}I$

$$E_a = \sum_{j=1}^{n} \sum_{l=1}^{\infty} l x^{j,l} \partial_{x^{j,l+a}}.$$

- The vector fields E_a are independent of the ch

  coordinates on M only for $a \geq -1$.

- Virasoro type algebra ($a, b \geq -1$):

$$[E_a, E_b] = (a - b)E_{a+b}.$$

- The conformal weight of $f \in \mathcal{A}(\mathcal{P}M)$ can be def

$$E_0 f = hf.$$

Differential forms on $\mathcal{P}M$

- A differential form on $\mathcal{P}U$ is a finite sum of the

$$\omega = \omega_{i_1,k_1;\ldots;i_p,k_p} dx^{i_1,k_1} \wedge \cdots \wedge dx^{i_p,k_p},$$

where $1 \leq i_1, \ldots, i_p \leq \dim M$,

$$k_1, \ldots, k_p \geq 0,$$

$$\omega_{i_1,k_1;\ldots;i_p,k_p} \in \mathcal{A}(\mathcal{P}U).$$

Fermionic charge and conformal weight of ω

They are defined by the following rules:

- Each $x^{i,k}$ has fermionic degree 0 and conformal

- Each $dx^{i,k}$ has fermionic degree 1 and conformal

- We have the following table:

	$x^{i,k}$	$dx^{i,k}$
c	0	1
h	k	$k+1/2$

BPS inequality

- The following inequality is satisfied:

$$h(\omega) \geq \frac{1}{2}c(\omega).$$

- The equality holds if and only if

$$\omega = \omega_{i_1,\ldots,i_p}(x^1,\ldots,x^n)dx^{i_1} \wedge \cdots \wedge dx^{i_p} \in \Omega$$

- This explains our definition of the conformal we

Filtered vector bundles on M associated with Ω

- Denote by $\Omega^{c,r}(\mathcal{P}M)$ the space of forms on $\mathcal{P}M$ w

  charge c and conformal weight r.

- $\Omega^{c,r}(\mathcal{P}M)$ is isomorphic to the space of sections

  vector bundle $V_{c,r}$ on M.

- $V_{c,r}$ has natural filtrations.

Example

$\Omega^{1,5/2}(\mathcal{P}M)$ is locally generated by:

$$\{x^{j_1,1}x^{j_2,1}dx^{j_3,0}, x^{j_1,2}dx^{j_1,0}, x^{j_1,1}dx^{j_2,1}, dx^{j_1,}$$

They transform as follows:

$$y^{i_1,1}y^{i_2,1}dy^{i_3,0} = \frac{\partial y^{i_1}}{\partial x^{j_1}}\frac{\partial y^{i_2}}{\partial x^{j_2}}\frac{\partial y^{i_3}}{\partial x^{j_3}}x^{j_1,1}x^{j_2,1}dx^{j_3,0},$$

$$y^{i_1,2}d\tilde{x}^{i_2,0} = \frac{1}{2}\frac{\partial^2 y^{i_1}}{\partial x^{j_1}\partial x^{j_2}}\frac{\partial y^{i_2}}{\partial x^{j_3}} \cdot x^{j_1,1}x^{j_2,1}dx^{j_3,0} + \frac{\partial y^{i_1}}{\partial x^{j_1}}$$

$$y^{i_1,1}dy^{i_2,1} = \frac{\partial y^{i_1}}{\partial x^{j_1}}\frac{\partial^2 y^{i_2}}{\partial x^{j_2}\partial x^{j_3}}x^{j_1,1}x^{j_2,1}dx^{j_3,0} + \frac{\partial y^{i_1}}{\partial x^{j_1}}\frac{\partial y}{\partial x}$$

$$dy^{i_1,2} = \frac{1}{2}\frac{\partial^3 y^{i}}{\partial x^{j_0}\partial x^{j_1}\partial x^{j_2}}x^{j_1,1}x^{j_2,1}dx^{j_0,0} + \frac{\partial^2 y^{i}}{\partial x^{j_1}\partial x^{j_2}}x$$

$$+\frac{\partial^2 y^{i}}{\partial x^{j_1}\partial x^{j_2}}x^{j_1,2}dx^{j_2,0} + \frac{\partial y^{i}}{\partial x^{j_1}}dx^{j_1,2}.$$

Hence $V_{1,5/2}(M)$ has a forward filtration with

$$Gr^0 V_{1,5/2}(M) \cong S^2 T^* M \otimes \Lambda^1 T^* M,$$

$$Gr^1 V_{1,5/2}(M) \cong S^1 T^* M \otimes \Lambda^1 T^* M,$$

$$Gr^2 V_{1,5/2}(M) \cong S^1 T^* M \otimes \Lambda^1 T^* M,$$

$$Gr^3 V_{1,5/2}(M) \cong \Lambda^1 T^* M.$$

As a smooth vector bundle,

$$V_{c,r}(M) \cong \bigoplus_{(m_i)_{i\geq 1},(n_i)_{i\geq 0}} \Big(\bigotimes_{i\geq 1} S^{m_i}T^*M \otimes \bigotimes_{i\geq 0} \Lambda^{n_i}T^*$$

where $\sum_{i\geq 0} n_i = c$, $\sum_{i\geq 1} im_i + \sum_{i\geq 0} n_i(i + \frac{1}{2}) = r$.

We have in $K(M)[[y, q]]$:

$$\sum_{c,r} (-y)^c q^r V_{c,r}(M, E) = \bigotimes_{i\geq 1} \Big(\Lambda_{-yq^{i-1/2}} T^*M \otimes S_{q^i} T^*$$

The exterior differential operator on $\mathcal{P}M$

- Define the exterior differential $d : \Omega(\mathcal{P}M) \to \Omega$

  the finite-dimensional case:

$$d = dx^{i,k} \wedge \partial_{x^{i,k}}.$$

- It is independent of the choices of local coordin

- d increase the fermionic charge by 1, and increa

  formal weight by $\frac{1}{2}$.

-

$$d^2 = 0,$$

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{|\omega_1|}\omega_1 \wedge$$

De Rham cohomology of $\mathcal{P}M$

- This is just the cohomology of the differential gr

$(\Omega^*(\mathcal{P}M), \wedge, d)$.

Theorem 1 *The inclusion map* $\Omega(M) \rightarrow \Omega(\mathcal{P}M)$

isomorphism. I.e., it induces an isomorphism of the

groups.

The proof

We use the standard argument by a homotopy oper

Lemma 4 *We have the following formula for the Li*

of the generalized Euler vector fields E_a *on* $\Omega(\mathcal{P}M)$

$$L_{E_a} = [d, i_{E_a}] = \sum_{j=1}^{n} \sum_{l=1}^{\infty} l dx^{j,l} \wedge i_{\partial_{x^{j,l+a}}} + \sum_{j=1}^{n} \sum_{l=1}^{\infty} l x^{j,l}$$

- We use L_0 to prove the Theorem.

- The eigenvalues of L_0 are ≥ 0.

- Zero eigenforms are just forms on M.

- Because L_0 commutes with d, one can restrict to
  of L_0.

Other operators on the spaces of forms on $\mathcal{P}M$

For $a \geq 0$, define

- $L_a = L_{E_a} = \sum_{j=1}^{n} \sum_{l=1}^{\infty} l dx^{j,l} \wedge i_{\partial_{x^{j,l+a}}} + \sum_{j=1}^{n} \sum_{l=}^{\infty}$

- $Q_a = dx^{j,l} \wedge \partial_{x^{j,l+a}}$

- $J_a = dx^{j,l} \wedge i_{\partial_{x^{j,l+a}}}$

- In particular, $Q_0 = d$, and the eigenvalues of
fermionic charge.

- L_a, Q_a and J_a are independent of the choices o
dinates $\{x^i\}$.

Lie algebra generated by such operators

$$[Q_a, Q_b]_+ = 0,$$

$$[J_a, J_b] = 0,$$

$$[L_a, L_b] = (a - b)L_{a+b},$$

$$[L_a, Q_b] = -bQ_{a+b},$$

$$[L_a, J_b] = -bJ_{a+b},$$

$$[J_a, Q_b] = Q_{a+b}.$$

Twisted algebra

We can also consider operators $L_a^{\pm} = L_a \pm \frac{1}{2}(a+1)$.

have

$$[L_a^{\pm}, L_b^{\pm}] = (a-b)L_{a+b}^{\pm},$$
$$[L_a^{\pm}, Q_b] = -(b \mp \frac{1}{2}(a+1))Q_{a+b},$$
$$[L_a^{\pm}, J_b] = -bJ_{a+b}.$$

Other geometric objects studied in this framew

1. Path bundles

2. Path connections

3. Characteristic classes of path bundles

4. Mathai-Quillen constructions for path bundles

More geometric objects to be studied:

1. Induced Riemannian metrics on formal path spac

2. Induced symplectic structures

3. Dirac operators, etc.

Thank you very much!