

Standing waves with critical frequency for nonlinear Schrödinger equations

!!!!!!! Happy Paul's 70th Birthday !!!!!!!!

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We consider the following equation

$$(1) \quad \varepsilon^2 \Delta u - V(x)u + f(u) = 0, \quad u > 0 \quad \text{on} \quad \mathbf{R}^n$$

with $\lim_{|x| \rightarrow \infty} u(x) = 0$ and sufficiently small $\varepsilon > 0$.

$$V \in C(\mathbf{R}^n), \quad f \in C(\mathbf{R}), \quad V \geq 0$$

Nonlinear Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2} \Delta \psi - V(x)\psi + f(\psi) = 0, \quad (t, x) \in \mathbf{R} \times \mathbf{R}^n$$

Standing wave

$$\psi(x, t) = \exp\left(-\frac{iEt}{\hbar}\right)u(x), \quad u(x) \in \mathbb{R}$$

Corresponding equation for u when $f(e^{i\theta}t) = e^{i\theta}f(t)$

$$\frac{\hbar^2}{2} \Delta u - (V(x) - E)u + f(u) = 0 \quad \text{in } \mathbf{R}^n$$

Semiclassical states

$$\varepsilon^2 \Delta u - (V(x) - E)u + f(u) = 0 \quad \text{in } \mathbf{R}^n$$

$$\varepsilon^2 \equiv \hbar^2/2 \rightarrow 0$$

Normalization $u(x) \rightarrow u(\varepsilon x)$

\Downarrow

$$\Delta u - (V(\varepsilon x) - E)u + f(u) = 0, u > 0 \text{ on } \mathbf{R}^n \text{ with } \lim_{|x| \rightarrow \infty} u(x) = 0$$

For each $d > 0$,

$$V(\varepsilon x) - E \rightarrow V(x_0) - E$$

uniformly on $B(\frac{x_0}{\varepsilon}, d)$ as $\varepsilon \rightarrow 0$.

If u_ε is a solution satisfying $\frac{1}{C} \leq \|u_\varepsilon\|_{C^{1,\alpha}(B(\frac{x_0}{\varepsilon}, d))} \leq C$ for each $d > 0$,

$u_\varepsilon(\cdot + x_0/\varepsilon) \rightarrow u$ in C_{loc}^1 and

$$\Delta u - (V(x_0) - E)u + f(u) = 0, u > 0$$

Almost optimal existence result by Berestycki-Lions.

Some criticality of V + some conditions on f

(stronger than or equal to Berestycki-Lions conditions on f)

⇓

Existence of solution concentrating around critical points of V .

Liapunov-Schmidt reduction method

Pioneering work and development by

Floer-Weinstein, Yonggeun Oh, Ambrosetti, Vieri Benci, B., Yanyan Li, Zhaoli Liu, Juncheng Wei, Oshita,

Variational approach

Pioneering work and development by

Paul H. Rabinowitz, Patricio Felmer, del Pino, Gui, B. Zhi-Qiang Wang, Kazunaga Tanaka, Jeanjean, Oshita, ...

Typical nonlinearity $f(u) = u^p$, $1 < p < \frac{n+2}{n-2}$

No solutions for small $\varepsilon > 0$ if $\inf_{x \in \mathbf{R}^n} V(x) - E < 0$

No solutions for limiting problem if $V(x_0) - E = 0$

[Gidas-Spruck]

A unique (up to a translation) solution U of limiting problem for $V(x_0) - E > 0$

If $V(x_0) - E = 0$, no solution u_ε for small $\varepsilon > 0$
such that $\frac{1}{C} \leq \|u_\varepsilon\|_{C^{1,\alpha}(B(\frac{x_0}{\varepsilon}, d))} \leq C$

(We change $V(x) - E \rightarrow V$ for simplicity)

If $V(x) \geq V(x_0) = 0$, the functional

$$\Gamma_\varepsilon(u) = \frac{1}{2} \int_{\mathbf{R}^n} |\nabla u|^2 + V(\varepsilon x) u^2 dx - \frac{1}{p+1} \int_{\mathbf{R}^n} u^{p+1} dx$$

has a (local) mountain pass structure.

Indeed, for each isolated zero set $A \subset \mathbf{R}^n$ and $\delta > 0$, there exists a local mountain pass solution u_ε such that $\lim_{\varepsilon \rightarrow \infty} \|u_\varepsilon\|_{L^\infty} = 0$, $\liminf_{\varepsilon \rightarrow \infty} \varepsilon^{-\frac{2}{p+1}} \|u_\varepsilon\|_{L^\infty} > 0$ and $u_\varepsilon(x) \leq C \exp(-\frac{c}{\varepsilon} \text{dist}(\varepsilon x, A^\delta))$, where $A^\delta = \{x \mid \text{dist}(x, A) \leq \delta\}$.

? Asymptotic profile of u_ε as $\varepsilon \rightarrow 0$

$$\Delta u_\varepsilon - V(\varepsilon x)u_\varepsilon + (u_\varepsilon)^p = 0$$

Direct limit

$$\Delta U + U^p = 0, \quad U > 0 \text{ in } \mathbf{R}^n$$

No solutions

$$\Delta u_\varepsilon - V(\varepsilon x)u_\varepsilon + (u_\varepsilon)^p = 0$$

↓

$$(\text{renormalization, } w_\varepsilon(x) \equiv (\frac{a(\varepsilon)}{\varepsilon})^{2/(p-1)}u_\varepsilon(\frac{a(\varepsilon)}{\varepsilon}x))$$

↓

$$\Delta w_\varepsilon - V(a(\varepsilon)x)(\frac{a(\varepsilon)}{\varepsilon})^2 w_\varepsilon + (w_\varepsilon)^p = 0$$

Take an appropriate $a(\varepsilon)$ so that $V(a(\varepsilon)x)(\frac{a(\varepsilon)}{\varepsilon})^2 \rightarrow V_0$ as $\varepsilon \rightarrow 0$ and there exists a solution of

$$\Delta U - V_0 U + U^p = 0$$

Examples

Flat cases

$$V = 0 \text{ on } \bar{\Omega} \Rightarrow a(\varepsilon) = 1$$

$$\Delta U + U^p = 0 \text{ in } \Omega, U = 0 \text{ on } \partial\Omega$$

Infinite cases

$$V(x) \approx \exp(-|x|^{-\alpha}) \text{ near } 0 \Rightarrow a(\varepsilon) = (\log \varepsilon^{-2})^{-1/\alpha}$$

$$\Delta U + U^p = 0 \text{ in } B(0, 1), U = 0 \text{ on } \partial B(0, 1)$$

Finite cases

$$V(x) \approx |x|^\alpha \text{ near } 0 \Rightarrow a(\varepsilon) = \varepsilon^{2/(\alpha+2)}$$

$$\Delta U - |x|^\alpha U + U^p = 0 \text{ in } \mathbf{R}^n$$

Very slow cases

$$V(x) \approx (-\log |x|)^{-1} \text{ near } 0 \Rightarrow a(\varepsilon) = g^{-1}(\varepsilon^2), g(x) = -x^2 / \log x$$

$$\Delta U - U + U^p = 0 \text{ in } \mathbf{R}^n$$

A rich variety of solutions following from
a rich variety of decaying behavior of V near zeros

Algebraically small solutions whose interaction is exponentially small



It should be possible to glue together the localized solutions

[B. and Oshita, 2004] Yes for general critical points if $f(t) = t^p \in C^4(\mathbb{R})$ and there exist *limiting problems* whose solutions are nondegenerate

(Lyapunov-Schmidt reduction method)

[Ding and Tanaka, 2003], [Cao and Noussair, 2004]

flat case - variational method

[Cao and Peng, 2006] exponentially (same order) decaying case - variational method

[Cao, Noussair and Yan, 2008]

different scales without the restriction $f(t) = t^p \in C^4(\mathbb{R})$

- Lyapunov-Schmidt reduction method

[Sato, 2007] different scales(no requiring the existence of limiting profile) for local minimum points of V and $f(t) = t^p$

- minimization on infinite dimensional torus

Aim

Develop a variational method and a hybrid method combining the variational method and reduction method to glue together localized solutions with different energy scales in a possibly general setting

(V1) $V \in C(\mathbb{R}^n)$; $\liminf_{|x| \rightarrow \infty} V(x) > 0 = \min_{x \in \mathbb{R}^n} V(x)$;

(V2-1) there exist disjoint bounded open sets Ω_i with smooth boundary $\partial\Omega_i$, $i = 1, \dots, l$, satisfying $0 = \inf_{x \in \Omega_i} V(x) < \min_{x \in \partial\Omega_i} V(x)$;

(f1) $f \in C^1(\mathbb{R})$, $f(t) = 0$ for $t \leq 0$ and there exist some $\mu > 1$ and $C > 0$ satisfying $|f(t)| \leq Ct^\mu$ for $t \in (0, 1)$;

(f2-1) there exists some $p \in (1, \frac{n+2}{n-2})$ for $n \geq 3$ and $p \in (1, \infty)$ for $n = 1, 2$ such that $\liminf_{t \rightarrow 0^+} \frac{1}{t^{p+1}} \int_0^t f(s) ds > 0$;

(f3) there exists $\mu_2 > 1$ such that $(\mu_2 + 1) \int_0^t f(s) ds \leq f(t)t$ for $t > 0$.

[B. and Wang, 2003] Suppose that (V1),(V2-1),(f1), (f2-1), (f3) hold. Then, for sufficiently small $\varepsilon > 0$, there exists a positive solution $u_\varepsilon^i, i = 1, \dots, l$ such that for any sufficiently small $d > 0$, there exist $C, c > 0$ satisfying

$$u_\varepsilon^i(x) \leq C \exp\left(-\frac{c}{\varepsilon} \text{dist}(x, (A_i)^d)\right)$$

(Here $B^d \equiv \{x \in \mathbf{R}^n \mid \text{dist}(x, B) \leq d\}$) satisfying

$$\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon\|_{L^\infty(\Omega_i)} = 0, \quad \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2/(\mu-1)} \|u_\varepsilon\|_{L^\infty(\Omega_i)} > 0.$$

[A related work of Sirakov]

[B. and Oshita] Suppose that (V1),(V2-1),(f1), (f2-1), (f3) hold. Then, for sufficiently small $\varepsilon > 0$, there exists a positive solution u_ε such that

(i) for any sufficiently small $d > 0$, there exist $C, c > 0$ satisfying

$$u_\varepsilon(x) \leq C \exp\left(-\frac{c}{\varepsilon} \text{dist}(x, (A_1 \cup \dots \cup A_l)^d)\right);$$

(Here $A^d \equiv \{x \in \mathbf{R}^n \mid \text{dist}(x, A) \leq d\}$)

(ii) for $i = 1, \dots, l$, it holds that

$$\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon\|_{L^\infty(\Omega_i)} = 0, \quad \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2/(\mu-1)} \|u_\varepsilon\|_{L^\infty(\Omega_i)} > 0.$$

(V2-2) there exist disjoint bounded open sets Ω_i with smooth boundary $\partial\Omega_i$, $i = 1, \dots, k$, satisfying

$$m_i = \inf_{x \in \Omega_i} V(x) < \min_{x \in \partial\Omega_i} V(x)$$

and

$$0 = m_1 = \dots = m_l < m_{l+1}, \dots, m_k;$$

(f2-2) there exists some $p \in (1, \frac{n+2}{n-2})$ for $n \geq 3$ and $p \in (1, \infty)$ for $n = 1, 2$ such that $\limsup_{t \rightarrow \infty} \frac{|f(t)| + |f'(t)t|}{t^p} < \infty$.

[B. and Oshita] Suppose that (V1) and (V2-2) hold, and that (f1), (f2-1), (f2-2), (f3-1) hold. Then, for sufficiently small $\varepsilon > 0$, there exists a positive solution u_ε such that

(i) for any sufficiently small $d > 0$, there exist $C, c > 0$ satisfying

$$u_\varepsilon(x) \leq C \exp(-c \text{dist}(x, (A_1 \cup \dots \cup A_k)^d) / \varepsilon);$$

(ii) for $i = 1, \dots, l$, it holds that

$$\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon\|_{L^\infty(\Omega_i)} = 0, \quad \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2/(\mu-1)} \|u_\varepsilon\|_{L^\infty(\Omega_i)} > 0;$$

and for $i = l + 1, \dots, k$, it holds that for a least energy solution U of $\Delta U - m_i U + f(U) = 0$ and some $x_\varepsilon^i \in \Omega_i$ with $\lim_{\varepsilon \rightarrow 0} x_\varepsilon^i = x_i$, a transformed solution $u_\varepsilon(\varepsilon x + x_\varepsilon^i)$ converges (up to a subsequence) uniformly to $U(x)$ on each bounded set in \mathbb{R}^n .

(V3) there are $x_1, \dots, x_m \in \mathbb{R}^n$ and disjoint bounded open sets $\Omega_{k+1}, \dots, \Omega_{k+m}$ with smooth boundary $\partial\Omega_{k+j}$ such that $x_j \in \Omega_{k+j}$, $V \in C^2(\Omega_{k+j})$, $\nabla V(x) \neq 0$ for $x \in \Omega_{k+j} \setminus \{x_j\}$, $\inf_{x \in \Omega_{k+j}} V(x) > 0$ and x_j is a non-degenerate critical point of V for $j \in \{1, \dots, m\}$;

(f4) for any $a \in \{V(x_1), \dots, V(x_m)\}$, the problem

$$\Delta u - au + f(u) = 0, \quad u > 0 \quad \text{in } \mathbb{R}^n, \quad u \in H^{1,2}(\mathbb{R}^n)$$

has a radially symmetric solution U_a which is non-degenerate in $H_{\mathbb{R}}^{1,2}(\mathbb{R}^n) \equiv \{w \in H^{1,2}(\mathbb{R}^n); w(x) = w(|x|)\}$, and $f \in C_{\text{loc}}^{1,\gamma}(\mathbb{R})$ for some $\gamma \in (0, 1)$.

[B. and Oshita] We assume that (V1), (V2-2) and (V3) hold. Suppose that (f1), (f2-1),(f2-2), (f3) and (f4) hold. Then for sufficiently small $\varepsilon > 0$, there exists a positive solution u_ε such that

- (i) for any sufficiently small $d > 0$, there exist $C, c > 0$ satisfying

$$u_\varepsilon(x) \leq C \exp\left(-\frac{c}{\varepsilon} \text{dist}(x, (A_1 \cup \dots \cup A_k \cup \{x_1, \dots, x_m\})^d)\right);$$

- (ii) the same behavior around zero $A_i, i = 1, \dots, k$;
- (iii) for each $i = 1, \dots, m$, there exist $y_\varepsilon^i \in \mathbb{R}^n$ such that $\lim_{\varepsilon \rightarrow 0} y_\varepsilon^i = x_i$ and that $u_\varepsilon(\varepsilon x + y_\varepsilon^i)$ converges uniformly (up to a subsequence) to $U_{V(x_i)}$ on each bounded set in \mathbb{R}^n . Here $U_{V(x_i)}$ is a function given in (f4).

Setting and preliminaries

$\mathcal{Z} := \{x \in \mathbf{R}^n \mid V(x) = 0\}$, $A_0 := \mathcal{Z} \setminus \cup_{i=1}^k A_i$ and $\Omega_0 \supset A_0$ a bounded open set with a smooth boundary such that for $\delta > 0$ and $\Omega_i^\delta = \{x \in \mathbf{R}^n \mid \text{dist}(x, \Omega_i) \leq \delta\}$, $\Omega_i^{2\delta} \cap \Omega_j^{2\delta} = \emptyset$ for each $0 \leq i \neq j \leq k + m$, and that $\partial\Omega_i^{\delta'}$ is smooth for each $0 \leq i \leq k + m$ and $\delta' \in [0, 2\delta]$.

For $\lambda \ll 1$, we define $g(x, t)$ so that $|g(x, t) - g(x, t')| \leq \lambda|t - t'|$ for $x \notin \cup_{i=0}^{k+m} \Omega_{i,\varepsilon}$, $g(x, t) = f(t)$ for small $t > 0$ and some more properties.

$$G(x, t) \equiv \int_0^t g(x, s) ds, \quad \|u\|_\varepsilon^2 \equiv \int_{\mathbf{R}^n} |\nabla u|^2 + V_\varepsilon(x)u^2 dx.$$

For $u \in H_\varepsilon \equiv \overline{(C_0^\infty, \|\cdot\|_\varepsilon)}$,

$$\Gamma_\varepsilon(u) \equiv \frac{1}{2}\|u\|_\varepsilon^2 - \int_{\mathbf{R}^n} G(x, u) dx, \quad \Gamma_\varepsilon \in C^2(H_\varepsilon)$$

For $u_i \in H^1(\Omega_{i,\varepsilon})$ (here $\Omega_{i,\varepsilon} = \frac{1}{\varepsilon}\Omega_i$), let

$$X_i^\varepsilon(u_i) = \{u \in H_0^1(\Omega_{i,\varepsilon}^\delta) \mid u = u_i \text{ on } \Omega_{i,\varepsilon}\}.$$

Proposition 0.1. *For each $u_i \in H^1(\Omega_{i,\varepsilon})$, $i \in \{0, \dots, k+m\}$, there exists a unique minimizer $P_i(u_i)$ of Γ_ε on $X_i^\varepsilon(u_i)$, which satisfies the following:*

(i) $w = P_i(u_i) \in H_0^1(\Omega_{i,\varepsilon}^\delta)$ solves

$$\begin{cases} \Delta w - V_\varepsilon w + f_\lambda(w) = 0 & \text{in } \Omega_{i,\varepsilon}^\delta \setminus \Omega_{i,\varepsilon} \\ w = u_i & \text{on } \partial\Omega_{i,\varepsilon} \\ w = 0 & \text{on } \partial\Omega_{i,\varepsilon}^\delta, \end{cases}$$

(ii) $P_i : H^1(\Omega_{i,\varepsilon}) \rightarrow H_0^1(\Omega_{i,\varepsilon}^\delta)$ is of class C^1 ,

(iii) there exists a positive constant C , independent of small $\varepsilon > 0$ such that

$$\|P_i(u_i)\|_\varepsilon \leq C\|u_i\|_{\varepsilon,\Omega_{i,\varepsilon}} \text{ for all } u_i \in H^1(\Omega_{i,\varepsilon}), i \in \{0, \dots, k+m\}.$$

For $\vec{u} = (u_0, \dots, u_{k+m})$, $u_i \in H^1(\Omega_{i,\varepsilon})$, let

$$X_*^\varepsilon(\vec{u}) = \{u \in H_\varepsilon \mid u = u_i \text{ on } \Omega_{i,\varepsilon}, i = 0, \dots, k+m\}.$$

Proposition 0.2. *For each $\vec{u} = (u_0, \dots, u_{k+m}) \in H^1(\Omega_{0,\varepsilon}) \times \dots \times H^1(\Omega_{k+m,\varepsilon})$, there exists a unique minimizer $\varphi(\vec{u})$ of Γ_ε on $X_*^\varepsilon(\vec{u})$, which satisfies the following:*

(i) $w = \varphi(\vec{u})$ solves

$$\begin{cases} \Delta w - V_\varepsilon w + f_\lambda(w) = 0 & \text{in } (\Omega_{0,\varepsilon} \cup \dots \cup \Omega_{k,\varepsilon})^c \\ w = u_i & \text{on } \partial\Omega_{i,\varepsilon} \ (i = 0, \dots, k), \end{cases}$$

(ii) $\varphi : H^1(\Omega_{0,\varepsilon}) \times \dots \times H^1(\Omega_{k,\varepsilon}) \rightarrow H_\varepsilon$ is of class C^1 ,

(iii) there exists a positive constant C , independent of small $\varepsilon > 0$ such that

$$\|\varphi(\vec{u})\|_\varepsilon \leq C\|\vec{u}\|_\varepsilon.$$

Let $\tilde{\varphi}(\vec{u}) = \varphi(\vec{u}) - \sum_{i=0}^{k+m} P_i(u_i)$. Then it follows that $\tilde{\varphi}(\vec{u}) \in X_*^\varepsilon(\vec{0})$. Now we obtain the following estimates for $\tilde{\varphi}(\vec{u})$.

Proposition 0.3. *For any $R > 0$ and $\varepsilon_0 > 0$, there exist constants $C, c > 0$ such that*

$$\|\tilde{\varphi}(\vec{u})\|_\varepsilon \leq Ce^{-c/\varepsilon}$$

for $\varepsilon \in (0, \varepsilon_0)$ and $\|\vec{u}\|_\varepsilon \leq R$.

Let $I_\varepsilon(\vec{u}) = \Gamma_\varepsilon(\varphi(\vec{u}))$. Then from Proposition 0.3, we conclude that for any $R, \varepsilon_0 > 0$, there exist constants $C, c > 0$ such that

$$\left| I_\varepsilon(\vec{u}) - \sum_{i=0}^{k+m} \Gamma_\varepsilon(P_i(u_i)) \right| \leq Ce^{-c/\varepsilon}$$

for $\|\vec{u}\|_\varepsilon \leq R$ and $\varepsilon \in (0, \varepsilon_0)$.

Proposition 0.4. *The following hold.*

(i) *A vector function $\vec{u} = (u_0, \dots, u_{k+m}) \in H^1(\Omega_{0,\varepsilon}) \times \dots \times H^1(\Omega_{k+m,\varepsilon})$ is a critical point of I_ε if and only if $\varphi(\vec{u})$ is a critical point of Γ_ε .*

(ii) *The functional $\vec{u} \mapsto I_\varepsilon(\vec{u})$ satisfies (PS) condition if Γ_ε does.*

(iii) *For any $R > 0$, $i = 0, \dots, k+m$ and $\varepsilon_0 > 0$, there exist constants $C, c > 0$ such that*

$$\left| \frac{\partial I_\varepsilon}{\partial u_i}(u_0, \dots, u_{k+m}) - \frac{d\Gamma_\varepsilon(P_i(u_i))}{du_i} \right| \leq C e^{-c/\varepsilon} \quad \text{for } \varepsilon \in (0, \varepsilon_0)$$

and $\|\vec{u}\|_\varepsilon \leq R$.

We define $\Gamma_\varepsilon^j(u) = \Gamma_\varepsilon(P_j(u))$ for $u \in H^1(\Omega_{j,\varepsilon})$ and $j \in \{1, \dots, k+m\}$.

Proposition 0.5. *For each $j \in \{1, \dots, k+m\}$, the following hold.*

- (i) *A function u_j is a critical point of Γ_ε^j if and only if $P_j(u_j) \in H_0^1(\Omega_{j,\varepsilon}^\delta)$ is a critical point of Γ_ε on $H_0^1(\Omega_{j,\varepsilon}^\delta)$.*
- (ii) *The functional Γ_ε^j on $H^1(\Omega_{j,\varepsilon})$ satisfies (PS) condition if Γ_ε on $H_0^1(\Omega_{j,\varepsilon}^\delta)$ does.*

Let $\alpha = 2\frac{\mu+1}{\mu-1}$. By Proposition 0.1 (iii), we can choose a constant $M > 1$, independent of small $\varepsilon > 0$, such that

$$\|u_i\|_{\varepsilon, \Omega_{i,\varepsilon}} \leq \|P_i(u_i)\|_{\varepsilon} \leq M\|u_i\|_{\varepsilon, \Omega_{i,\varepsilon}} \text{ for all } u_i \in H^1(\Omega_{i,\varepsilon})$$

and $i \in \{0, \dots, k+m\}$.

Proposition 0.6. *Let $i \in \{0, \dots, l\}$. For sufficiently small $\varepsilon > 0$, it holds that $\Gamma_{\varepsilon}(u) \geq 6M^2\varepsilon^{2\alpha}$ for all $u \in H_0^1(\Omega_{i,\varepsilon}^{\delta})$ with $\|u\|_{\varepsilon} = 4M\varepsilon^{\alpha}$.*

Proposition 0.7. *Let $i \in \{0, \dots, l\}$. For sufficiently small $\varepsilon > 0$, it holds that $|\Gamma_{\varepsilon}(u)| \leq 5M^2\varepsilon^{2\alpha}$ for all $u \in H_0^1(\Omega_{i,\varepsilon}^{\delta})$ with $\|u\|_{\varepsilon} \leq 3M\varepsilon^{\alpha}$, and $\Gamma_{\varepsilon}(u) \geq 0$ for all $u \in H_0^1(\Omega_{i,\varepsilon}^{\delta})$ with $\|u\|_{\varepsilon} \leq 5M\varepsilon^{\alpha}$.*

Note that we take $\alpha = 2\frac{\mu+1}{\mu-1}$.

Proposition 0.8. *For sufficiently small $\varepsilon > 0$,*

$$\|I'_\varepsilon(\vec{u})\| \geq \frac{1}{2}\varepsilon^\alpha$$

if $2\varepsilon^\alpha \leq \|P_i(u_i)\|_\varepsilon \leq 3M\varepsilon^\alpha$ for some $i \in \{0, \dots, k\}$.

Proposition 0.9. *Let $E > 0$ be a given constant. Then for sufficiently large $R_1 > 0$, there exists a constant ε_0 such that if $I_\varepsilon(\vec{u}) \leq E$, $\varepsilon \in (0, \varepsilon_0)$, $\frac{R_1}{4M} \leq \|P_i(u_i)\|_\varepsilon \leq R_1$ for some $i = 1, \dots, k$, then $\|I'_\varepsilon(\vec{u})\| \geq 1$.*

Scheme of the Proof for $m_1 = \dots = m_k = 0$.

Let $G_\varepsilon^i = \{\gamma \in C([0, 1], H_0^1(\Omega_{i,\varepsilon}^\delta)) \mid \gamma(0) = 0, \gamma(1) = h_i\}$, $\Gamma_\varepsilon(h_i) < 0$, and

$$c_\varepsilon^i := \inf_{\gamma \in G_\varepsilon^i} \max_{t \in [0,1]} \Gamma_\varepsilon(P_i(\gamma(t))) \geq 6M^2\varepsilon^{2\alpha}, i \in \{1, \dots, k\}$$

We find a good path $\gamma_i \in C([0, 1], H_0^1(\Omega_{i,\varepsilon}^\delta))$ so that

$$\max_{t \in [0,1]} \Gamma(\gamma_i(t)) \leq c_\varepsilon^i + \exp\left(-\frac{1}{\varepsilon}\right).$$

Now we define $\vec{\gamma}(s) = (0, \gamma_\varepsilon^1(s_1), \dots, \gamma_\varepsilon^k(s_k))$ with $s = (s_1, \dots, s_k) \in [0, 1]^k$. We define

$$d_\varepsilon := \max_{s \in [0,1]^k} I_\varepsilon(\vec{\gamma}(s)).$$

Then $d_\varepsilon = c_\varepsilon^1 + \dots + c_\varepsilon^k + O(e^{-c/\varepsilon})$ for some $c > 0$.

Proposition 0.10. *For sufficiently small $\varepsilon > 0$, there exists a critical point $u_\varepsilon \in H_\varepsilon$ of Γ_ε such that $u_\varepsilon \in \left(\Gamma_\varepsilon^{d_\varepsilon} \setminus \Gamma_\varepsilon^{d_\varepsilon - \varepsilon^{2\alpha}} \right)$*
 $u_\varepsilon \in \left\{ \|P_0(u)\|_\varepsilon \leq 5M\varepsilon^\alpha, \varepsilon^\alpha \leq \|P_i(u)\|_\varepsilon \leq 2R, i = 1, \dots, k \right\}$.

PROOF

If not, through by the gradient flow, we can find paths

$$\vec{\gamma}_* = (\gamma_*^0, \dots, \gamma_*^k) \in C([0, 1]^k, \Pi_{i=0}^k H_0^1(\Omega_{i,\varepsilon}))$$

such that

$$I(\vec{\gamma}_*) \leq d_\varepsilon - \varepsilon^{2\alpha}/2.$$

On the other hand,

$$\begin{aligned} I(\vec{\gamma}_*) &\geq \sum_{i=1}^k \Gamma(P_i(\gamma_*^i)) + O(\exp(-\frac{c}{\varepsilon})) \geq c_\varepsilon^1 + \dots + c_\varepsilon^k + O(\exp(-\frac{c}{\varepsilon})) \\ &= d_\varepsilon + O(\exp(-\frac{c}{\varepsilon})); \quad \text{contradiction} \end{aligned}$$

Scheme of the Proof for the last gluing result

Simple case. $x_1 = 0$ and $\Omega_{k+1} = B_\sigma(0)$. $(\mathbf{u}, u_{k+1}) \in H^1(\Omega_{0,\varepsilon}) \times \cdots \times H^1(\Omega_{k+1,\varepsilon})$, $\mathbf{u} = (u_0, \cdots, u_k) \in H^1(\Omega_{0,\varepsilon}) \times \cdots \times H^1(\Omega_{k,\varepsilon})$. We write $\Omega_\varepsilon = \Omega_{k+1,\varepsilon}$, $P = P_{k+1}$. For a given \mathbf{u} , we define $Q_\varepsilon(u_{k+1}) = I_\varepsilon(\mathbf{u}, u_{k+1})$, $\psi(u_{k+1}) = \varphi(\mathbf{u}, u_{k+1})$, $K_\varepsilon(u_{k+1}) = \Gamma_\varepsilon(P(u_{k+1}))$.

The first step: we find a solution Ψ_ε of the localized problem $K'_\varepsilon(u) = 0$.

The second step: we find a solution $u_{k+1} = \Phi_\varepsilon(\mathbf{u})$ of $Q'_\varepsilon(u_{k+1}) = 0$ in an exponentially small neighborhood of the solution Ψ_ε to the localized problem.

The third step: to find a critical point of $I_\varepsilon(\mathbf{u}, \Phi_\varepsilon(\mathbf{u}))$, we consider the following functional

$$\bar{I}_\varepsilon(\mathbf{u}) \equiv I_\varepsilon(\mathbf{u}, \Phi_\varepsilon(\mathbf{u})) - K_\varepsilon(\Psi_\varepsilon).$$

The last step : we apply the variational method as before to find a critical point of \bar{I}_ε .

What happen when V vanishes at infinity

$$\varepsilon^2 \Delta u - Vu + u^p = 0, \quad u > 0, \quad \text{in } \mathbf{R}^n$$

[Ambrosetti-Felli-Malchiodi, Ambrosetti-Malchiodi-Ruiz, B.-Wang] If $\liminf_{|x| \rightarrow \infty} V(x)|x|^2 > 0$, we can construct standing waves with finite energy.

If V has compact support, for $p \in (1, n/(n-2)]$, there exists no positive solution for

$$\varepsilon^2 \Delta u - Vu + u^p = 0, \quad u > 0, \quad \text{in } \mathbf{R}^n$$

[Chang-Yin, Moroz-Van Schaftingen] No decay restriction on $V \geq 0$ when $p \in (n/(n-2), (n+2)/(n-2))$, $n \geq 3$

What is an optimal threshold of decay rate of V for existence and nonexistence

[Ambrosetti and Malchiodi, 2007, Concentration phenomena for NLS: recent results and new perspectives, Contem. Math.]

[Bae-B.] Suppose that

$$\lim_{|x| \rightarrow \infty} V(x)|x|^2 = 0$$

for $1 < p < n/(n-2)$ if $n \geq 3$ and $p > 1$ if $n = 1, 2$, and that $\lim_{|x| \rightarrow \infty} V(x)|x|^2 \ln |x| = 0$ for $p = n/(n-2)$ and $n \geq 3$, then there exist no positive solutions for any small $\varepsilon > 0$.

Moreover there exists a potential V satisfying $\lim_{|x| \rightarrow \infty} V(x)|x|^2 \ln |x| > 0$ such that for some small $\varepsilon > 0$, there exists a solution of the problem

$$\varepsilon^2 \Delta u - Vu + u^{n/(n-2)} = 0, \quad u > 0 \quad \text{in } \mathbf{R}^n \setminus B(0, 1).$$

**A goal on the standing waves through
by development of variational methods**

Construct solutions under the Beresticki-Lions condition
on nonlinearity and the optimal decay condition on V