Lecture dedicate to the 70th birthday of professor Paul Rabinowitz

Index and Stability of Symmetric Periodic Orbits in Hamiltonian Systems with Application to Figure-Eight Orbit

Xijun Hu
Department of Mathematics
Shandong University, P.R.C.

Shanzhong SUN
Department of Mathematics
Capital Normal University, P.R.C.

Period solution of n-body problems

- n particles with masses $m_i > 0$, position $q_i \in \mathbf{R}^d$

$$m_i \ddot{q}_i = \frac{\partial U}{\partial q_i}, \quad i = 1, ..., n, \tag{1}$$

$$U(q) = \sum_{1 \le i < j \le n} \frac{m_i m_j}{\|q_i - q_j\|}. \tag{2}$$

- Euler-Lagrange equation of

$$\mathcal{A}(q(t)) = \int_0^T [\sum_{i=1}^{n} \frac{m_i \|\dot{q}_i(t)\|^2}{2} + U(q(t))]dt$$

on $W^{1,2}(\mathbf{R}/T\mathbf{Z}, \hat{\mathcal{X}})$,

$$\hat{\mathcal{X}} := \{q| \sum_{i=1}^{n} m_i q_i = 0, \ q_i \neq q_j, \ \forall i \neq j\}$$

- Find critical point of $\mathcal{A}(q(t))$, minimizer under topological constrain and symmetry constrain.

Linear stability

- Corresponding Hamiltonian systems

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{3}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \tag{4}$$

$H(p,q) = \sum_{i=1}^{n} \frac{\|p_i\|^2}{2m_i} - U(q)$

- $\mathrm{Sp}(2n) = \{M \in \mathrm{GL}(2n, \mathbf{R}) \,|\, M^T J M = J\}$, where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$.

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$$\dot{z}(t) = JH'(t, z(t)) \tag{5}$$

$$z(0) = z(T) \tag{6}$$

- Its fundamental solution $\gamma \equiv \gamma(t)$ is

$$\dot{\gamma}(t) = JH''(t, z(t))\gamma(t) \qquad (7)$$

$$\gamma(0) = I_{2n}. \qquad (8)$$

- Fundamental solution $\gamma(t) \in \mathrm{Sp}(2n)$, $t \in [0, T]$

- Spectral stability $\sigma(\gamma(T)) \in \mathbf{U}$

- Linear stability $\|\gamma(T)^k\|$ is bounded for $k \in \mathbf{N}$

- Linear stability implies $\gamma(T)$ splits into two dimensional rotations.

  This from Y.Long, normal from, basic normal form analysis or paper of W. Ballman, G.Thorbergsson and W.Ziller

- Difference of Spectral and linear stability

- First integral: Momentum, angle momentum, energy

- Reduction the system

- Spectral stability is same, linear stability from the essential part

- Problem for the angle momentum?

The Figure-Eight orbit

- Fixed period T, the Klein group $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ with generators σ and τ acts on $\mathbf{R}/T\mathbf{Z}$ and on $\mathbf{R}^2$ as follows:
  $\sigma \cdot t = t + \frac{T}{2},\ \tau \cdot t = -t + \frac{T}{2}$,
  $\sigma \cdot (x, y) = (-x, y),\ \tau \cdot (x, y) = (x, -y)$.

- (Chenciner and Montgomery) There exists an "eight"-shaped planar loop $q : (\mathbf{R}/T\mathbf{Z}, 0) \to (\mathbf{R}^2, 0)$ with the following properties:

(i) for each t,

$$q(t) + q(t + T/3) + q(t + 2T/3) = 0;$$

(ii) $q(t)$ is equivariant with respect to the actions of $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ on $\mathbf{R}/T\mathbf{Z}$ and $\mathbf{R}^2$ above:

$$q(\sigma \cdot t) = \sigma \cdot q(t) \text{ and } q(\tau \cdot t) = \tau \cdot q(t);$$

The Figure-Eight orbit

(iii) the loop $x : \mathbf{R}/T\mathbf{Z} \to \hat{\mathcal{X}}$ defined by

$$x(t) = (q(t + 2T/3), q(t + T/3), q(t))$$

is a zero angular momentum T-periodic solution of the planar three-body problem with equal masses.

- Figure-Eight is minimizer in the D_6-invariant loop space

- linear stable by Kapela and Simó, also Roberts use computer assisted proof

- Can't understand why it is stable

- Motivated by Maslov-type index. Y.Long,...... We study the linear stability from variation property

Symmetry period orbits in n-body problems

- **Type I: (Cyclic Symmetry)** $Q, S \in \mathrm{Sp}(2n) \cap O(2n)$, $SJ = JS$ and $S^m = Q$.

  $E = \{z \in W^{1,2}(\mathbf{R}/T\mathbf{Z}, \mathbf{R}^{2n}) \mid z(t) = Qz(t+T)\}$.

  $\mathbf{Z}_m$-group action with generator $g \in \mathbf{Z}_m$:

$$\begin{array}{rcl} g : E & \rightarrow & E, \\ z(t) & \mapsto & Sz(t + \dfrac{T}{m}), \end{array}$$

- hence $g^m = id$.

- Hamiltonian function $H(t, z) \in \mathbf{C}^2(\mathbf{R} \times \mathbf{R}^{2n}, \mathbf{R})$ satisfies $H(t - T/m, Sz) = H(t, z)$ ($H(Sz) = H(z)$ in autonomous case),

- $f(z) = \int_0^T [(-J\frac{dz(t)}{dt}, z(t)) - H(t, z(t))]dt$ is $\mathbf{Z}_m$-invariant.

Symmetry period orbits in n-body problems

- **Type II: (Brake Symmetry)** Let $S, N \in O(2n)$, and satisfy $SJ = JS$, $N^2 = id_{2n}$, $NJ = -JN$, $N = N^T$, $NS^T = SN$.

$$E = \{z \in W^{1,2}([0,T], \mathbf{R}^{2n}) \mid z(0) = Sz(T)\}$$

time-reversal $\mathbf{Z}_2$-group action given by then

$$\begin{aligned} g : E &\to E, \\ z(t) &\mapsto Nz(T-t), \end{aligned}$$

- $H(t,z)$ satisfies $H(T-t, Nz) = H(t,z)$ ($H(Nz) = H(z)$ in autonomous case).

- functional $f(z)$ is $\mathbf{Z}_2$-invariant. $V^{\pm}(SN)$ and $V^{\pm}(N)$ are Lagrangian subspaces of $(\mathbf{R}^{2n}, \omega)$

Symmetry Hamiltonian systems

- These two group actions are motivated by the periodic solutions of the n-body problems appearing in recent literature A. Chenciner, Chen, D. L. Ferrario, S. Terracini,..............

- find critical point of $f(z)$ in $\mathbf{Z}^m$ invariant loop space by Palis principle

- Hamiltonian equation on the fundamental domain with corresponding boundary condition.

- Type I $x(0) = Sx(T/m)$, Type II $x(0) \in V^+(SN)$, $x(T/2) \in V^+(N)$

- boundary condition given by $(x(0), x(T)) \in \Lambda$, Λ is lagrangian subspace of $(\mathbf{R}^{2n} \oplus \mathbf{R}^{2n}, -\omega \oplus \omega)$

Maslov index

- Maslov index of a path of Lagrangian subspaces $V(t)$ with respect to a fixed Lagrangian subspace Λ (Cappell, Lee, Miller)

- $\Sigma_\Lambda = \{V \in Lag(2n) | dim V \cap \Lambda \neq 0\}$

- Maslov index $\mu(\Lambda, V(t))$ is intersection number of $e^{-\varepsilon J} V(t)$ with Σ_Λ, $0 < \varepsilon \ll 1$

- Positive direction is given by $e^{J(t-t_0)} V(t_0)$

- $V(t) = Gr(\gamma(t))$ is Lagrangian subspace $(\mathbf{R}^{2n} \oplus \mathbf{R}^{2n}, -\omega \oplus \omega)$

- $\mu(z) = \mu(\Lambda, V(t))$

Bott-type iteration formula

- In 1956, Bott got his celebrated iteration formula for the Morse index of closed geodesics, and it was generalized by Ballmann, Thorbergsson, Ziller,.......

- The precise iteration formula of general Hamiltonian system was established by Long.

- the iteration could be regarded as a special group action (Type I) $Q = S = I_{2n}$.

- brake symmetry iteration formula has studied by Long, Liu, Zhang, Zhu. It is special case of Type II

Bott-type iteration formula

- **Theorem 1.** Let z be a solution fundamental solution $\gamma(t)$. for type I symmetry

$$\mu(Gr(Q^T), Gr(\gamma(t)), t \in [0, T]) = \sum_{i=1}^{m} \mu(Gr(\exp(\frac{i}{m} 2\pi\sqrt{-1}) S^T), Gr(\gamma(t)), t \in [0, T/m]), \quad (9)$$

Type II symmetry

$$\mu(Gr(S^T), Gr(\gamma(t)), t \in [0, T]) = \mu(V^+(N), \gamma(t)V^+(SN), t \in [0, \frac{T}{2}]) + \mu(V^-(N), \gamma(t)V^-(SN), t \in [0, \frac{T}{2}]). \quad (10)$$

- We have noticed that the k-th iteration formula for brake symmetry is studied by Liu, Zhang by a different way.

Relation of Morse index and Maslov index

- n-body problem is also a second order system. Its solution, as the critical point of the action functional (on the symmetry loop space), has also Morse index.

- The relation between Morse index of solution of Lagrangian system and Maslov index of corresponding solution in Hamiltonian is an intriguing problem, has studied by many author, Duistermaat,...

- especially for the period case by An, Long, Viterbo,...

- No one is suitable for our use

Relation of Morse index and Maslov index

- Boundary condition of type I

  $$x(0) = \bar{S}x(T/m), \bar{S} \in O(n)$$

  Type II

  $$x(0) \in V_1, x(T/2) \in V_2,$$

  V_1, V_2 are subspace of $\mathbf{R}^n$

- Corresponding boundary condition in Hamiltonian systems, Type I

  $\Lambda = Gr(S)$, with $S = \begin{pmatrix} \bar{S} & 0 \\ 0 & \bar{S} \end{pmatrix}$

- Type II, let $\bar{\Lambda}_i = V_i \oplus V_i^{\perp} \in \mathbf{R}^{2n}$, $i = 1, 2$,

  $$x(0) \in \bar{\Lambda}_1, x(T) \in \bar{\Lambda}_2$$

Relation of Morse index and Maslov index

- **Theorem 2.** For a critical point x of lagrangian function, with $\gamma(t)$ is the fundamental solution of the corresponding solution in Hamiltonian system

- under boundary condition Type I,

  $$m^-(x) + \nu_1(\bar{S}) = \mu(Gr(S^T), Gr(\gamma(t))),$$

  where $\nu_1(\bar{S}) = \dim\ker(\bar{S} - I_n)$.

- Under boundary condition Type II,

  $$m^-(x) + \dim V_1^\perp \cap V_2^\perp = \mu(\bar{\Lambda}_2, \gamma(t)\bar{\Lambda}_1).$$

  .

Relation with the Maslov-type index

- Masov-type index (Conley, Ekeland, Long, Zehnder,.....) for symplectic matrix path is a successful theory in study the stability of period solution in Hamiltonian systems

- $\mathrm{Sp}(2n)^0_\omega = \{M \in \mathrm{Sp}(2n) | \det(M - I_{2n}) = 0\}$, $\omega \in \mathbf{U}$

- $\gamma(t) \in \mathrm{Sp}(2n)$, $i_\omega(\gamma)$ is the intersection number of $e^{-\varepsilon J}\gamma(t)$ with $\mathrm{Sp}(2n)^0_\omega$ (minus n if $\omega = 1$)

- Let $\tilde{\gamma}(t) = S\gamma(t)$, $\xi(t) \in \mathrm{Sp}(2n)$ be any path connected I_{2n} to S

- $\mu(Gr(\omega S^T), Gr(\gamma)) = i_\omega(\tilde{\gamma} * \xi) - i_\omega(\xi)$

stability criteria

- Let $e(M)$ the total algebraic multiplicity of all eigenvalues of M on $\mathbf{U}$.

- $M \in \mathrm{Sp}(2n)$, for any symplectic path η from I_{2n} to M, define $\mathcal{D}_\omega(M)$ for $\omega \in \mathbf{U}$ by

$$\mathcal{D}_\omega(M) = i_\omega(\eta) - i_1(\eta). \tag{11}$$

Following book of Long, this definition is independent of the choice of η

- For function $g(w)$ on $[a, b]$, define its variation by

$$var(g(w), [a, b]) = \max\{\sum_{j=0}^{k-1} |g(w_{j+1}) - g(w_j)|,$$
$$a = w_0 < \cdots < w_k = b \text{ is any partition}\}.$$

stability criteria

- $$e(M)/2 \geq var(\mathcal{D}_{\exp(\sqrt{-1}\theta)}(M), \theta \in [0, \pi]),$$
  by book of Long

- For period solution with type I symmetry,
  $$\gamma(T) = (S\gamma(T/m))^m$$

- let
  $$f(\theta) = \mu(Gr(\exp(\sqrt{-1}\theta)S^T), Gr(\gamma_z(t)), \\ t \in [0, T/m]) + \mathcal{D}_{\exp(\sqrt{-1}\theta)}(S),$$

- **Theorem 3.**
  $$e(\gamma_z(T))/2 \geq var(f(\theta), \theta \in [0, \pi]). \quad (12)$$

linear instability criteria

- Observation: $M \in \mathrm{Sp}(2n)$ is linearly stable, then

$$\det(e^{-\varepsilon J}M - I_{2n}) > 0. \tag{13}$$

- **Theorem 4.** For period solution with Type I symmetry, then the solution is linearly unstable if $\mu(Gr(S^T), Gr(\gamma_z(t)), t \in [0, T/m])$ is odd.

- we need to consider the affect of first integral if it has

- a simple criteria could given to judge the linear instability of closed geodesics

Application to Figure-Eight orbits

- $$D_6 =< g_1, g_2 \,|\, g_1^6 = I_6,\ g_2^2 = I_6,\ g_1 g_2 = g_2 g_1^{-1} > .$$

- g_1 generator $\mathbf{Z}_6$ group is type I, g_2 type II

- **fact 1.** Figure-Eight nondegenerate and is local minimizer in the $\mathbf{Z}_2$, $\mathbf{Z}_3$ invariant loop space

- **fact 2.** the symplectic Jordan form corresponding to the angular momentum of the monodromy matrix is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

- **Theorem 5.** The Figure-Eight is linear stable under the condition of the above fact.

The Figure-Eight orbits

- We had verified the fact by matlab

- some question proposed by Chenciner:
  1. Prove the Figure-Eight is the $\mathbf{Z}_3$ minimizer with a topology constrain

- 2. Figure-Eight is minimizer on the $\mathbf{Z}_6$ invariant loop space

- 3. Figure-Eight is minimizer on D_3($\mathbf{Z}_3$ with the brake symmetry) invariant loop space

- the symplectic Jordan form corresponding to the angular momentum also studied by Chenciner, Féjoz, and Montgomery (numerical for detail form)

Idea of Prof Theorem 5.

- Configuration space

  $\mathcal{X} = \{x = (x_1, x_2, x_3) \in (\mathbf{R}^2)^3 \,|\, x_1 + x_2 + x_3 = 0\}$,

- $\mathbf{Z}_6$ group generator g_1 on $\mathcal{X}$ is

$$(\tilde{g}_1 \circ u)(t) \;=\; \tilde{S}u(t + T/6) \qquad (14)$$

- $$\tilde{S} = \begin{pmatrix} 1/2 & 0 & \sqrt{3}/2 & 0 \\ 0 & -1/2 & 0 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 0 & 1/2 & 0 \\ 0 & \sqrt{3}/2 & 0 & -1/2 \end{pmatrix},$$

- Set $S = \begin{pmatrix} \tilde{S} & 0 \\ 0 & \tilde{S} \end{pmatrix}$

- Set $M = S\gamma(T/6)$, $\gamma(T) = M^6$

Symplectic normal form

- Configuration space

- $x(t)$ is a period T solution of the Newton system, then $h^{-2/3}x(ht)$ is also a solution with period T/h

- energy is negative, differential with h get the corresponding normal form $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

- normal form for angle momentum of M is $\begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}$, $b = 1, -1, 0$.

- $b = -1$ by fact 2. the essential matrix M_2 is 2×2

idea of proof

- $\forall \omega \in \mathbf{U}$ define

  $\mu(\omega) := \mu(Gr(\omega S^T), Gr(\gamma(t)), t \in [0, T/6]),$

- By Theorem 1. and 2., fact 1, $\mu(\omega) \geq 0$, $\mu(1) = \mu(-1) = 0$, $\mu(\exp(2\pi\sqrt{-1}/3)) = 1$

- $\mathcal{D}_\omega(S) = 0, \omega \in \mathbf{U}^+ \backslash \{\exp(\pi\sqrt{-1}/3), \exp(2\pi\sqrt{-1}/3$ and

  $\mathcal{D}_{\exp(\pi\sqrt{-1}/3)}(S) = \mathcal{D}_{\exp(2\pi\sqrt{-1}/3)}(S) = -1.$

- detail analysis could get Theorem 5.

Lagrangian solutions

- (1772 Lagrange) three bodies form an equilateral triangle, each body travels along a specific Keplerian orbit

- Sun-Jupiter-Trojan asteroids system

- The stability had studied by many authors: Gascheau, Routh, Danby, Roberts, Meyer, Schmidt, Martínez, Samà, Simó......

- $\beta = \frac{27(m_1m_2+m_1m_3+m_2m_3)}{(m_1+m_2+m_3)^2}$

- Linear stable if $\beta < 1$, eccentricity $e = 0$. Numerical for general

- minimizer under topology constrain

- Morse index is zero

Main Theorems

- ϕ_k Morse index of k-th iteration of the Lagrangian solution in the variational problem

- **Theorem A**(Hu-Sun) For the elliptic Lagrangian solution $x(t)$,

$$
\begin{aligned}
2 &\leq \phi_2 \leq 4 && (15) \\
\phi_2 &\leq e(\gamma(T))/2. && (16)
\end{aligned}
$$

- $\phi_2 = 4$, spectrally stable;

- $\phi_2 = 3$, linear unstable;

- $\phi_2 = 2$, spectrally stable if $\exists k \geq 3$, such that $\phi_k > 2(k-1)$.

- $\phi_k = 2(k-1)$, for all $k \in \mathbf{N}$, linear unstable.

Affect of First integral

- **Theorem**(Meyer and Schmidt)

$$\gamma(t) = \gamma_1(t) \diamond \gamma_2(t),$$

  $\gamma_1(t)$ is basic solution of Kepler solution, $\gamma_2(t)$ is the essential part.

- Solution is linear stable if $\gamma_2(t)$ is linear stable

- First integral of energy is clear

- $$\gamma_1(T) = P^{-1}(N_1(1,1) \diamond I_2)P.$$

  where $N_1(1,1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $P \in \mathrm{Sp}(2n)$

- Fixed energy, all solution is periodic with same period

- **Theorem**(Gordon) The planar Kepler problem with prime period T is the minimizer of the action functional on the subspace of $W^{1,2}(\mathbf{R}/T\mathbf{Z}, \mathbf{R}^2)$-loops with winding number ± 1 with respect to the origin.

- Local minimizer, Morse index is zero

$$i_1(\gamma_1) = 0$$

- For the Keplerian solution

$$\gamma_1(T) = P^{-1}(N_1(1,1) \diamond I_2)P.$$

- $i_\omega(\gamma_1) = 2$, for $\omega \in \mathbf{U}$, $\omega \neq 1$

- Iteration formula is clear

Stability of Lagrangian solution

- **Theorem** (Venturelli, also Long, Zhang and Zhou) fix an element $(k_1, k_2, k_3) \in H_1(\hat{\mathcal{X}}) \cong \mathbf{Z}^3$. If $(k_1, k_2, k_3) = (1, 1, 1)$ or $(-1, -1, -1)$, the minimizers among the loops in this homology class are the elliptic Lagrangian solutions with prime period T.

- The important of prime period is pointed by Long

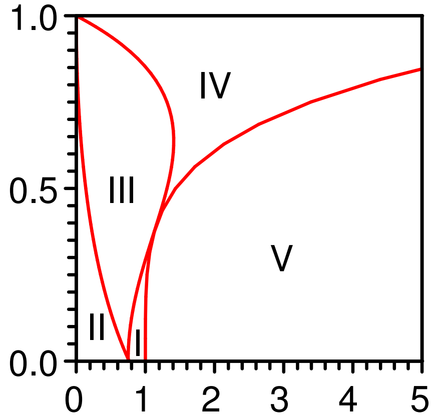
- $\phi_1 = 0$

- $i_1(\gamma_2) = 0$, $i_{-1}(\gamma_2) \leq 2$

- $\phi_2 = i_{-1}(\gamma_2) + 2$

- Prof of Theorem A

- I, II are linear stable, III is hyperbolic-elliptic, IV is hyperbolic with real eigenvalue, V is hyperbolic with complex eigenvalues.



- The region for $\gamma_2(2T)$ to be degenerate on boundary III

- $\phi_2 = 4$ on II, $\phi_2 = 3$ on III, $\phi_2 = 2$ on I, V, and IV

- $\phi_2 = 3$ on left boundary of III, $\phi_2 = 2$ on right boundary of III.

- $\phi_k = 2(k-1)$ on boundary of V, IV

- Normal formal or basic normal form is clear on each region