A local mountain pass type result for a system of nonlinear Schrödinger equations

Kazunaga Tanaka
(Waseda University, Tokyo, Japan)

Joint work with Norihisa Ikoma

0. Introdcution

In this talk we consider a singular perturbation problem for a system of nonlinear Schrödinger equations:

$$\begin{cases} -\varepsilon^2\Delta u_1 + V_1(x)u_1 = \mu_1 u_1^3 + \beta u_1 u_2^2, & \text{in } \mathbf{R}^N, \\ -\varepsilon^2\Delta u_2 + V_2(x)u_2 = \mu_2 u_2^3 + \beta u_1^2 u_2, & \text{in } \mathbf{R}^N, \\ u_1(x) > 0, \quad u_2(x) > 0 & \text{in } \mathbf{R}^N, \\ u_1(x),\ u_2(x) \in H^1(\mathbf{R}^N). \end{cases} \qquad (S)$$

Here $N = 2, 3$, μ_1, $\mu_2 > 0$, $\beta \in \mathbf{R}$ are constants, and $V_1(x)$, $V_2(x) : \mathbf{R}^N \to \mathbf{R}$ are bounded continuous positive functions, and $\varepsilon > 0$ is a small parameter.

We consider the situation:

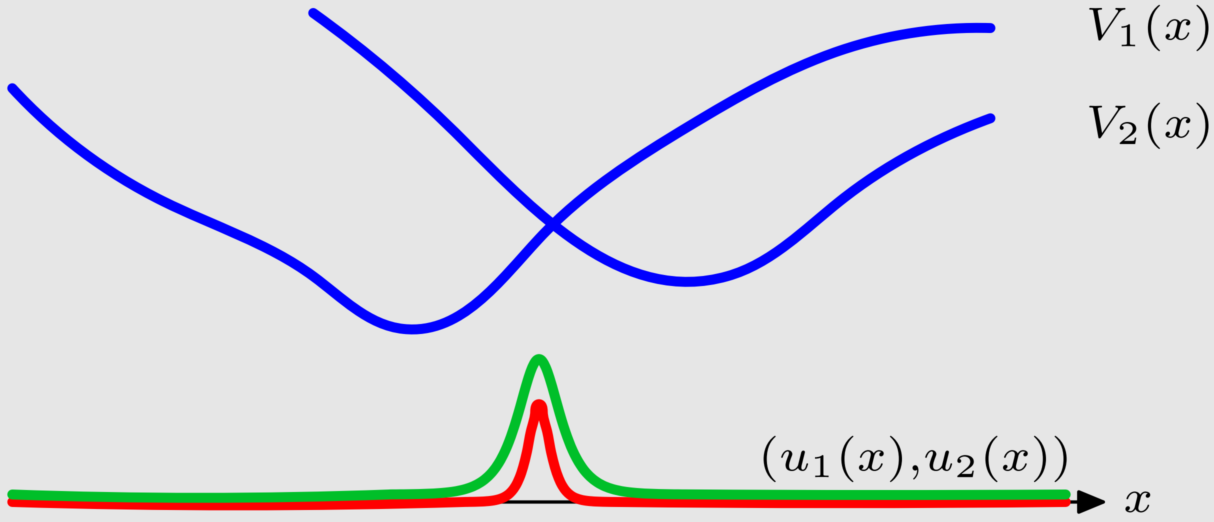
$$0 < \beta < \sqrt{\mu_1\mu_2} \quad \text{and} \quad \beta \quad \text{is relatively small.}$$

Remark. Our problem (S) has **semi-trivial solutions**, i.e., solutions of type $(u_1(x), 0)$ or $(0, u_2(x))$, where $u_1(x)$ and $u_2(x)$ solve the scalar equation:

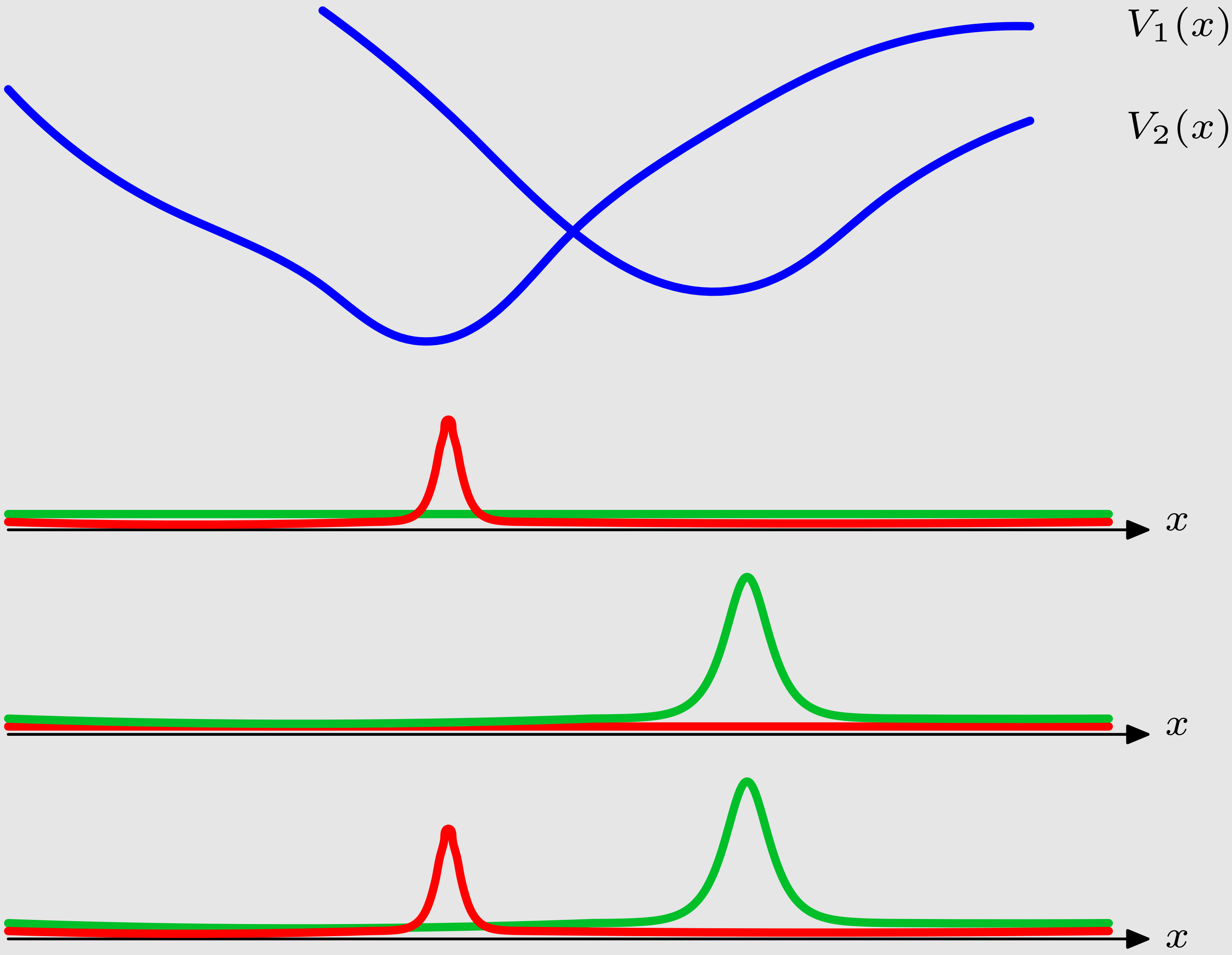
$$-\varepsilon^2 \Delta u + V_i(x) u = \mu_i u^3 \quad \text{in } \mathbf{R}^N .$$

We call solutions $(u_1(x), u_2(x))$ with $u_1 \not\equiv 0$ and $u_2 \not\equiv 0$ as **non-trivial vector solutions**.

Our aim is to find a family of concentrating solutions of (S), whose limit is a non-trivial vector solution.



Other solutions



Plan of my talk.

1. Singular perturbation problem for scalar equations
   - Local mountain pass for scalar equations:
2. Known results for the case: $V_i(x) \equiv V_i$
   - Results of Lin-Wei, Ambrosetti-Corolado, Sirakov
3. Singular perturbation problem for systems
   - Setting of problem
   - Main result
4. Idea of a proof

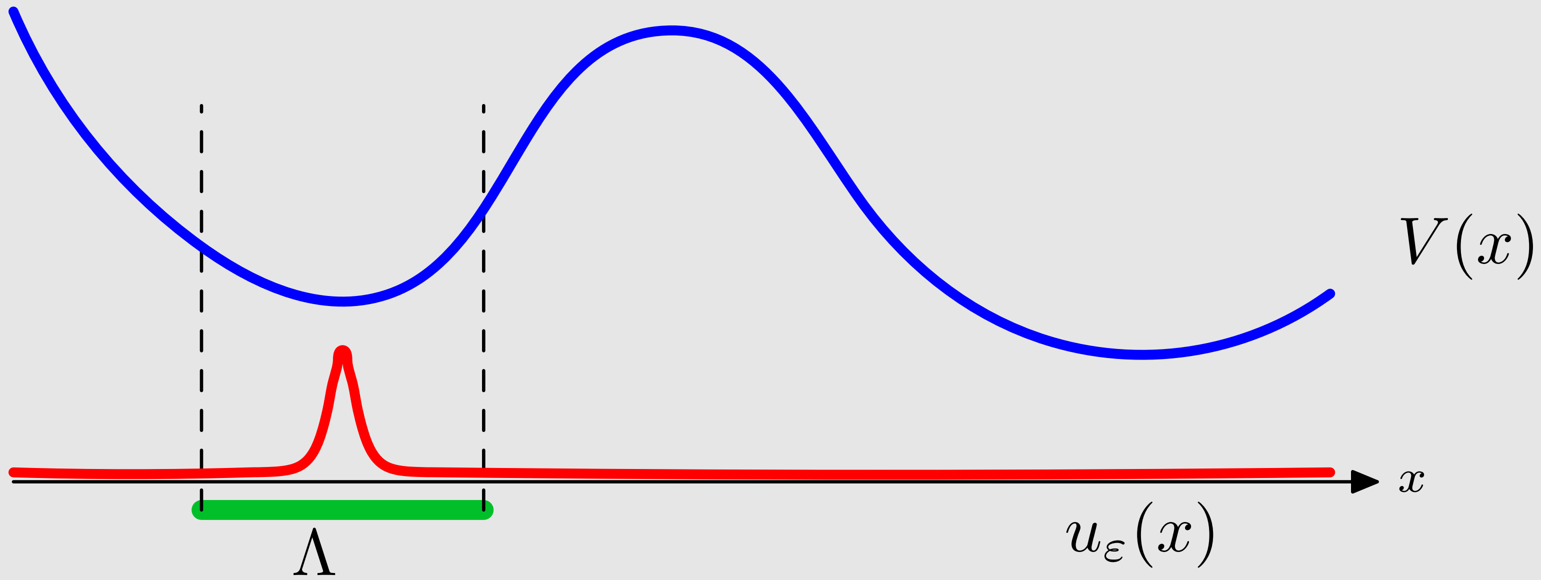
1. Singular perturbation problem for scalar equations

Singular perturbation problems for scalar problems are well-studied.

$$-\varepsilon^2\Delta u + V(x)u = f(u),\ u(x) > 0,\ u(x) \in H^1(\mathbf{R}^N). \tag{1}$$

A partial list of contributors:

Ambrosetti, Badiale, Bartsch, Byeon, Cao, Cingolani, D'Aprile, Dancer, del Pino, Felmer, Floer, Grossi, Gui, Jeanjean, Kang, Li, Lin, Liu, Malchiodi, Martínez, Montenegro, Ni, Nirenberg, Oh, Pistoia, Rabinowitz, Wang, Wei, Weinstein, Yan



Here we state the result precisely for a case:

$$-\varepsilon^2 \Delta u + V(x)u = u^3.$$

Suppose that there exists a bounded open set $\Lambda \subset \mathbf{R}^N$ such that

$$\inf_{x\in\Lambda} V(x) < \inf_{x\in\partial\Lambda} V(x).$$

We denote $K = \{x \in \Lambda \mid V(x) = \inf_{x\in\Lambda} V(x)\}$.

Setting $v(x) = u(\varepsilon x)$, we introduce a rescaled problem:

$$-\Delta v + V(\varepsilon x)v = v^3. \tag{2}$$

The corresponding functional is

$$I_\varepsilon(v) = \frac{1}{2}\|\nabla v\|_2^2 + \frac{1}{2}\int_{\mathbf{R}^N} V(\varepsilon x)v^2\,dx - \frac{1}{4}\int_{\mathbf{R}^N} |v|^4\,dx.$$

There exists a family $(v_\varepsilon(x))$ of solutions of (2): after taking a subsequence $\varepsilon_n \to 0$

$$P_{\varepsilon_n} \to P_0 \in K,$$

$$v_{\varepsilon_n}(x + P_{\varepsilon_n}/\varepsilon_n) = u_{\varepsilon_n}(\varepsilon_n x + P_{\varepsilon_n}) \to \omega_{P_0}(x),$$

$$I_{\varepsilon_n}(v_{\varepsilon_n}) \to I_{P_0}(\omega_{P_0}) = C_0 V(P_0)^{(4-N)/2}.$$

Here $\omega_{P_0}(x)$ is a least energy solution of the limit problem: $-\Delta\omega + V(P_0)\omega = \omega^3$.

We remark that accumulating point P_0 is also characterized as

$$d(P_0) = \inf_{P \in \Lambda} d(P),$$

where $d(P)$ $(P \in \mathbf{R}^N)$ is the least energy level of non-trivial solutions for

$$-\Delta v + V(P)v = v^3 \quad \text{in } \mathbf{R}^N.$$

Remark. If P_0 is a global minimum of $V(x)$, a family $(v_\varepsilon(x))$ can be obtained as a minimizer of a minimizing problem:

$$\inf\{I_\varepsilon(v) \mid v \in \mathcal{N}_\varepsilon\},$$

where $\mathcal{N}_\varepsilon = \{u \in H^1(\mathbf{R}^N) \mid u \neq 0,\ I'_\varepsilon(u)u = 0\}$.

There are several ways to show the existence of such a family:

- **Lyapunov-Schmidt reduction:** Floer-Weinstein (86), Oh (88, 90)
- **Variational methods:** Rabinowitz (92),...

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Remark. To apply Lyapunov-Schmidt reduction method, uniqueness and non-degeneracy of solutions of limit problems and critical points of $V(x)$ are required.

There are many efforts to relax uniqueness and non-degeneracy assumptions.

Local mountain pass method: del Pino-Felmer (96)
They succeeded to construct a family of solutions concentrating to a local minima of the potential $V(x)$ without non-degeneracy nor uniqueness of the solutions of the limit problem.

Generalizations: Jeanjean-T. (04), Byeon-Jeanjean (07), Byeon-Jeanjean-T. (08)

Generalization to a saddle point setting: del Pino-Felmer(04)

These ideas help us to study singular perturbation problem for systems of nonlinear Schrödiger equations.

2. Known results for the case: $V_i(x) \equiv V_i$.

There are many works for the following problem with constant coefficients, which appears as a limit problem in our study.

$$\begin{cases} -\Delta u_1 + V_1 u_1 = \mu_1 u_1^3 + \beta u_1 u_2^2, & \text{in } \mathbf{R}^N, \\ -\Delta u_2 + V_2 u_2 = \mu_2 u_2^3 + \beta u_1^2 u_2, & \text{in } \mathbf{R}^N, \end{cases} \tag{3}$$

where V_1, $V_2 > 0$ are independent of x.

Lin-Wei (05, 06), Ambrosetti-Colorado (07), Bartsch-Wang (06), Bartsch-Wang-Wei (07), Busca-Sirakov (00), Maia-Montefusco-Pellacci (06), Pompionio (06), Sirakov (07), Wei-Weth (08)

Professor Wang gave a good lecture about this topic. Especially sign and size of β are important in the study of (3).

We remark that **when $\beta > 0$ any positive solution of (3) is radially symmetric** (Busca-Sirakow 00).
We restrict ourselves to a case: $\beta > 0$ and β is relatively small.
Here relatively small means that β satisfies $0 < \beta^2 < \mu_1\mu_2$ and the following condition.

(∗) Let $\omega_i(x)$ be a ground state solution of $-\Delta\omega + V_i\omega = \mu_i\omega^3$. Then both of the following linear operators are **positive definite** in $H_r^1(\mathbf{R}^N)$:

$$-\Delta + V_1 - \beta\omega_2^2, \quad -\Delta + V_2 - \beta\omega_1^2.$$

In other words, critical points correspoinding semi-trivial solutions $(\omega_1, 0)$, $(0, \omega_2)$ are non-degenerate and their **Morse index is 1**.

This condition is introduced in Ambrosetti-Corolado (07). We remark that $(*)$ holds for small $\beta > 0$.

Our assumption: Throughout this talk, we assume

$$(*) \text{ and } 0 < \beta^2 < \mu_1 \mu_2.$$

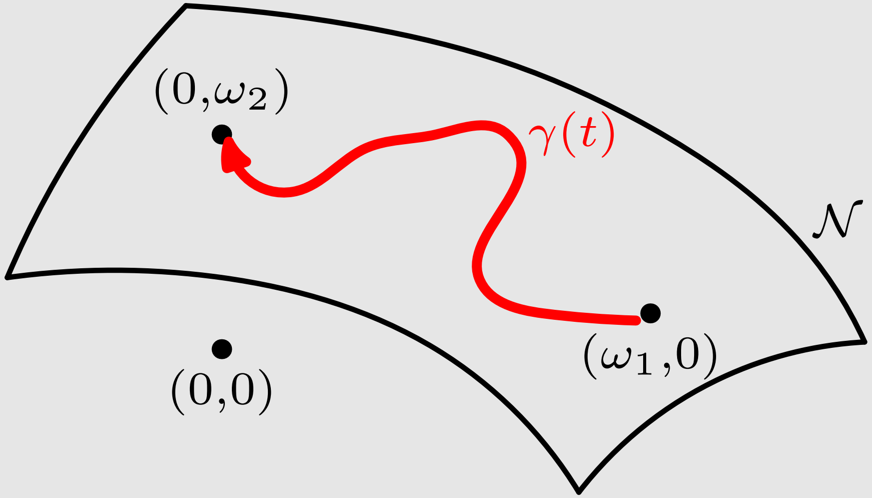
Under these conditions, via variational methods they find a non-trivial vector solution $U = (u_1, u_2)$ satisfying

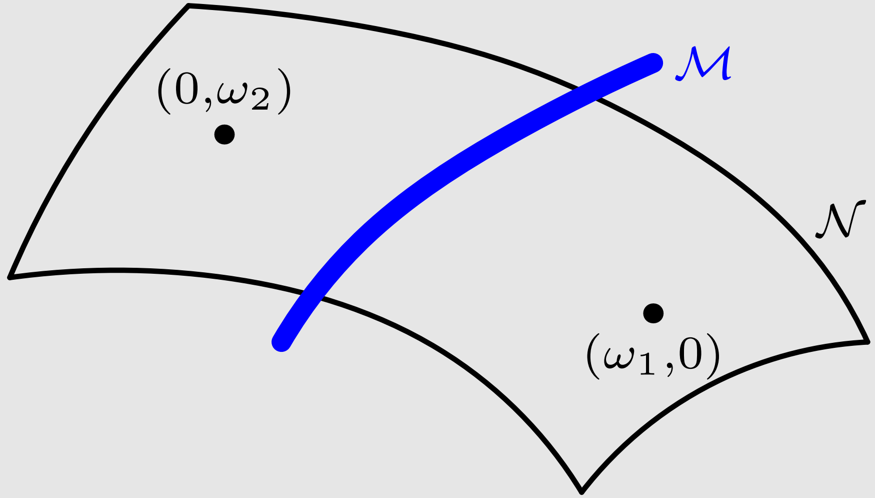
$$\max\{I(\omega_1, 0), I(0, \omega_2)\} < I(U) < I(\omega_1, 0) + I(0, \omega_2).$$

Here $I(u_1, u_2)$ is a functional corresponding to (3):

$$\begin{aligned} I(u_1, u_2) =& \frac{1}{2} \int_{\mathbf{R}^N} |\nabla u_1|^2 + V_1 u_1^2 + |\nabla u_2|^2 + V_2 u_2^2 \, dx \\ & - \frac{1}{4} \int_{\mathbf{R}^N} \mu_1 u_1^4 + \mu_2 u_2^4 + 2\beta u_1^2 u_2^2 \, dx. \end{aligned}$$

I give a mension to their approaches.





a) **Approach by Ambrosetti and Corolado (07)**

They introduce Nehari manifold:

$$\mathcal{N} = \{U = (u_1, u_2) \in H_r^1 \times H_r^1 \mid U \neq (0,0), \text{ and } I'(U)U = 0\}.$$

They apply the mountain pass theorem on $\mathcal{N}$:

$$b_{\mathcal{N}} = \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} I(\gamma(s)),$$

where $\Gamma = \{\gamma(s) \in C([0,1], \mathcal{N}) \mid \gamma(0) = (\omega_1, 0),\ \gamma(1) = (0, \omega_2)\}$.

b) **Approach by Lin-Wei (05) and Sirakov (07)**

Lin-Wei and Sirakov introduce a manifold $\mathcal{M}$ of Nehari type of codimension 2:

$$\mathcal{M} = \{(u_1, u_2) \in H_r^1 \times H_r^1 \mid u_1 \neq 0,\ u_2 \neq 0,\ \text{and}$$
$$I'(u_1, u_2)(u_1, 0) = 0,\ I'(u_1, u_2)(0, u_2) = 0\}.$$

They consider $b_{\mathcal{M}} = \inf_{U \in \mathcal{M}} I(U)$.

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They consider $b_{\mathcal{M}} = \inf_{U \in \mathcal{M}} I(U)$.

Fact: $b_{\mathcal{M}} = b_{\mathcal{N}}$.

$b(V_1, V_2) (\equiv b_{\mathcal{M}} = b_{\mathcal{N}})$ can be characterized as the least energy level for non-trivial vector solutions:

$$b(V_1, V_2) = \inf\{I(U) \mid U \text{ is a non-trivial vector solution of (3)}\}.$$

Uniqueness and non-degeneracy of least energy vector solutions.

1) Uniqueness for small $\beta > 0$: Ikoma (to appear), Wei-Yao (to appear).
2) Non-degeneracy of non-trivial vector solutions except for countably many values of β, i.e,

$$\exists \{\beta_n\}_{n=1}^{\infty} \text{ such that solutions are non-degenerate for } \beta \notin \{\beta_n\}_{n=1}^{\infty}.$$

Dancer-Wei (08).

3. Singular perturbation problem for systems

$$\begin{cases} -\varepsilon^2 \Delta u_1 + V_1(x) u_1 = \mu_1 u_1^3 + \beta u_1 u_2^2, & \text{in } \mathbf{R}^N, \\ -\varepsilon^2 \Delta u_2 + V_2(x) u_2 = \mu_2 u_2^3 + \beta u_1^2 u_2, & \text{in } \mathbf{R}^N. \end{cases} \qquad (S)$$

The following papers deal with singular perturbation problems for systems of NLS,

Lin-Wei (06), Pompionio (06),
Maia-Montefusco-Pellacci (06)

a) Setting of problem

Now we set up our problem for a system (S). We assume the following conditions:

Assumption 1. There exists a compact set $A \subset \mathbf{R}_+^2$ such that

(i) For any $(V_1, V_2) \in A$, the problem

$$\begin{cases} -\Delta u_1 + V_1 u_1 = \mu_1 u_1^3 + \beta u_1 u_2^2 & \text{in } \mathbf{R}^N, \\ -\Delta u_2 + V_2 u_2 = \mu_2 u_2^3 + \beta u_1^2 u_2 & \text{in } \mathbf{R}^N. \end{cases}$$

satisfies the condition $(*)$.

(ii) $(V_1(P), V_2(P)) \in A$ for all $P \in \mathbf{R}^N$.

Under the assumption 1, the least energy level for vector solutions

$$m(P) = b(V_1(P), V_2(P))$$

is well-defined for each $P \in \mathbf{R}^N$.

Assumption 2. There exists a bounded open set $\Lambda \subset \mathbf{R}^N$ such that

$$\inf_{P\in\Lambda} m(P) < \inf_{P\in\partial\Lambda} m(P).$$

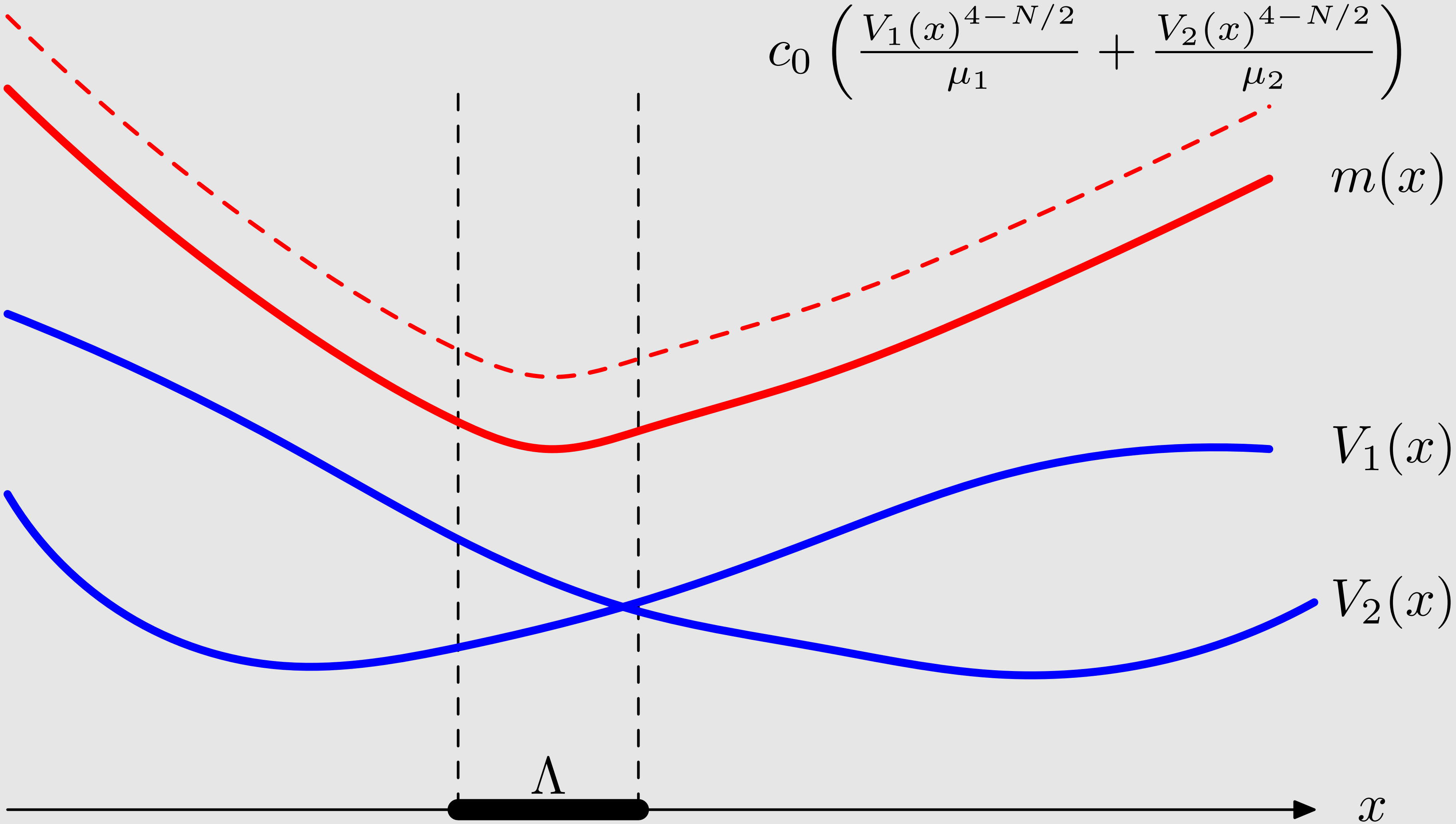
We also set $m_0 = \inf_{P\in\Lambda} m(P)$ and

$$K = \{P \in \Lambda \mid m(P) = m_0\}$$

Remark. When $\beta \to 0$, we can show that

$$\begin{aligned} b(V_1, V_2) \to &\text{ (least energy level for } -\Delta\omega + V_1\omega = \mu_1\omega^3) \\ &+ \text{(least energy level for } -\Delta\omega + V_2\omega = \mu_2\omega^3) \\ &= c_0 \left(\frac{V_1^{(4-N)/2}}{\mu_1} + \frac{V_2^{(4-N)/2}}{\mu_2} \right). \end{aligned}$$

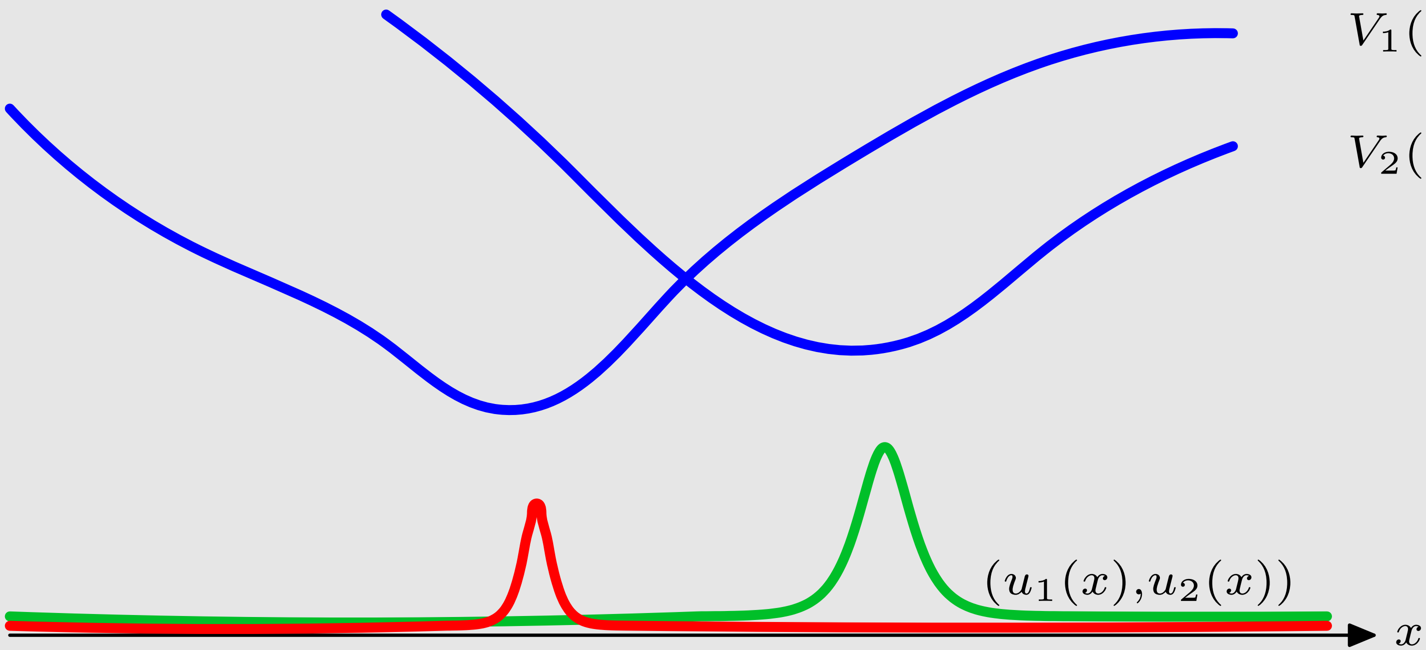
An example of $V_1(x)$, $V_2(x)$, $m(x) = b(V_1(x), V_2(x))$.



We try to find a family of solutions (U_ε) concentrating in Λ.

Before we state our main result, we give some remarks.

Remark 1. If $\beta = 0$, there does not exist a family of solutions concentrating to a point in Λ in general.

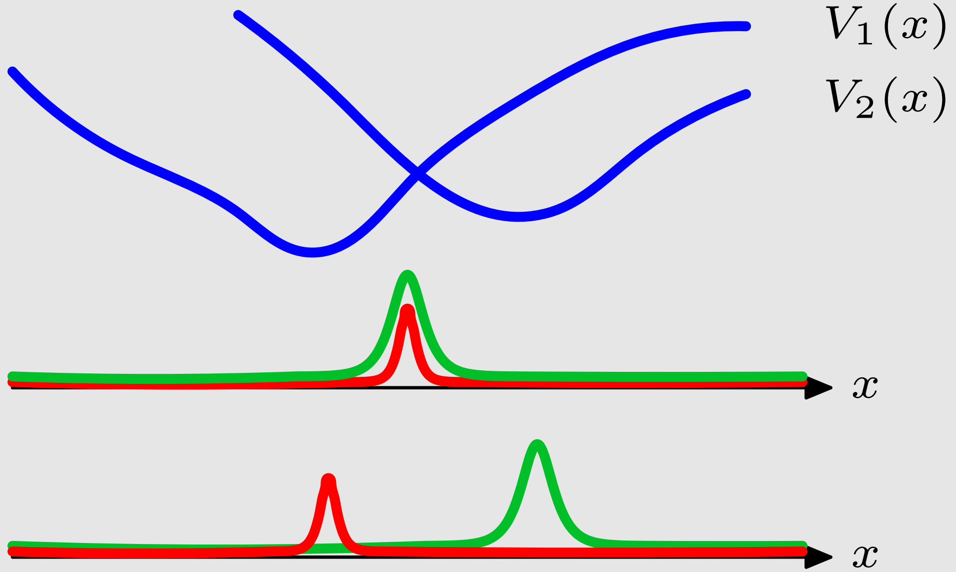


Remark 2. Even if P_0 is a global minimum of $m(P)$, the minimizer $U_\varepsilon(x)$ of

$$\inf\{I_\varepsilon(U) \mid U \in \mathcal{M}_\varepsilon\},$$

does not give a desired family in general. Here

$$\mathcal{M}_\varepsilon = \{U = (u_1, u_2) \mid u_1 \neq 0,\ u_2 \neq 0,$$
$$\text{and } I'_\varepsilon(u_1, u_2)(u_1, 0) = 0, I'_\varepsilon(u_1, u_2)(0, u_2) = 0\}.$$



Result of Lin-Wei (06):

Under a suitable condition on the behavior of $V_i(x)$ at infinity, Lin and Wei studied

- the existence of the minimizer $I_\varepsilon(U_\varepsilon) = \inf_{U \in \mathcal{M}_\varepsilon} I_\varepsilon(U)$
- the behavior of the minimizer $U_\varepsilon(x) = (u_{1\varepsilon}(x), u_{2\varepsilon}(x))$ as $\varepsilon \to 0$.

Among other results, they showed

$$I_\varepsilon(U_\varepsilon) \to \min\left\{ \inf_{P \in \mathbf{R}^N} m(P), \ \inf_{P \in \mathbf{R}^N} d_1(P) + \inf_{P \in \mathbf{R}^N} d_2(P) \right\}.$$

Here $d_i(P)$ $(P \in \mathbf{R}^N)$ is the least energy level of $-\Delta u + V_i(P)u = \mu_i u^3$, i.e.,

$$d_i(P) = c_0 \frac{V_i(P)^{(4-N)/2}}{\mu_i} \qquad (i = 1, 2).$$

Moreover they showed

- If $\inf_{P\in\mathbf{R}^N} m(P) < \inf_{P\in\mathbf{R}^N} d_1(P) + \inf_{P\in\mathbf{R}^N} d_2(P)$, then both components $u_{1\varepsilon}(x)$, $u_{2\varepsilon}(x)$ concentrate same point $P_0 \in K$.

- If $\inf_{P\in\mathbf{R}^N} m(P) > \inf_{P\in\mathbf{R}^N} d_1(P) + \inf_{P\in\mathbf{R}^N} d_2(P)$, then for $i \in \{1, 2\}$, $u_{i\varepsilon}(x)$ concentrates at P_i after extracting a subsequence, where P_i satisfies

$$d_i(P_i) = \inf_{P\in\mathbf{R}^N} d_i(P_i), \quad \text{i.e.,} \quad V_i(P_i) = \inf_{P\in\mathbf{R}^N} V_i(P_i).$$

scalar problem		system of NLS
$V(x)$	$\longleftrightarrow$	$m(x) = b(V_1(x), V_2(x))$

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Difficulties:

- Non-degeneracy of critical points of $m(x)$ is not known.
- Uniqueness and non-degeneracy of least energy solution of the limit problem is not known.

$$\begin{cases} -\Delta u_1 + V_1(P)u_1 = \mu_1 u_1^3 + \beta u_1 u_2^2, & \text{in } \mathbf{R}^N, \\ -\Delta u_2 + V_2(P)u_2 = \mu_2 u_2^3 + \beta u_1^2 u_2, & \text{in } \mathbf{R}^N, \\ u_1(x) > 0,\ u_2(x) > 0 \quad \text{in } \mathbf{R}^N. \end{cases}$$

- Even for the global minimizer of $m(x)$, the global minimizer U_ε of $I_\varepsilon(U)$ on $\mathcal{M}_\varepsilon$ does not give a desired solution in general.

b) Main result

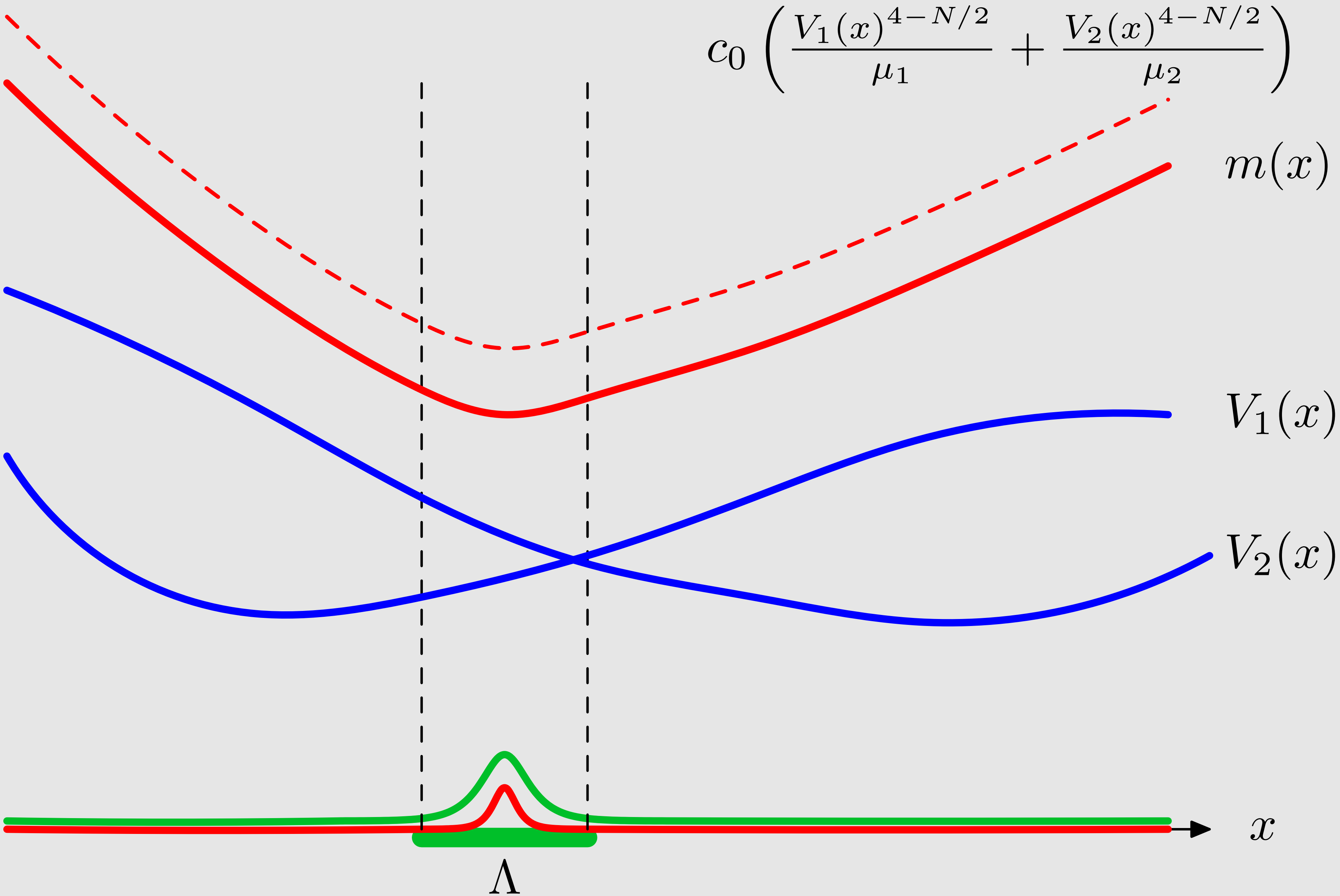
We state our main result for rescaled problem:

$$\begin{cases} -\Delta u_1 + V_1(\varepsilon x)u_1 = \mu_1 u_1^3 + \beta u_1 u_2^2 & \text{in } \mathbf{R}^N, \\ -\Delta u_2 + V_2(\varepsilon x)u_2 = \mu_2 u_2^3 + \beta u_1^2 u_2 & \text{in } \mathbf{R}^N. \end{cases} \qquad (S)$$

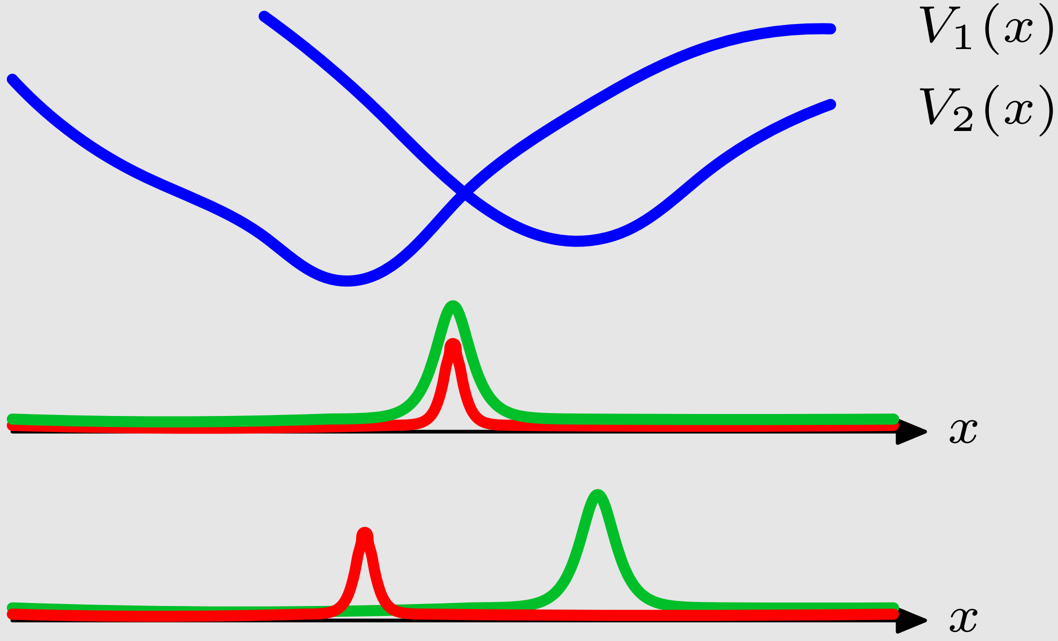
Theorem. Suppose that Assumptions 1 and 2 hold. Then (S) has a family of non-trivial positive vector solutions (U_ε) satisfying the following property: After taking a subsequence $\varepsilon_n \to 0$ there exist a sequnece $(P_{\varepsilon_n}) \subset \Lambda$ such that

$$P_{\varepsilon_n} \to P_0 \in K, \quad U_{\varepsilon_n}(x - P_{\varepsilon_n}/\varepsilon_n) \to \Omega_{P_0}(x) \quad \text{strongly.}$$

Here $\Omega_{P_0}(x)$ is a solution of the limit problem and satisfies $I_{P_0}(\Omega_{P_0}) = m(P_0)$.


$c_0\left(\frac{V_1(x)^{4-N/2}}{\mu_1}+\frac{V_2(x)^{4-N/2}}{\mu_2}\right)$
$m(x)$
$V_1(x)$
$V_2(x)$
x
Λ

Remark. Under the situation like picture, 2 type of solutions coexist.



4. Idea of a proof

As stated in Remark 1, the minimizing method in $\mathcal{M}_\varepsilon$ does not work even if P_0 is a global minimizer of $m(P)$.

We use an idea from Byeon and Jeanjean (07), which was used to prove the existence of concentrating solutions for NLS:

$$-\varepsilon^2\Delta u + V(x)u = f(u) \quad \text{in } \mathbf{R}^N.$$

Their condition on the nonlinearity $f(u)$ is very general; they assume just Berestycki-Lions' type conditions.

They develop a mountain pass type argument in the whole space $H^1(\mathbf{R}^N)$. Their keys of argument is the following points:

(a) Mountain pass characterization of the solution of the limit equation.

(b) Uniform estimate of $\|I'(u)\|$ from below in a neighborhood of a set of expected solutions: for some $\rho > 0$

$$\|I'(u)\| \geq \rho \quad \text{for } u \in (X_\varepsilon^d \setminus X_\varepsilon^{d/2}),$$

where $X_\varepsilon^d = \{\omega(x - \frac{x_0}{\varepsilon}) + \varphi(x) \mid \text{dist}\,(x_0, K) < \delta,\ ||\varphi||_{H^1} \leq d\}$.

This type of ideas are also used in Séré (92, 93), Coti Zelati-Rabinowitz (91, 92),... for construction of multi-bump type solutions for Hamiltonian systems and nonlinear elliptic PDEs.

We use their idea in the Nehari type manifold of codimension 2:

$$\mathcal{M}_\varepsilon = \{U \mid U = (u_1, u_2),\ u_1 \neq 0,\ u_2 \neq 0,\ \text{and}$$
$$I'_\varepsilon(u_1, u_2)(u_1, 0) = 0,\ I'_\varepsilon(u_1, u_2)(0, u_2) = 0\},$$

where $I_\varepsilon(u_1, u_2)$ is a functional corresponding to (S). That is,

$$I_\varepsilon(u_1, u_2) = \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u_1|^2 + V_1(\varepsilon x)u_1^2 + |\nabla u_2|^2 + V_2(\varepsilon x)u_2^2\, dx$$
$$- \frac{1}{4}\int_{\mathbf{R}^N} \mu_1 u_1^4 + \mu_2 u_2^4 + 2\beta u_1^2 u_2^2\, dx.$$

We also introduce a functional corresponding the limit problem at $P \in \mathbf{R}^N$, that is,

$$I_P(u_1, u_2) = \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u_1|^2 + V_1(P)u_1^2 + |\nabla u_2|^2 + V_2(P)u_2^2\, dx$$
$$- \frac{1}{4}\int_{\mathbf{R}^N} \mu_1 u_1^4 + \mu_2 u_2^4 + 2\beta u_1^2 u_2^2\, dx.$$

Let $m_0 = \inf_{P\in\Lambda} m(P)$. We use the following notation: for $P \in K$

$$\mathcal{S}_P = \{\Omega(x) = (\omega_1(x), \omega_2(x)) \in H_r^1 \times H_r^1 \mid \omega_1 \neq 0,\ \omega_2 \neq 0,\ I_P'(\Omega) = 0,\ I_P(\Omega) = m_0)\}.$$

We have

- $\bigcup_{P\in K} (\{P\} \times \mathcal{S}_P)$ is compact in $\mathbf{R} \times H_r^1$.
- there exist δ_0, $C_0 > 0$ independent of $P \in K$ and $\Omega \in \mathcal{S}_P$ such that

$$\Omega(x),\ |\nabla\Omega(x)| \leq C_0 \exp(-\delta_0 |x|) \quad \text{for all } \Omega \in \mathcal{S}_P,\ x \in \mathbf{R}^N.$$

For d, $\delta > 0$ small we define the following neighborhood of limiting solutions:

$$X_\varepsilon^d = \{\Omega(x - \frac{\widetilde{P}}{\varepsilon}) + \Phi(x) \mid P \in K,\ \widetilde{P} \in \mathbf{R}^N,\ |\widetilde{P} - P| < \delta,$$
$$\Omega \in \mathcal{S}_P,\ \|\Phi\|_{H^1} \le d\},$$

The following proposition is a key of proof of our Theorem.

Proposition. There exist d, $\delta > 0$ such that for some $\rho > 0$

$$\inf\{I_\varepsilon(U) \mid U \in (X_\varepsilon^d \setminus X_\varepsilon^{d/2}) \cap \mathcal{M}_\varepsilon\} \ge m_0 + \rho$$

for sufficiently small $\varepsilon > 0$.

To prove Proposition, we need to show the following two properties:

(A) $\liminf_{\varepsilon\to 0}\{I_\varepsilon(U) \mid U \in X_\varepsilon^d \cap \mathcal{M}_\varepsilon\} = m_0$.

(B) There exist $\nu > 0$ and $\rho > 0$ such that

$$\|I_\varepsilon'(U)\| \geq \rho$$

for $U \in (X_\varepsilon^d \setminus X_\varepsilon^{d/2}) \cap \mathcal{M}_\varepsilon$ with $I_\varepsilon(U) \leq m_0 + \nu$.

We remark that

- Property (B) follows from a version of concentration-compactness principle.
- The proof of property (A) is a little bit technical and we use the properties of $\mathcal{S}_P$ $(P \in K)$.