Multiple positive solutions for semilinear Schrödinger equations

(joint work with Lishan Lin and Shaowei Chen)

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Happy Birthday to Professor Paul Rabinowitz

1 Introduction and main results

In this talk, we study the existence of multi-bump solutions for the time independent semilinear Schrödinger equation

$$-\triangle u + (1 + \epsilon a(x))u = |u|^{p-2}u, \quad u \in H^1(\mathbb{R}^N), \tag{1}$$

where $N \geq 1$, $2 < p < 2^*$, 2^* is the critical Sobolev exponent defined by $2^* = \frac{2N}{N-2}$ if $N \geq 3$ and $2^* = \infty$ if $N = 1$ or $N = 2$, and $\epsilon > 0$ is a parameter. Assumptions on $a : \mathbb{R}^N \to \mathbb{R}$ will be formulated later.

This kind of equation arises in many fields of physics. For the following nonlinear Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\hbar^2\triangle\psi + \widehat{V}\psi - g(x,|\psi|)\psi, \tag{2}$$

where i is the imaginary unit, Δ is the Laplacian operator, and $\hbar > 0$ is the Planck constant, a standing wave solution is a solution of the form

$$\psi(x,t) = e^{-iEt/\hbar}u(x), \quad u(x) \in \mathbb{R}.$$

Thus, looking for a standing wave ψ of (2) is equivalent to finding a solution u of the equation

$$-\hbar^2\triangle u + V(x)u = f(x,u), \tag{3}$$

where $V(x) = \widehat{V}(x) - E$ and $f(x,u) = g(x,|u|)u$. The function V is called the potential of (3). If $g(x,|u|) = |u|^{p-2}$ then (3) can be written as

$$-\hbar^2\triangle u + V(x)u = |u|^{p-2}u. \tag{4}$$

In the case in which $\hbar = 1$ and $V(x) = 1 + \epsilon a(x)$, (4) is reduced to (1). Since we are interested in bound states, we require that $u \in H^1(\mathbb{R}^N)$.

The nonlinear Schrödinger equation (2) models some phenomena in physics, for example, in nonlinear optics, in plasma physics, and in condensed matter physics, and the nonlinear term simulates the interaction effect, called Kerr effect in nonlinear optics, among a large number of particles. The case where $p = 4$ and $N = 3$ is of particularly physical interest, and in this case the equation is called the Gross-Pitaevskii equation.

The limiting equation of (1) as $\epsilon \to 0$ is

$$-\triangle u + u = |u|^{p-2}u, \quad u \in H^1(\mathbb{R}^N). \tag{5}$$

It is well known that (5) has a unique positive radial solution, denoted by ω, which decays exponentially at ∞. This ω will serve as a building block to construct multi-bump solutions of (1). Let $n \geq 1$ be an integer. For sufficiently separated $y_1, y_2, \cdots, y_n$ in $\mathbb{R}^N$, a solution of (1) which is close to $\sum_{i=1}^{n} \omega(\cdot - y_i)$ in a sense which will be made clear later is called an n-bump solution. We are interested here in constructing multi-bump solutions of (1).

To state the main result on (1), we need the following condition on the function a.

(A) $a \in C(\mathbb{R}^N)$, $a(x) > 0$ **in** $\mathbb{R}^N$, $\lim_{|x| \to \infty} a(x) = 0$ **and** $\lim_{|x| \to \infty} \ln(a(x))/|x| = 0$.

Our main result is the following theorem.

Theorem 1. *Let a satisfy* **(A)**. *Then for any positive integer n there exists $\epsilon(n) > 0$ such that for $0 < \epsilon < \epsilon(n)$,* (1) *has an n-bump positive solution. As a consequence, for any positive integer n, there exists $\epsilon_1(n) > 0$ such that for $0 < \epsilon < \epsilon_1(n)$,* (1) *has at least n positive solutions.*

There have been enormous studies on the solutions of (3) as $\hbar \to 0$, which exhibit a concentration phenomenon and are called semi-classical states. Most of the former researches were focused on the case $\inf_{x\in\mathbb{R}^N} V(x) > 0$. In this case and for $N = 1$ and $p = 4$, Floer and Weinstein in [JFA, 1986], using Lyapunov-Schmidt reduction argument, constructed for the first time semi-classical states which concentrate near a nondegenerate critical point of the potential V. Their result was extended to higher dimensions by Oh [CPDE, 1988; CMP, 1990], using also the Lyapunov-Schmidt reduction argument. For a potential V without any nondegenerate critical point, Rabinowitz [ZAMP, 1992] obtained existence result for (3) with $\hbar$ small, provided that $0 < \inf_{x\in\mathbb{R}^N} V(x) < \liminf_{|x|\to\infty} V(x)$, using a global variational argument. Del Pino and Felmer [CVPDE, 1996; JFA, 1997; AIHPAN, 1998] established existence of multi-peak solutions having exactly k maximum points provided that there are k disjoint open bounded sets Λ_i such that $\inf_{x\in\partial\Lambda_i} V(x) > \inf_{x\in\Lambda_i} V(x)$, each Λ_i having one peak concentrating at its bottom. There have been also recent results on existence of solutions concentrating on manifolds; see, Ambrosetti and Malchiodi [Progress in Mathematics, No. 240, Birkhäuser, 2005], Ambrosetti, Malchiodi and Ni [CMP, 2003], del Pino, Kowalczyk, and Wei [CPAM].

Byeon and Wang [ARMA, 2002; CVPDE, 2003] were the first to study semi-classical states of (3) with critical frequency, that is, $\inf_{x \in \mathbb{R}^N} V(x) = 0$. They exhibit new concentration phenomena for bound states and their results were extended and generalized by Byeon and Oshita [CPDE, 2004], Cao and Noussair [JDE, 2004], Cao and Peng [MathAnn 2006].

The solutions we obtain in Theorem 1 do not concentrate near any point in the space. Instead, the bumps of the solutions we obtain are separated far apart and the distance between any pair of bumps goes to infinity as $\epsilon \to 0$, and each bump has a fixed profile as $\epsilon \to 0$. This is in sharp contrast to the concentration phenomenon described above. It was shown by Kang and Wei [ADE, 2000] that, at a strict local maximum point x_0 of $V(x)$ and for any positive integer k, (4) has a positive solution with k interacting bumps concentrating near x_0, while at a nondegenerate local minimum point of $V(x)$ such solutions do not exist. In our case, $1 + \epsilon a(x)$ has a maximum point and we do not have solutions concentrating near this point, but we have solutions with arbitrary many bumps near ∞ which is a strict minimum point of the potential $1 + \epsilon a(x)$. Similar phenomenon has been observed for a Maxwell-Schrödinger system by D'Aprile and Wei in [CVPDE, 2005], where the optimal configuration of the bumps was described. Here we do not know whether the bumps obtained in this paper obey an optimal configuration as in [D'Aprile and Wei, CVPDE, 2005].

Existence of multi-bump solutions has been studied also for other class of equations. Coti Zelati and Rabinowitz in [CPAM, 1992] constructed multi-bump solutions for Schrödinger equations of the form

$$-\Delta u + V(x)u = f(x, u), \quad x \in \mathbb{R}^N, \tag{6}$$

where V and f are T_i periodic in x_i. The building blocks are one-bump solutions at the mountain pass level and the existence of such solutions as well as multi-bump solutions is guaranteed by a nondegenerate assumption of the solutions near the mountain pass level. Under the same nondegenerate assumption, Coti Zelati and Rabinowitz in [JAMS, 1991] constructed multi-bump solutions for periodic Hamiltonian systems. Multi-bump solutions have also been obtained for asymptotically periodic Schrödinger equations by Alama and Li [IUMJ 1992].

The conditions we impose on the potential $a(x)$ are generic conditions. We neither require $a(x)$ to be periodic nor require $a(x)$ to have non-degenerate critical points. In fact, from the condition **(A)**, we know that the potential $a(x)$ may have only one critical point and every critical point of $a(x)$ may be degenerate.

In a similar way, we obtain existence of multi-bump positive solutions of the equation

$$-\triangle u + u = (1 - \epsilon a(x))|u|^{p-2}u, \quad u \in H^1(\mathbb{R}^N), \tag{7}$$

where $N \geq 1$, $2 < p < 2^*$, $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent if $N \geq 3$ and $2^* = \infty$ if $N = 1$ or $N = 2$, and $\epsilon > 0$ is a parameter.
We now formulate the assumptions on a.

(A) $a \in C(\mathbb{R}^N)$, $a(x) > 0$ for $x \in \mathbb{R}^N$, $\lim_{|x|\to\infty} a(x) = 0$, and there exist $c > 0$ and $\sigma > 0$ such that $a(x) \geq ce^{-\sigma|x|}$.

(B) $a \in C(\mathbb{R}^N)$, $a(x) > 0$ for $x \in \mathbb{R}^N$, $\lim_{|x|\to\infty} a(x) = 0$, and for any $\sigma > 0$ there exists $c > 0$ such that $a(x) \geq ce^{-\sigma|x|}$.

Theorem 2. *Let* a *satisfy* **(A)**. *If* $n \in \mathbb{N}$ *satisfies*

$$n < 1 + \frac{p-2}{2\sigma(p-1)},$$

then there exists $\epsilon(n) > 0$ *such that for* $0 < \epsilon < \epsilon(n)$, (7) *has an* n*-bump positive solution.*

As a consequence of Theorem 2, we have the following result.

Corollary 3. *Let* a *satisfy* **(B)**. *Then for any* $n \in \mathbb{N}$, *there exists* $\epsilon(n) > 0$ *such that for* $0 < \epsilon < \epsilon(n)$, (7) *has an* n*-bump positive solution. Therefore, as* $\epsilon \to 0$, (7) *has more and more multi-bump positive solutions.*

As a problem closely related to (7), we also consider the prescribed scalar curvature equation

$$-\triangle u = (1 - \epsilon K(|x|))u^{\frac{N+2}{N-2}}, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \tag{8}$$

where $N \geq 3$, $\epsilon > 0$ is a parameter, and K satisfies the following assumptions.

(C) $K \in C([0,\infty))$, $K(r) > 0$ for $r > 0$, $\lim_{r\to 0} K(r) = 0$, $\lim_{r\to\infty} K(r) = 0$, and there exist $c > 0$ and $\mu > 0$ such that $K(r) \geq cr^{\mu}$ for $r > 0$ small and $K(r) \geq cr^{-\mu}$ for r large.

(D) $K \in C([0,\infty))$, $K(r) > 0$ for $r > 0$, $\lim_{r\to 0} K(r) = 0$, $\lim_{r\to\infty} K(r) = 0$, and for any $\mu > 0$ there exists $c > 0$ such that $K(r) \geq cr^{\mu}$ for $r > 0$ small and $K(r) \geq cr^{-\mu}$ for r large.

Theorem 4. *Let K satisfy* **(C)**. *If $n \in \mathbb{N}$ satisfies*

$$n < 1 + \frac{N-2}{\mu(N+2)},$$

then there exists $\epsilon(n) > 0$ such that for $0 < \epsilon < \epsilon(n)$, (8) *has an n-tower positive solution.*

Here, by an n-tower positive solution of (8) we mean a radial solution which is sufficiently close to $\sum_{i=1}^{n} U_{\lambda_i}$ in the $\mathcal{D}^{1,2}(\mathbb{R}^N)$ norm

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^{\frac{1}{2}},$$

where $\lambda_i > 0$ $(i = 1, 2, \cdots, n)$ are such that $\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i}$ are large enough for all $i \neq j$ and

$$U_\lambda(x) = \frac{\lambda^{\frac{N-2}{2}}[N(N-2)]^{\frac{N-2}{4}}}{(1 + \lambda^2|x|^2)^{\frac{N-2}{2}}}.$$

As a consequence of Theorem 4, we have the following result.

Corollary 5. *Let K satisfy* **(D)**. *Then for any $n \in \mathbb{N}$, there exists $\epsilon(n) > 0$ such that for $0 < \epsilon < \epsilon(n)$,* (8) *has an n-tower positive solution. Therefore, as $\epsilon \to 0$,* (4) *has more and more multi-tower positive solutions.*

Note that the assumptions **(B)** and **(D)** can be satisfied by quite general functions. For example, for any $\alpha > 0,\ c_1 > 0,\ c_2 > 0$, functions of the form

$$a(x) = \frac{c_1}{c_2 + |x|^\alpha}$$

satisfy the assumption **(B)**, and functions of the form

$$K(r) = \begin{cases} \dfrac{c_1}{c_2 + |\ln r|^\alpha}, & r > 0 \\ 0, & r = 0 \end{cases}$$

satisfy the assumption **(D)**.

Equations of the type of (8) arise in the scaler curvature problem in differential geometry. If (M, g_0) is a Riemannian manifold of dimension $N \geq 3$, with scaler curvature S_0, then to find a conformal metric $g_1 = u^{\frac{4}{N-2}} g_0$ having scaler curvature S_1 is equivalent to find a solution u to the equation

$$-4\frac{N-1}{N-2}\Delta_{g_0} u + S_0 u = S_1 u^{\frac{N+2}{N-2}}. \qquad (9)$$

Up to a positive constant, if (M, g_0) is the standard sphere then the stereographic projection $\pi : S^N \to \mathbb{R}^N$ converts (9) into

$$-\Delta u = S u^{\frac{N+2}{N-2}}, \quad x \in \mathbb{R}^N, \qquad (10)$$

where $S(x) = S_1(\pi^{-1}(x))$, and if (M, g_0) is the standard $\mathbb{R}^N$ then (9) is just (10). If $S(x)$ is a perturbation of 1 and has the form $S(x) = 1 - \epsilon K(|x|)$ and if we require u to be in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ then (10) becomes (8).

2 Proof of Theorem 2

We begin with introducing some notations. In the Hilbert space $H^1(\mathbb{R}^N)$, we shall use the usual inner product

$$\langle u, v\rangle = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v + uv$$

and the induced norm $\|\cdot\|$. Let $|\cdot|_p$ be the usual norm of $L^p(\mathbb{R}^N)$. We shall use C and C_i to represent positive constants which may be variant even in the same line. Let $n \in \mathbb{N}$. We shall use $\sum_{i<j}$ and $\sum_{i\neq j}$ to represent summation over all subscripts i and j satisfying $1 \leq i < j \leq n$ and $1 \leq i \neq j \leq n$, respectively.

Recall that, for $2 < p < 2^*$, the equation

$$-\triangle u + u = |u|^{p-2}u, \quad u \in H^1(\mathbb{R}^N) \tag{11}$$

has a unique positive radial solution $w \in C^\infty(\mathbb{R}^N)$ which satisfies, for some $c > 0$,

$$w(r)r^{\frac{N-1}{2}}e^r \to c > 0, \;\; w'(r)r^{\frac{N-1}{2}}e^r \to -c, \quad \text{as } r = |x| \to \infty,$$

and each positive solution of (11) has the form $w_y := w(\cdot - y)$ for some $y \in \mathbb{R}^N$. We shall use w_y as building blocks to construct multi-bump solutions of (1). For $y_i, y_j \in \mathbb{R}^N$, the identity

$$\int_{\mathbb{R}^N} w_{y_i}^{p-1} w_{y_j} = \langle w_{y_i}, w_{y_j} \rangle = \int_{\mathbb{R}^N} w_{y_i} w_{y_j}^{p-1}$$

will be frequently used in the sequel.

The following lemma is a consequence of Bahri and P.L. Lions.

Lemma 6. *There exists a positive constant* $c > 0$ *such that as* $|y_i - y_j| \to \infty$,

$$\int_{\mathbb{R}^N} w_{y_i}^{p-1} w_{y_j} \sim c|y_i - y_j|^{-\frac{(N-1)}{2}} e^{-|y_i - y_j|}.$$

For $\lambda > 0$, define

$$\Omega_\lambda = \{(y_1, \cdots, y_n) \in (\mathbb{R}^N)^n \mid |y_i - y_j| > \lambda \text{ for } i \neq j\}$$

if $n \geq 2$ and $\Omega_\lambda = \mathbb{R}^N$ if $n = 1$. For $y = (y_1, \cdots, y_n) \in \Omega_\lambda$, denote

$$u_y(x) = \sum_{i=1}^{n} w_{y_i}, \qquad M = \{u_y |\, y \in \Omega_\lambda\},$$

$$T_y = \text{span}\left\{ \frac{\partial w_{y_i}}{\partial x_\alpha} \mid \alpha = 1, 2, \cdots, N,\ i = 1, 2, \cdots, n \right\},$$

and

$$W_y = \{v \in H^1(\mathbb{R}^N) \mid \langle v, v_1 \rangle = 0,\ \forall v_1 \in T_y\}.$$

Two orthogonal projections:

$$P_y : H^1(\mathbb{R}^N) \to T_y, \qquad Q_y : H^1(\mathbb{R}^N) \to W_y.$$

Solutions of (7) correspond to critical points of the functional

$$J_\epsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) - \frac{1}{p}\int_{\mathbb{R}^N} P_\epsilon |u|^p, \quad u \in H^1(\mathbb{R}^N),$$

where $P_\epsilon(x) = 1 - \epsilon K(x)$. The equation $\nabla J_\epsilon(u) = 0$ is equivalent to

$$P_y \nabla J_\epsilon(u) = 0, \qquad Q_y \nabla J_\epsilon(u) = 0.$$

Set $K = (p-1)(-\triangle + 1)^{-1}$. For $y \in \Omega_\lambda$ and $\varphi \in H^1(\mathbb{R}^N)$, define

$$A_y\varphi = \varphi - \sum_{j=1}^{n} K(w_{y_j}^{p-2}\varphi) + L_y\varphi,$$

where

$$L_y\varphi = \sum_{i\neq j}\sum_{\alpha=1}^{N} \left\langle K(w_{y_j}^{p-2}\varphi), \frac{\partial w_{y_i}}{\partial x_\alpha}\right\rangle \left\|\frac{\partial w_{y_i}}{\partial x_\alpha}\right\|^{-2} \frac{\partial w_{y_i}}{\partial x_\alpha}.$$

Note that $A_y(W_y) \subset W_y$ for any $y \in \Omega_\lambda$.

Lemma 7. *There exist $\lambda_0 > 0$ and $\eta_0 > 0$ such that for $\lambda > \lambda_0$ and $y \in \Omega_\lambda$, $A_y|_{W_y} : W_y \to W_y$ is invertible and*

$$\|(A_y|_{W_y})^{-1}\| \le \eta_0.$$

Lemma 8. *Let $v \in H^1(\mathbb{R}^N)$. If $\epsilon \to 0$, $v \to 0$, and $\lambda \to \infty$, then*

$$\sup_{y\in\Omega_\lambda,\ \varphi\in H^1(\mathbb{R}^N),\ \|\varphi\|=1} \|A_y\varphi - \big(\varphi - K\big(P_\epsilon|u_y + v|^{p-2}\varphi\big)\big)\| \to 0.$$

Lemma 9. *There exist* $\epsilon_0 > 0$ *and* $\Lambda_0 > 0$ *such that for* $0 < \epsilon < \epsilon_0$ *and* $\lambda > \Lambda_0$, *there exists a* C^1 *map*

$$v_{\lambda,\epsilon} : \Omega_\lambda \longrightarrow H^1(\mathbb{R}^N),$$

depending on λ *and* ϵ, *such that*

(a) *for any* $y \in \Omega_\lambda$, $v_{\lambda,\epsilon,y} \in W_y$;

(b) *for any* $y \in \Omega_\lambda$, $Q_y \nabla J_\epsilon(u_y + v_{\lambda,\epsilon,y}) = 0$, *where* $Q_y : H^1(\mathbb{R}^N) \longrightarrow W_y$ *is the orthogonal projection onto* W_y;

(c) $\lim_{\lambda\to\infty,\ \epsilon\to 0} \|v_{\lambda,\epsilon,y}\| = 0$ *uniformly in* $y \in \Omega_\lambda$; $\lim_{|y|\to\infty} \|v_{\lambda,\epsilon,y}\| = 0$ *uniformly in* $\epsilon \in (0, \epsilon_0)$ *if* $n = 1$.

Lemma 10. *For* $0 < \epsilon < \epsilon_0$ *and* $\lambda > \Lambda_0$, *if* $y^0 = (y_1^0, \cdots, y_n^0) \in \Omega_\lambda$ *is a critical point of* $J_\epsilon(u_y + v_{\lambda,\epsilon,y})$, *then* $u_{y^0} + v_{\lambda,\epsilon,y^0}$ *is a critical point of* J_ϵ.

To prove Theorem 2, we need first to estimate $J_\epsilon(u_y + v_{\lambda,\epsilon,y})$. Denote

$$c_0 := \frac{1}{2}\|w\|^2 - \frac{1}{p}|w|_p^p.$$

Then

$$\begin{aligned} J_\epsilon(u_y + v_{\lambda,\epsilon,y}) =& \frac{1}{2}\|u_y + v_{\lambda,\epsilon,y}\|^2 - \frac{1}{p}\int_{\mathbb{R}^N}(1-\epsilon a)|u_y + v_{\lambda,\epsilon,y}|^p \\ =& nc_0 - \frac{1}{p}|u_y|_p^p + \frac{n}{p}|w|_p^p + \sum_{i<j}\int_{\mathbb{R}^N} w_{y_i}^{p-1} w_{y_j} + \frac{\epsilon}{p}\int_{\mathbb{R}^N} a u_y^p \\ & + O\left(\left(\sum_{i<j}\int_{\mathbb{R}^N} w_{y_i}^{p-1} w_{y_j}\right)^{\frac{2(p-1)}{p}}\right) + O\left(\epsilon^2 \int_{\mathbb{R}^N} a u_y^p\right) \\ & + O(\|v_{\lambda,\epsilon,y}\|^2). \end{aligned}$$

Lemma 11.

$$\|v_{\lambda,\epsilon,y}\| = O\left(\left(\sum_{i<j}\int_{\mathbb{R}^N} w_{y_i}^{p-1} w_{y_j}\right)^{\frac{p-1}{p}}\right) + O\left(\epsilon\left(\int_{\mathbb{R}^N} a u_y^p\right)^{\frac{p-1}{p}}\right).$$

Lemma 12.

$$\begin{aligned} J_\epsilon(u_y + v_{\lambda,\epsilon,y}) =& n c_0 - \frac{1}{p}|u_y|_p^p + \frac{n}{p}|w|_p^p + \sum_{i<j}\int_{\mathbb{R}^N} w_{y_i}^{p-1} w_{y_j} + \frac{\epsilon}{p}\int_{\mathbb{R}^N} a u_y^p \\ &+ O\left(\left(\sum_{i<j}\int_{\mathbb{R}^N} w_{y_i}^{p-1} w_{y_j}\right)^{\frac{2(p-1)}{p}}\right) + O\left(\epsilon^2\int_{\mathbb{R}^N} a u_y^p\right). \end{aligned}$$

We are now ready to prove Theorem 2. Let $n \in \mathbb{N}$ and we first consider the case $n \geq 2$. Define

$$d = \sup_{y \in (\mathbb{R}^N)^n} \frac{1}{p} \int_{\mathbb{R}^N} a u_y^p. \tag{12}$$

Then for any ϵ satisfying

$$0 < \epsilon < \epsilon_1 := \min \left\{ \epsilon_0, \ \frac{p-2}{3pd} |w|_p^p \right\}$$

there exist $\mu^* = \mu^*(\epsilon) > \mu = \mu(\epsilon) > \Lambda_0$ such that, for $z \in \mathbb{R}^N$ with $|z| \in [\mu(\epsilon), \mu^*(\epsilon)]$,

$$\frac{3pd\epsilon}{p-2} \leq \int_{\mathbb{R}^N} w^{p-1} w_z \leq \frac{4pd\epsilon}{p-2}. \tag{13}$$

We shall prove that, for $\epsilon > 0$ sufficiently small, $J_\epsilon(u_y + v_{\mu,\epsilon,y})$ achieves its maximum at some point in $\Omega_{\mu(\epsilon)}$, which produces an n-bump positive solution of (7). Define

$$M_\epsilon := \sup\{ J_\epsilon(u_y + v_{\mu,\epsilon,y}) \mid y \in \Omega_{\mu(\epsilon)} \}.$$

Lemma 13. *Assume $n \geq 2$. Then there exists $\epsilon_2 \in (0, \epsilon_1)$ such that for $0 < \epsilon < \epsilon_2$,*

$$M_\epsilon > \sup\{ J_\epsilon(u_y + v_{\mu,\epsilon,y}) \mid y \in \Omega_{\mu(\epsilon)} \text{ and } |y_i - y_j| \in [\mu(\epsilon), \mu^*(\epsilon)] \text{ for some } i \neq j \}.$$

Proof. For $\epsilon > 0$ small enough, if $y = (y_1, \cdots, y_n) \in \Omega_{\mu(\epsilon)}$ and $|y_i - y_j| \in [\mu(\epsilon), \mu^*(\epsilon)]$ for some $i \neq j$, then by Lemma 12, (12), and (13), we obtain

$$\begin{aligned} J_\epsilon(u_y + v_{\mu,\epsilon,y}) \leq & nc_0 - \frac{p-2}{p} \sum_{i<j} \int_{\mathbb{R}^N} w_{y_i}^{p-1} w_{y_j} + \frac{\epsilon}{p} \int_{\mathbb{R}^N} a u_y^p \\ & + C \left(\sum_{i<j} \int_{\mathbb{R}^N} w_{y_i}^{p-1} w_{y_j} \right)^{\frac{2(p-1)}{p}} + C\epsilon^2 \\ \leq & nc_0 - 3d\epsilon + d\epsilon + C\epsilon^{\frac{2(p-1)}{p}} \leq nc_0 - d\epsilon. \end{aligned} \tag{14}$$

On the other hand, for ϵ small enough such that the fifth and the seventh terms on the right side of the equality from Lemma 12 satisfy

$$\frac{\epsilon}{p}\int_{\mathbb{R}^N} a u_y^p + O\left(\epsilon^2 \int_{\mathbb{R}^N} a u_y^p\right) > 0,$$

we have

$$\liminf_{y\in\Omega_{\mu(\epsilon)},\ |y_i-y_j|\to+\infty \text{ for all } i\neq j} J_\epsilon(u_y + v_{\mu,\epsilon,y}) \geq n c_0. \tag{15}$$

From (14) and (15), we obtain the result.

For any $0 < \epsilon < \epsilon_2$, let $y^k(\epsilon) = (y_1^k(\epsilon), \cdots, y_n^k(\epsilon)) \in \Omega_{\mu(\epsilon)}$, $k = 1, 2, \cdots$, be a maximizing sequence for $J_\epsilon(u_y + v_{\mu,\epsilon,y})$. Then Lemma 13 implies that

$$\inf_k \min_{i \neq j} |y_i^k(\epsilon) - y_j^k(\epsilon)| \geq \mu^*.$$

Therefore, for any $0 < \epsilon < \epsilon_2$ and $1 \leq i \leq n$, passing to a subsequence if necessary, we may assume either $\lim_{k\to\infty} y_i^k(\epsilon) = y_i^0(\epsilon) \in \mathbb{R}^N$ with $|y_i^0(\epsilon) - y_j^0(\epsilon)| \geq \mu^*$ for $i \neq j$ or $\lim_{k\to\infty} |y_i^k(\epsilon)| = \infty$. Define, for $0 < \epsilon < \epsilon_2$,

$$\Pi(\epsilon) = \{1 \leq i \leq n \mid |y_i^k(\epsilon)| \to \infty, \text{ as } k \to \infty\}.$$

We shall prove that $\Pi(\epsilon) = \emptyset$ for $\epsilon > 0$ sufficiently small and thus $J_\epsilon(u_y + v_{\mu,\epsilon,y})$ achieves its maximum at $(y_1^0(\epsilon), \cdots, y_n^0(\epsilon))$ in $\Omega_{\mu(\epsilon)}$.

Lemma 14. *Assume $n \geq 2$. Then there exists $\epsilon(n) \in (0, \epsilon_2)$ such that for $\epsilon \in (0, \epsilon(n))$, $\Pi(\epsilon) = \emptyset$.*

Proof. We argue by contradiction and assume that $\Pi(\epsilon) \neq \emptyset$ along a sequence $\epsilon_m \to 0$. Without loss of generality, we may assume $\Pi(\epsilon_m) = \{1, \cdots, j_n\}$ for all $m \in \mathbb{N}$ and for some $1 \leq j_n < n$. The case in which $j_n = n$ can be handled similarly. For convenience of notations, we shall denote $\epsilon = \epsilon_m$, $y_i^k = y_i^k(\epsilon_m)$, $y^k = (y_1^k, \cdots, y_n^k)$ and $y_*^k = (y_{j_n+1}^k, \cdots, y_n^k)$ for $k = 1, 2, \cdots$. Then, as $k \to \infty$,

$$|y_1^k| \to \infty, \cdots, |y_{j_n}^k| \to \infty,$$

and

$$y_*^k \to y_*^0 := (y_{j_n+1}^0, \cdots, y_n^0).$$

In view of Lemma 12 and (13), we see that

$$J_\epsilon(u_{y^k} + v_{\mu,\epsilon,y^k}) = nc_0 - \frac{1}{p}|u_{y^k}|_p^p + \frac{n}{p}|w|_p^p + \sum_{i<j} \int_{\mathbb{R}^N} w_{y_i^k}^{p-1} w_{y_j^k} + \frac{\epsilon}{p} \int_{\mathbb{R}^N} a u_{y^k}^p + O\left(\epsilon^{\frac{2(p-1)}{p}}\right),$$

and

$$J_\epsilon(u_{y_*^k} + v_{\mu,\epsilon,y_*^k}) = (n - j_n)c_0 - \frac{1}{p}|u_{y_*^k}|_p^p + \frac{n - j_n}{p}|w|_p^p + \sum_{j_n+1 \le i < j \le n} \int_{\mathbb{R}^N} w_{y_i^k}^{p-1} w_{y_j^k} + \frac{\epsilon}{p} \int_{\mathbb{R}^N} a u_{y_*^k}^p + O\left(\epsilon^{\frac{2(p-1)}{p}}\right).$$

Therefore,

$$\begin{aligned}
&J_\epsilon(u_{y^k}+v_{\mu,\epsilon,y^k}) - J_\epsilon(u_{y_*^k}+v_{\mu,\epsilon,y_*^k}) \\
&\quad = j_n c_0 - \frac{1}{p}|u_{y^k}|_p^p + \frac{1}{p}|u_{y_*^k}|_p^p + \frac{j_n}{p}|w|_p^p + \sum_{i<j}\int_{\mathbb{R}^N} w_{y_i^k}^{p-1} w_{y_j^k} \\
&\qquad - \sum_{j_n+1\le i<j\le n}\int_{\mathbb{R}^N} w_{y_i^k}^{p-1} w_{y_j^k} + \frac{\epsilon}{p}\int_{\mathbb{R}^N} a(u_{y^k}^p - u_{y_*^k}^p) + O\left(\epsilon^{\frac{2(p-1)}{p}}\right),
\end{aligned} \tag{16}$$

which implies

$$J_\epsilon(u_{y^k}+v_{\mu,\epsilon,y^k}) - J_\epsilon(u_{y_*^k}+v_{\mu,\epsilon,y_*^k}) \le j_n c_0 + \frac{\epsilon}{p}\int_{\mathbb{R}^N} a(u_{y^k}^p - u_{y_*^k}^p) + C\epsilon^{\frac{2(p-1)}{p}}.$$

Letting $k\to\infty$, in view of $|y_i^k|\to\infty$ for $i=1,\cdots,j_n$, we see that

$$M_\epsilon \le J_\epsilon(u_{y_*^0}+v_{\mu,\epsilon,y_*^0}) + j_n c_0 + C\epsilon^{\frac{2(p-1)}{p}}. \tag{17}$$

On the other hand, since, according to the assumption,

$$n < 1 + \frac{p-2}{2\sigma(p-1)},$$

we can choose δ such that

$$0 < \delta < \frac{p-2-2\sigma(n-1)(p-1)}{2(1+\sigma(n-1))(p-1)}. \tag{18}$$

By Lemma 6 and (13), there exist $C_i > 0$, $i = 1, 2$, such that $\mu = \mu(\epsilon)$ satisfies

$$C_1\epsilon \le \mu^{-\frac{N-1}{2}} e^{-\mu} \le C_2\epsilon,$$

which implies for ϵ small enough

$$(1-\delta)\ln\frac{1}{\epsilon} < \mu < (1+\delta)\ln\frac{1}{\epsilon}. \tag{19}$$

Define

$$\overline{y}_s^\epsilon = ((4s-2n-2)(1-p^{-1})\mu, 0, \cdots, 0) \in \mathbb{R}^N, \quad s = 1, 2, \cdots, n.$$

The open balls $B(\overline{y}_s^\epsilon, 2(1-p^{-1})\mu)$ $(s = 1, 2, \cdots, n)$ are mutually disjoint. Thus there are j_n integers from $\{1, 2, \cdots, n\}$, denoted by $s_1 < s_2 < \cdots < s_{j_n}$, such that

$$|\overline{y}_{s_i}^\epsilon - y_j^0| \geq 2(1-p^{-1})\mu, \quad i = 1, \cdots, j_n, \quad j = j_n + 1, \cdots, n.$$

Denote $\overline{y}_{s_i}^\epsilon$ by y_i^ϵ, $i = 1, 2, \cdots, j_n$. Then, clearly,

$$|y_i^\epsilon| \leq 2(n-1)(1-p^{-1})\mu, \quad i = 1, \cdots, j_n, \tag{20}$$

$$|y_i^\epsilon - y_j^\epsilon| \geq 2(1-p^{-1})\mu, \quad 1 \leq i < j \leq j_n, \tag{21}$$

and

$$|y_i^\epsilon - y_j^0| \geq 2(1-p^{-1})\mu, \quad i = 1, \cdots, j_n, \quad j = j_n + 1, \cdots, n. \tag{22}$$

Therefore,

$$(y_1^\epsilon, \cdots, y_{j_n}^\epsilon, y_{j_n+1}^0, \cdots, y_n^0) \in \Omega_\mu.$$

Denote $y^\epsilon = (y_1^\epsilon, \cdots, y_{j_n}^\epsilon, y_{j_n+1}^0, \cdots, y_n^0)$. We see that

$$J_\epsilon(u_{y^\epsilon} + v_{\mu,\epsilon,y^\epsilon}) - J_\epsilon(u_{y_*^0} + v_{\mu,\epsilon,y_*^0}) \geq j_n c_0 + \frac{\epsilon}{p}\int_{\mathbb{R}^N} a(u_{y^\epsilon}^p - u_{y_*^0}^p) - C\epsilon^{\frac{2(p-1)}{p}(1-\delta)}.$$

Now, the assumption **(A)** together with (19) and (20) yields

$$\frac{\epsilon}{p}\int_{\mathbb{R}^N} a(u_{y^\epsilon}^p - u_{y_*^0}^p) \geq \frac{\epsilon}{p}\int_{\mathbb{R}^N} a w_{y_1^\epsilon}^p \geq \frac{\epsilon}{p}\int_{|x-y_1^\epsilon|\leq 1} a w_{y_1^\epsilon}^p$$
$$\geq C\epsilon e^{-\sigma(|y_1^\epsilon|+1)} \geq C\epsilon e^{-2\sigma(n-1)(1-p^{-1})(1+\delta)\ln\frac{1}{\epsilon}} = C\epsilon^{1+2\sigma(n-1)(1-p^{-1})(1+\delta)}.$$

Since (18) implies

$$1 + 2\sigma(n-1)(1-p^{-1})(1+\delta) < \frac{2(p-1)}{p}(1-\delta),$$

we then arrive at, for ϵ small enough,

$$M_\epsilon \geq J_\epsilon(u_{y_*^0} + v_{\mu,\epsilon,y_*^0}) + j_n c_0 + C\epsilon^{1+2\sigma(n-1)(1-p^{-1})(1+\delta)}. \tag{23}$$

But (23) contradicts (17). Thus there exists $\epsilon(n) > 0$ such that if $0 < \epsilon < \epsilon(n)$ then $\Pi(\epsilon) = \emptyset$ and $J_\epsilon(u_y + v_{\mu,\epsilon,y})$ achieves its maximum at some point $(y_1^0, \cdots, y_n^0) \in \Omega_{\mu(\epsilon)}$.

Proof of Theorem 2. Combining the last two lemmas above gives the result. □

3 Proof of Theorem 4

Now, we turn to consider equation (8)

$$-\triangle u = (1 - \epsilon K(|x|))u^{\frac{N+2}{N-2}}, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^N).$$

In order to obtain results on radial solutions for equation (8), we make the following transformation (see, [Catrina-Wang, CPAM, 2001] and, [Korevaar-Mazzeo-Pacard-Schoen, InventMath, 1999])

$$u(x) = |x|^{-\frac{N-2}{2}} w(-\ln|x|), \quad x \in \mathbb{R}^N. \tag{24}$$

Let $y = -\ln|x|$. Then u is a radial solution of (8) if and only if w is a solution of the equation

$$-w''(y) + \frac{(N-2)^2}{4} w(y) = (1 - \epsilon K(e^{-y}))(w(y))^{\frac{N+2}{N-2}}, \quad y \in \mathbb{R}.$$

Through the dilations

$$v(y) = \left(\frac{2}{N-2}\right)^{\frac{N-2}{2}} w\left(\frac{2}{N-2}y\right), \quad a(y) = K(e^{-\frac{2}{N-2}y}), \tag{25}$$

the last equation becomes

$$-v''(y) + v(y) = (1 - \epsilon a(y))(v(y))^{\frac{N+2}{N-2}}, \quad y \in \mathbb{R}. \tag{26}$$

Proof of Theorem 4. It follows from Theorem 2. □

Our approach also applies to the equations

$$-\triangle u = (1 - \epsilon K(|x|))|x|^{\frac{(N-2)q-(N+2)}{2}} u^q, \quad x \in \mathbb{R}^N \setminus \{0\}, \tag{27}$$

and

$$-\triangle u + \epsilon V(|x|)u = |x|^{\frac{(N-2)q-(N+2)}{2}} u^q, \quad x \in \mathbb{R}^N \setminus \{0\}, \tag{28}$$

where $N \geq 3$ and $q > 1$.

Thank you !