Free boundary problems of the Euler equation: local well-posedness and hydrodynamical instabilities

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Droplet in Vacuum (1F)

- Euler's equation (E): $v_t + v \cdot \nabla v = -\nabla p, \quad x \in \Omega_t$

— $\Omega_t \subset\subset \mathbf{R}^n$: moving fluid domain

— p: pressure

— Incompressibility: $\nabla \cdot v = 0$

- Free boundary problem in vacuum:

1. $v|_{\partial\Omega_t}$: velocity of the moving boundary

2. $p|_{\partial\Omega_t} = \alpha\kappa$: surface tension, $\alpha \in [0, 1]$

κ : mean curvature of $\partial\Omega_t \subset \mathbf{R}^n$

Interface Problem (2F)

- $\Omega_t^{\pm}$: moving domains of two fluids

—$S_t = \partial\Omega_t^{+} = \partial\Omega_t^{-}$: moving interface

— Euler's equation (E): $\rho_{\pm}(\partial_t v_{\pm} + v_{\pm} \cdot \nabla v_{\pm}) = -\nabla p_{\pm}, \quad x \in \Omega_t^{\pm}$

— $\rho_{\pm}$: constant densities of two fluids

—Incompressibility $\nabla \cdot v = 0$

- Free boundary problem:

1. $v_{+}^{\perp}|_{S_t} = v_{-}^{\perp}|_{S_t}$: interface velocity

2. $(p_{+} - p_{-})|_{S_t} = \kappa_{+}$: surface tension

- **Instabilities** in the absence of surface tension: ill-posedness

  — **(1F)**: *possible* Rayleigh-Taylor instability. **[RT]**

  — **(2F)**: Kelvin-Helmholtz instability. **[KH]**

  * **(1F)**: if $\nabla \times v = 0$, then no **[RT]**

  * Surface tension: regularizing force, suppress **[RT]** or **[KH]**

- **Difficulties**:

  * Fluid domain unknown

  * Choice of coordinates: Lagrangian coordinates?

Reference I. Without surface tension

- Nalimov 74, Yoshihara 82: **(1F)**, irrotation, 2-d, small data $\rightarrow$ local well posedness

- Ebin 87: **(1F)**, ill-posedness with Rayleigh-Taylor instability

- Wu 97, 99 **(1F)** irrotational $\rightarrow$ local well-posedness; 06 **(2F)** irrotational

- Beyer & Gunther 00: **(1F)**, irrotational $\rightarrow$ linearization

- Christodoulou & Lindblad 00, Lindblad 03, ... : **(1F)**, energy estimates, local well-posedness

- Coutand & Shkoller 05, Zhang & Zhang 06: **(1F)**, local well-posedness

Reference II. With surface tension

- Yoshihara 83, Iguchi 01 **(1F)** 2d irrotational well-posedness
- Ambrose 03: **(2F)** 2-d irrotational $\rightarrow$ well-posedness
- Beyer & Gunther 98, 05: **(1F)** irrotational $\rightarrow$ well-posedness;
- Masmoudi & Ambrose: 05: **(1F)** 2-d irrotational, 06: **(2F)** 3-d irrotational, well-posedness
- Coutand & Shkoller & Cheng 05, 08: **(1F), (2F)** well-posedness

••• Shinbrot 76, Nishida 79, Sulem & *et. al.* 81, Kano & Nishida 79, 86, Craig 85, Ogawa & Tani: 02, Schneider & Wayne 02, Lannes 04: 2-d, Schweizer: 05, Beale, Hou, & Lowengrub 93, Hou, Lowengrub, & Shelley 97, Christianson, Hur, & Staffinani ....

Irrotational and **simply connected** case: $\nabla \times v = 0$

(a) $\exists \phi(t, x)$, $x \in \Omega_t$, s.t. $\nabla \phi = v$

(b) $\nabla \cdot v = 0 \Rightarrow \Delta \phi = 0$

(c) $\frac{\partial \phi}{\partial N} = v \cdot N = v^{\perp}$ on $\partial \Omega_t$

- The problem is reduced to $\partial \Omega_t$

  * Explicit pressure is avoided

  $*$ Nonlocal operators are involved

A first look at the pressure

$$-\Delta p = \partial_i v^j \partial_j v^i \qquad \text{in} \quad \Omega_t$$

- Fixed boundary + non-penetrating B.C. on $\partial\Omega$: $v \cdot N = 0$

  (E)$\cdot N|_{\partial\Omega} \Rightarrow \nabla_N p|_{\partial\Omega} = -\nabla_v v \cdot N = v \cdot \nabla_v N \Rightarrow$ regularity: $\nabla p \sim v$

- **(1F)** w\o surface tension: $p = 0$ on $\partial\Omega_t \Rightarrow$

  reg. of $\nabla p \le$ (reg. of $\partial\Omega_t - \frac{1}{2}$) and (reg. of v)

- **(1F)** + surface tension: $p = \kappa$ on $\partial\Omega_t \Rightarrow$

  reg. of $\nabla p \le$ (reg. of $\partial\Omega_t - \frac{5}{2}$) and (reg. of v)

- Surface tension vs. No surface tension

  (a) Physically, a force regularizing $\partial\Omega_t$

  (b) Lower regularity of ∇p

- Regularity of $\partial\Omega_t$: best descried by Lagrangian coordinates?

  * Lagrangian coordinates $\rightarrow$ reg. $\partial\Omega_t =$ reg. $v - \frac{1}{2}$

  * How to close the regularity count?

conservation of energy

$$v_t + v \cdot \nabla v = -\nabla p, \quad x \in \Omega_t \tag{E}$$

* $v|_{\partial\Omega_t}$: boundary velocity, $p|_{\partial\Omega_t} = \kappa$: mean curvature of $\partial\Omega_t$

- $\int_{\Omega_t} (E) \cdot v dx \implies$

$$E_0(t) = \int_{\Omega_t} \frac{1}{2}|v|^2 dx + \alpha S(\partial\Omega_t) = E_0(0)$$

- $S(\partial\Omega_t)$: surface area $\Rightarrow$ regularity control on $\partial\Omega_t$?

- Lagrangian formulation?

Lagrangian formulation

- Action:

$$\mathcal{A}(u) = \int \int_{\Omega_0} |u_t|^2 dy dt - \int \alpha S(\partial\Omega_t) dt$$

$u(t,\cdot) : \Omega_0 \to \Omega_t$: Lagrangian coordinate map, $u_t = v(t,u)$.

$u(t,\cdot) \in$ configuration space $\mathcal{M}$:

$$\mathcal{M} = \{u : \Omega_0 \to \mathbf{R}^n : det \nabla u = 1\} \subset L^2(\Omega_0, \mathbf{R}^n)$$

$$T_u\mathcal{M} = \{W : \Omega_0 \to \mathbf{R}^n : \nabla \cdot (W \circ u^{-1}) = 0\}$$

$$(T_u\mathcal{M})^{\perp} = \{(\nabla q) \circ u : \Omega_0 \to \mathbf{R}^n : q|_{u(\partial\Omega)} = 0\}$$

- $S(\partial\Omega_t)$ depends only on $\partial\Omega_t = u(\partial\Omega_0)$

• Euler-Lagrangian equation:

$$u_{tt} + \alpha S'(u) = \text{(LM)} \triangleq \text{Lagrangian multiplier} \qquad \text{(E)}$$

* $S'(u) \in T_u\mathcal{M}$: div free in Eulerian coordinates

* (LM) $\in (T_u\mathcal{M})^{\perp} \Rightarrow$ (LM) $= (\nabla p_0) \circ u$, $p_0|_{\partial\Omega} = 0$

$$\mathcal{D}_t u_t + \alpha S'(u) = 0, \qquad \text{(E)}$$

$\mathcal{D}$: covariant derivative on $\mathcal{M}$

Splitting of pressure

- (LM) $= -\nabla p_0$:

$$-\Delta p_0 = \partial_i v^j \partial_j v^i \qquad p_0|_{\partial\Omega} = 0$$

* p_0: volume preserving pressure

- $S' = -\nabla \tilde{p}$:

$$\Delta \tilde{p} = 0 \qquad \tilde{p}|_{\partial\Omega_t} = \kappa : \text{mean curvature}$$

* $\tilde{p}$: surface tension

* $\forall\, w$, $\nabla \cdot w = 0$: $\qquad < S', w >= \int_{\partial\Omega_t} \kappa w \cdot N dS = \int_{\Omega_t} \nabla \tilde{p} \cdot w dx$

- $p = p_0 + \alpha \tilde{p}$: $\qquad$ reg. of $\tilde{p}$ < reg. of p_0

Linear analysis I.

- Linearized Euler-Lagrangian equation:

$$W_{tt} + \alpha S''(u) W = \cdots$$

$W(t, \cdot)$: div free in Eulerian coordinates

- $S'' = A + l.o.t.$

* A self-adjoint on $L^2(\Omega_t)$ and $\forall\, w$, $\nabla \cdot w = 0$

$$< w, Aw >_{L^2(\Omega_t)} = \int_{\Omega_t} (w, Aw) dx \triangleq \int_{\partial\Omega_t} |\nabla^{tan} w^{\perp}|^2 dS$$

* $w^{\perp} = w \cdot N$: normal velocity of $\partial\Omega$

Splitting of velocity

- For vector field w on Ω_t, $\nabla \cdot w = 0$,

$$w = w^{irrot} + w^{rot}$$

$$\nabla \cdot w^{rot} = 0 = \nabla \cdot w^{irrot}$$

- $w^{rot}|_{\partial\Omega_t} \in T\,\partial\Omega_t$: internal rotations

- $(w^{irrot})^{\perp}|_{\partial\Omega_t} = w^{\perp}|_{\partial\Omega_t}$: motion of $\partial\Omega_t$, $w^{irrot} = \nabla\phi$

- $< w^{rot}, w^{irrot} >_{L^2(\Omega_t)} = 0$

Linear analysis II.

$$W_{tt} + \alpha S''(u) W = \cdots$$

$W(t, \cdot)$: div free in Eulerian coordinates

- $S'' = A + l.o.t.$

* A self-adjoint on $L^2(\Omega_t)$ and $\forall\, w$, $\nabla \cdot w = 0$

$$< w, Aw >_{L^2(\Omega_t)} = \int_{\partial \Omega_t} |\nabla^{tan} w^{\perp}|^2 dS \sim |w^{irrot}|^2_{H^{\frac{3}{2}}(\Omega_t)}$$

- $\partial_t \sim (\partial_x)^{\frac{3}{2}}$: only geometrically, may not be reflected in Lagrangian coordinates!

Energy estimates

$$u_{tt} + S'(u) = (\mathrm{LM}) \qquad \text{(E)}$$

$A^k u_t \cdot$ (E) & vorticity transport $\Rightarrow$

Theorem. *(**Shatah-Z**) Estimates (local in time) on*

$$v = u_t \circ u^{-1} \in H^{\frac{3}{2}k}(\Omega_t), \qquad \partial\Omega_t \in H^{\frac{3}{2}k+1}, \qquad p \in H^{\frac{3}{2}k-\frac{1}{2}}(\Omega_t)$$

- Estimate on $\partial\Omega_t$ in terms of κ, coordinate independent

- Interface problem **(2F)**.

(1F): Without surface tension

$$v_t + v \cdot \nabla v = -\nabla p, \quad x \in \Omega_t \qquad \text{(E)}$$

$v|_{\partial\Omega_t}$: boundary velocity, $\qquad p|_{\partial\Omega_t} = 0$

- No $A = S'' + l.o.t.$ or $\tilde{p}$
- $p = p_0$: ∇p_0 becomes a leading order term

Geodesic Flow

Action: $\mathcal{A} = \int \int_{\Omega_0} |u_t|^2 dydt$, $u(t, \cdot) \in \mathcal{M}$

$$\mathcal{M} = \{u : \Omega_0 \to \mathbf{R}^n : det\nabla u = 1\} \subset L^2(\Omega_0, \mathbf{R}^n)$$

$$T_u\mathcal{M} = \{W : \Omega_0 \to \mathbf{R}^n : \nabla \cdot (W \circ u^{-1}) = 0\}$$

- $\frac{\delta \mathcal{A}}{\delta u} = 0$: geodesic flow on $\mathcal{M}$

$$\mathcal{D}_t u_t = 0,$$

$\mathcal{D}$: covariant derivative on $\mathcal{M}$

Linear Analysis: Jacobi Fields

• Linearized equation:

$$\mathcal{D}_t^2 W + \mathcal{R}(u_t, W)u_t = 0$$

$\mathcal{R}$: curvature of $\mathcal{M}$, for $W \in T_u\mathcal{M}$,

$$< \mathcal{R}(u_t, W)u_t, W >= \int_{\partial\Omega_t} -\frac{\partial p}{\partial N}|W\cdot N|^2 dS + ... \triangleq < \mathcal{R}_0(W), W > + \dots$$

• $\mathcal{R}(u_t, \cdot)u_t$: unbounded $\sim$ 1st order differential operator for W no smoother than $\nabla_N p$

• **[RT]** occurs unless $-\frac{\partial p}{\partial N} > 0$ on $\partial\Omega_t$: **(RT)** sign condition

Energy estimates

- $J = \nabla\tilde{p}$ is almost a Jacobi field (linearized solution):

$$\mathcal{D}_t^2 J + \mathcal{R}(u_t, J)u_t = l.o.t.$$

- $\mathcal{D}_t\kappa = \Delta_{\partial\Omega_t} v^\perp + l.o.t.$

(RT) Assume $-\frac{\partial p}{\partial N} > a > 0$ on $\partial\Omega_t$ for $t \in [0, T_0]$

Theorem. *(**Shatah-Z**) Assume **(RT)**. Estimates on*

$$v = u_t \circ u^{-1} \in H^k(\Omega_t), \quad \partial\Omega_t \in H^k, \quad p \in H^{k+\frac{1}{2}}(\Omega_t).$$

Uniform estimates hold as surface tension $\alpha \to 0$.

- **(2F)** w/o surface tension $\Rightarrow R(u_t, \cdot)u_t < 0$ 2nd order $\Rightarrow$ **[KH]**.

• Surface tension vs. No surface tension **(1F)**:

* No surface tension:

reg. of $\partial\Omega_t =$ reg. of $v|_{\partial\Omega_t} + \frac{1}{2}$ reg. of $p =$ reg. of $v + \frac{1}{2}$

* With surface tension:

reg. of $\partial\Omega_t =$ reg. of $v|_{\partial\Omega_t} + \frac{3}{2}$ reg. of $p =$ reg. of $v - \frac{1}{2}$

• Lagrangian coordinates optimal?

* Example:

$$\Omega_t = B(1) \subset \mathbf{R}^2, \quad v(t,r,\theta) = \Theta(r)\frac{\partial}{\partial\theta}, \quad p(t,r,\theta) = \int_r^1 r'\Theta(r')^2 dr'$$

$$u(t,r_0,\theta_0) = (r_0, \theta_0 + t\Theta(r_0)), \qquad \text{supp}(\Theta) \subset\subset (0,1)$$

Construction of solutions (w. surface tension)

- Fix the moving domains with a nearby reference domain $\Omega_*^{\pm}$

* Represent $\partial\Omega_t$ by a modified mean curvature $\kappa_M : \partial\Omega_* \to \mathbf{R}$

* Determine normal velocity (& thus v^{ir}) by $\partial_t \kappa_M$

* Transfer v^{rot} to $v_*^{rot} : \Omega_* \to \mathbf{R}^n$ with $v_*^{rot}|_{\partial\Omega_*} \in T\partial\Omega_*$

1. Start with $(\kappa_M, \partial_t\kappa_M, v_*^{rot})$

2. Reconstruct (Ω_t, v) and u by solving elliptic equations

3. Solve for $(\tilde{\kappa}_M, \partial_t\tilde{\kappa}_M)$ from an equ derived from (L)

$$\partial_{tt}\tilde{\kappa}_M + '' |\partial^3| '' \tilde{\kappa}_M = Q(\kappa_M, \partial_t\kappa_M, v_*^{rot})$$

4. Construct $\tilde{v}_*^{rot}$ by symmetries (Noether's Theorem)

5. contraction mapping

Theorem. *(**Shatah-Z, 08**) (1F) is locally well-posed in the space* $\partial\Omega \in H^{\frac{3}{2}k+1}$ *and* $v \in H^{\frac{3}{2}k}(\Omega_t)$.

• w/o surface tension assuming **(RT)**: vanishing surface tension limit

• **(2F)** w. surface tension

Other cases of (1F)

$$v_t + v \cdot \nabla v = -\nabla p - g e_n, \quad x \in \Omega_t \tag{E}$$

* Ω_t: allowed to be infinite domain

* $\partial\Omega_t = S_t \cup B$, $d(S_t, B) \geq a > 0$

$$v \cdot N = 0 \text{ on } B, \quad p = \alpha\kappa \text{ on } S_t \tag{BC}$$

* B: rigid part of boundary (bottom, obstacles ...)

* S_t: free surface, asymptotically flat in space

* g: gravity

Framework

- Action:

$$\mathcal{A}(u) = \int \int_{\mathbf{R}^n} \frac{1}{2}|u_t|^2 dy dt - \alpha \int S(u(S_0)) dt - G(u)$$

* $u(t,\cdot)$: Lagrangian coordinate map

* Gravitational potential:

$$G(u) = \int_{u(\Omega_0)\cap x_n > 0} x_n dx - \int_{u(\Omega_0)\cap x_n > 0} x_n dx$$

configuration space $\mathcal{M}$:

$$\mathcal{M} = \{u : \Omega_0 \to \mathbf{R}^n : det \nabla u = 1,\ u(B) = B\} \subset L^2(\Omega_0, \mathbf{R}^n)$$

$$T_u\mathcal{M} = \{W : \Omega_0 \to \mathbf{R}^n : \nabla \cdot (W \circ u^{-1}) = 0,\ (W \cdot N)|_{u(S_0)} = 0\}$$

$$(T_u\mathcal{M})^\perp = \{(\nabla q) \circ u : \Omega_0 \to \mathbf{R}^n : q|_{u(\partial\Omega)} = 0\}$$

- Linearization:

$$\mathcal{D}_t^2 W + (\mathcal{R}(u_t, \cdot)u_t + D^2G(u))W + \alpha D^2 S(u) W = 0$$

- For $\alpha = 0$, **(RT)** from $\mathcal{R}(u_t, \cdot)u_t + D^2G(u)$.

Framework for (2F)

- Action:

$$\mathcal{A}(u) = \int \int_{\mathbf{R}^n} \frac{\rho}{2}|u_t|^2 dy dt - \int S(u) dt$$

* $u(t,\cdot) = (u_+, u_-) : \mathbf{R}^n \setminus S_0 \to \mathbf{R}^n \setminus S_t$: Lagrangian coordinates

- Configuration space:

$$\mathcal{M} = \{u_\pm : \Omega_0^\pm \to \Omega_t^\pm : \text{volume preserving diffeo. s.t.}$$
$$S \triangleq u_+(S_0) = u_-(S_0)\}$$

* In Eulerian cordinates:

$$T_u\mathcal{M} = \{v = (v_+, v_-) : \nabla \cdot v_\pm = 0,\ (v_+^\perp + v_-^\perp)|_S = 0\} \subset L^2(\rho dx)$$

$$(T_u\mathcal{M})^\perp = \{w = (\nabla\phi_+, \nabla\phi_-) : (\rho_+\phi_+ - \rho_-\phi_-)_S = 0\} \subset L^2(\rho dx)$$