Variational problems with linear growth in dimension 1

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1. Introduction

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain $L : \Omega \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$, and let $L(x,u,p)$ be C^1 in (u,p), convex in p and satisfying

$$|L_p(x,u,p)| \leq C(1+|p|). \qquad (L1)$$

We consider the critical points of the functional

$$u \to I(u) := \int_\Omega L(x,u,\nabla u)dx$$

for 1-dimensional Ω.

Example 1: Let $F(x,u) : \Omega \times \mathbf{R} \to \mathbf{R}$ and $L(x,u,p) = \sqrt{1+|p|^2} - F(x,u)$. The Euler-Lagrange equation of $I(u)$

$$-div(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}) = f(x,u), \quad x \in \Omega,$$
$$u = \phi, \quad x \in \partial\Omega, \qquad (1)$$

where $F(x,u) = \int_0^u f(x,s)ds$. (1) and its parabolic version have been extensively studied.

In geometry, the left hand side of (1) is the mean curvature of the graph

$$\Sigma = \{(x, u(x)) \in \mathbf{R}^{n+1} | x \in \Omega\}.$$

(1) Minimal surface: $f(x, u) = 0$.

(2) Prescribed mean curvature: $f(x, u) = H(x)$.

Example 2: Let $L(x, u, p) = |p| - F(x, u)$, then the Euler-Lagrange equation of $I(u)$ is the following 1-Laplacian equation

$$\begin{aligned} -div(\frac{\nabla u}{|\nabla u|}) = f(x, u), \quad x \in \Omega, \\ u = \phi \quad x \in \partial\Omega. \end{aligned} \tag{2}$$

The meaning of the solutions of (1) and (2) will be specified later.

Due to the condition (L1), the functional

$$I(u) = \int_\Omega L(x, u, \nabla u)dx$$

is well defined and C^1 in the Sobolev space $W^{1,1}(\Omega)$ if some growth conditions are satisfied. However, it is well known that the functional I may not have critical points in this space.

One considers the solutions in BV space:

$$BV(\Omega) = \{u \in L^1(\Omega) | \|Du\| < \infty\}$$

with the norm $\|u\| = \|u\|_1 + \|Du\|$, where

$$\|Du\| = \text{sup}\{ \int_\Omega u \cdot div\phi dx | \phi \in C_0^1(\Omega, \mathbf{R}^n), \\ |\phi(x)| \leq 1 \forall x \in \Omega\} := \int_\Omega |Du|dx.$$

For $u \in BV(\Omega)$, the trace $u|_{\partial\Omega}$ can be defined. For the 1-Laplacian equation (2), taking the boundary value into account, one uses the functional

$$I(u) = \int_\Omega |Du|dx + \int_{\partial\Omega} |u - \phi|dx - \int_\Omega F(x,u)dx.$$

For the mean curvature equation (1) we need

$$\begin{aligned} &\int_\Omega \sqrt{1 + |Du|^2}dx \\ := \sup\{&\int_\Omega (\phi_0 + u \cdot div\phi_1)dx \\ &|(\phi_0, \phi_1) \in C_0^1(\Omega, \mathbf{R}^{1+n}), \\ &|\phi_0(x)|^2 + |\phi_1(x)|^2 \leq 1 \forall x \in \Omega\} \end{aligned}$$

and

$$\begin{aligned} I(u) &= \int_\Omega \sqrt{1 + |Du|^2}dx + \int_{\partial\Omega} |u - \phi|dx - \int_\Omega F(x,u)dx \\ &= I_0(u) - \int_\Omega F(x,u)dx. \end{aligned}$$

Definition 1. $u \in BV(\Omega)$ *is called a solution of*

$$-div(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}) = f(x,u), \quad x \in \Omega,$$
$$u = \phi, \quad x \in \partial\Omega \tag{3}$$

if for $\forall v \in BV(\Omega)$,

$$I_0(v) - I_0(u) \geq \int_\Omega f(x,u)(v-u)dx. \tag{4}$$

Such a function is also called a critical point of the functional I. $u \in BV(\Omega)$ *is called a solution of*

$$-div(\frac{\nabla u}{|\nabla u|}) = f(x,u), \quad x \in \Omega,$$
$$u = \phi \quad x \in \partial\Omega \tag{5}$$

if for $\forall v \in BV(\Omega)$,

$$\int_\Omega (|Dv| - |Du|)dx + \int_{\partial\Omega} (|v-\phi| - |u-\phi|)dx$$
$$\geq \int_\Omega f(x,u)(v-u)dx. \tag{6}$$

Main Difficulties of Existence of Solutions:

(1) the functional I is not differentiable on $BV(\Omega)$;

(2) the working space $BV(\Omega)$ is not reflexive and its dual has not been completely understood. This makes the verification of P.S. condition very difficult.

Applying a non-smooth version of symmetric mountain pass theorem to the functional

$$\tilde{I}(u) = \begin{cases} I(u), & u \in BV(\Omega) \\ +\infty, & u \in L^p(\Omega) \setminus BV(\Omega) \end{cases}$$

the following theorem is proved:

Theorem 2. *(M. Marzocchi) Let* $f : \overline{\Omega} \times \mathbf{R} \to \mathbf{R}$ *be a continuous function satisfying:*
(1) $\exists \theta > 1$ *and* $R > 0$ *such that*

$$f(x,u)u \geq \theta F(x,u) > 0, \quad |u| \geq R; \qquad (f1)$$

(2) $\exists p < \frac{n}{n-1}$ *such that*

$$|f(x,u)| \leq C(1+|u|^p), p < \frac{n}{n-1} \quad u \in \mathbf{R} \quad (f2)$$

and

$$f(x,-u) = -f(x,u), \quad x \in \Omega, u \in \mathbf{R}; \qquad (f3)$$

then the equation

$$-div\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = f(x,u), \quad x \in \Omega, \qquad (7)$$
$$u = 0, \quad x \in \partial\Omega,$$

has a sequence of solutions $\{u_j\} \subset BV(\Omega)$ *such that* $I(u_j) \to +\infty$ *as* $j \to \infty$.

This result generalizes the well known theorem of Ambrosetti-Rabinowitz:

(1) $\exists \theta > 2$ and $R > 0$ such that

$$f(x,u)u \geq \theta F(x,u) > 0, \quad |u| \geq R; \qquad (f1)$$

(2) $\exists p < \frac{n+2}{n-2}$ such that

$$|f(x,u)| \leq C(1 + |u|^p), \quad u \in \mathbf{R} \qquad (f2)$$

and

$$f(x,-u) = -f(x,u), \quad x \in \Omega, u \in \mathbf{R}; \qquad (f3)$$

then the equation

$$\begin{aligned} -\triangle u = f(x,u), \quad x \in \Omega, \\ u = 0 \quad x \in \partial\Omega. \end{aligned} \qquad (8)$$

has a sequence of solutions $\{u_j\}$ such that $\|u_j\|_\infty \to +\infty$ as $j \to \infty$. The symmetric assumption (f3) is crucial.

Results without symmetry:
Rabinowitz, Bahri-Berestyski, Bahri-Lions et al, 1980s,

$$F(x, u) = |u|^{p+1} + l.o.t,$$

$p \leq (<) p^* = \frac{n+2}{n-2}$, (8) has infinitely many solutions;

Bahri-Berestyski, Long et al, 1980s-1990s,

$$V(x, u) = V_0(u) + l.o.t,$$

or

$$V(x, u) = V_0(x, u) + l.o.t.$$

with $V_0(x, -u) = V(x, u)$, $x \in S^1$, $u \in \mathbf{R}^n$ satisfying $\exists \theta > 2$ and $R > 0$ such that

$$V_{0u}(x, u)u \geq \theta V_0(x, u) > 0, \quad |u| \geq R; \quad (V1)$$

then

$$-u'' = V_u(x, u)$$

has infinitely many 2π-periodic solutions.

2. 1-Dimensional Problem

We start with periodic solutions of

$$-u'' = f(x, u), u \in \mathbf{R}. \tag{9}$$

Suppose f is continuous and 2π-periodic in x.
Theorem 3. *(Nehari 1961, Jacobowitz, 1976, Hartman, 1977, W.Y. Ding, 1982, $\cdots$) If f satisfies*

$$\lim_{|u|\to\infty} \frac{f(x, u)}{u} = +\infty \tag{$f4$}$$

and the initial problem of (9) is uniquely and globally solvable, then (9) has an unbounded sequence of 2π-periodic solutions.

The same conclusion holds for the Dirichlet and Neumann boundary conditions as well as the following p-Laplacian equation:

$$-(|u'|^{p-2}u')' = f(x,u), \quad p > 1 \tag{10}$$

if f satisfies the 'superlinear' condition

$$\lim_{|u|\to\infty} \frac{f(x,u)}{|u|^{p-2}u} = +\infty. \tag{$f4$)$'$}$$

The main feature of this result is that f may not be **odd** in u.

Two Approaches for (9):

(i) Fixed points approach: Using the Poincaré-Birkhoff fixed point theorem. The fixed points of the Poincaré map correspond to 2π-periodic solutions.

(ii) Variational approach: Find the critical points of the functional

$$I(u) = \frac{1}{2}\int_0^{2\pi} \dot{u}^2 dt - \int_0^{2\pi} F(x,u)dt.$$

In this talk we will show how to generalize Theorem 3 to the equations

$$-\left(\frac{u'}{\sqrt{1+u'^2}}\right)' = f(x,u), \qquad (11)$$

and

$$-\left(\frac{u'}{|u'|}\right)' = f(x,u), \qquad (12)$$

on BV space by variational method.

For the equations (11) and (12), the Poincaré map is **not well defined** and the associated functional is **not differentiable (or Lipschitz)** on BV space. There may be **no continuous solution**.

For simplicity we consider the periodic boundary condition in $BV(S^1)$: $u \in BV(S^1)$ if

$$\int_0^{2\pi} |Du| dx = \|Du\|$$

$$= \text{sup}\{\int_{S^1} u\phi' dx | \phi \in C^1(S^1), |\phi(x)| \leq 1 \forall x \in S^1\} < \infty,$$

$$\|u\| = \|u\|_1 + \|Du\|$$

and

$$\int_{S^1} \sqrt{1 + |Du|^2} dx := \text{sup}\{\int_{S^1} (\phi_0 + u \cdot \phi_1') dx$$
$$|\phi_0, \phi_1 \in C^1(S^1), |\phi_0(x)|^2 + |\phi_1(x)|^2 \leq 1 \forall x \in S^1\}.$$

A function $u \in BV(S^1)$ is called a 2π-periodic solution of (11) if

$$\begin{aligned}&\int_0^{2\pi} \sqrt{1+|Dv|^2}dx - \int_0^{2\pi} \sqrt{1+|Du|^2}dx \\ \geq &\int_0^{2\pi} f(x,u)(v-u)dx, \quad \forall v \in BV(S^1).\end{aligned} \tag{13}$$

$u \in BV(S^1)$ is called a 2π-periodic solution of (12) if

$$\begin{aligned}&\int_0^{2\pi} |Dv|dx - \int_0^{2\pi} |Du|dx \\ \geq &\int_0^{2\pi} f(x,u)(v-u)dx, \quad \forall v \in BV(S^1).\end{aligned} \tag{14}$$

Theorem 4. *Let $F(x,u) = \int_0^u f(x,s)ds$ be a C^1 function, 2π-periodic in x such that*

(1) f is 'superlinear',

$$sgn(u)f(x,u) \to +\infty \quad \text{as} |u| \to +\infty; \qquad (f4)$$

(2) there exists a constant $C > 0$ such that

$$|\frac{\partial F(x,u)}{\partial x}| \leq C(1 + |F(x,u)|), (x,u) \in S^1 \times \mathbf{R}. \qquad (f5)$$

Then there is a positive integer k_0 such that for $k \geq k_0$, the equation (11) has a 2π-periodic solution $u_k \in BV(S^1)$ satisfying

(1) u_k is C^2, satisfies (11) on $S^1 \backslash \{x_1, x_2, \cdots, x_{2k}\}$ and $\forall v \in C^1(S^1)$,

$$\int_0^{2\pi} \frac{u'}{\sqrt{1+u'^2}} v' dx = \int_0^{2\pi} f(x,u) v dx;$$

(2) for each x_i, the follwoings hold:

$$\lim_{x\to x_{2i-1}-0} u_k(x) > 0 > \lim_{x\to x_{2i-1}+0} u_k(x), \\ \lim_{x\to x_{2i-1}\pm 0} u_k'(x) = -\infty, \tag{15}$$

$$\lim_{s\to x_{2i}-0} u_k(x) < 0 < \lim_{x\to x_{2i}+0} u_k(x), \\ \lim_{x\to x_{2i}\pm 0} u_k'(x) = +\infty, \tag{16}$$

$$\lim_{x\to x_i-0} F(x,u_k(x)) = \lim_{x\to x_i+0} F(x,u_k(x)); \tag{17}$$

(3)

$$\lim_{k\to\infty} \inf_{x\in S^1} |u_k(x)| = +\infty; \tag{18}$$

(4) If the function f satisfies

$$f(x,u)u - F(x,u) \to +\infty \quad \textit{as} \quad |u| \to \infty, \tag{$f6$}$$

then

$$I(u_k) \to \infty \quad \textit{as} \quad k \to \infty. \tag{19}$$

Theorem 5. *With the same assumptions as that of Theorem 4, the same conclusion holds for*

$$-(\frac{u'}{|u'|})' = f(x,u). \qquad (20)$$

Remarks: 1. Marzocchi's theorem (Theorem 2) can be applied to (11) and (12) if $f(x,-u) = -f(x,u), u \in \mathbf{R}$.

2. The solutions u_k in Theorems 4-5 are not continuous, but in $BV(S^1)$. The existence of classical and non-classical solutions has been studied by many authors and various methods. Based on the phase analysis and bifurcation method, detail analysis of the numbers of solutions (11) depending on a parameter λ is given for $f = \lambda u^p$ and $f = \lambda e^u$ by Omari, Pan, respectively.

3. A Class of BV Solutions

Let $f : S^1 \times \mathbf{R} \to \mathbf{R}$ be continuous and 2π-periodic in x.

Proposition 6. *Let $u : S^1 \to \mathbf{R}$ be a function such that*

(1) u is C^2 and satisfies the equation

$$-\left(\frac{u'}{\sqrt{1+u'^2}}\right)' = f(x,u) \tag{21}$$

outside a finite set $\{x_1, x_2, \cdots, x_k\} \subset S^1$;

(2) for $x \in \{x_1, x_2, \cdots, x_k\}$, the limits $\lim_{s\to x\pm 0} u(s)$ exist and u satisfies:

$$\lim_{s\to x\pm 0} u'(s) = +\infty \textit{if} \lim_{s\to x-0} u(s) \le \lim_{s\to x+0} u(s) \tag{22}$$

and

$$\lim_{s\to x\pm 0} u'(s) = -\infty \textit{if} \lim_{s\to x-0} u(s) \ge \lim_{s\to x+0} u(s). \tag{23}$$

Then $u \in BV(S^1)$ *and satisfies*

$$\int_0^{2\pi} \frac{u'}{\sqrt{1+u'^2}} v' dx = \int_0^{2\pi} f(x,u) v dx, \qquad (24)$$

for $\forall v \in C^1(S^1)$*, and*

$$\begin{aligned} &\int_0^{2\pi} \sqrt{1+|Dv|^2} dx - \int_0^{2\pi} \sqrt{1+|Du|^2} dx \\ &\geq \int_\Omega f(x,u)(v-u) dx. \end{aligned} \qquad (25)$$

Hence u *is a solution of (21) in the sense of Definition 1.*

The following is the counterpart of Proposition 6 for 1-Laplacian equation.

Proposition 7. *Let* $\{x_1, x_2, \cdots, x_k\} \subset S^1$ *and* $x_1 < x_2 < \cdots, x_k \leq 2\pi$ *and* $\{u_1, u_2, \cdots, u_k\} \subset \mathbf{R}$ *satisfying*

$$(u_i - u_{i-1})(u_{i+1} - u_i) < 0 \tag{26}$$

and

$$\int_{x_i}^{x_{i+1}} f(x, u_i)dx = 2(-1)^i. \tag{27}$$

Then the function $u : S^1 \to \mathbf{R}$ *given by*

$$u(x) = u_i, \quad x \in [x_i, x_{i+1}) \tag{28}$$

is a BV solution of

$$-(\frac{u'}{|u'|})' = f(x, u). \tag{29}$$

Remarks: 1. (27) are not differential equations. We will see that the solutions can be obtained by solving a Hamiltonian system.

2. The number of unknowns is $2k$, and the number of equations is k. Thus it is overdetermined and there should be many solutions.

The key point in our approach to the problem is to use the Hamiltonian formalism. Using Legendre transformation,

$$H(x,u,v) = \sup_p (vp - L(x,u,p)),$$

one knows that the equations (21) and (27) are "**equivalent**" to the Hamiltonian systems:

$$-u' = H_v(x,u,v), \quad v' = H_u(x,u,v) \qquad (30)$$

and

$$-u' = K_v(x,u,v), \quad v' = K_u(x,u,v) \qquad (31)$$

where $H(x,u,v) = 1 - \sqrt{1-v^2} + F(x,u)$ and $K(x,u,v) = F(x,u)$, $(u,v) \in \mathbf{R} \times (-1,1)$. One should note that the Hamiltonian H and K are only defined for $v \in (-1,1)$. We need to specify the behaviors of solutions of (30) and (31) near $v = \pm 1$. According to Propositions 6-7 we need to find

$(u,v) : S^1 \to \mathbf{R} \times (-1,1)$ such that

(1) (u,v) is C^1 and satisfies (30) (or (31)) on the set $S^1 \setminus \{x_1, x_2, \cdots, x_k\}$;

(2) v is continuous and satisfies

$$v(x) = \lim_{s \to x} v(s) = \pm 1 \tag{32}$$

for each $x \in \{x_1, x_2, \cdots, x_k\}$;

(3) the limits $\lim_{s \to x \pm 0} u(s)$ exist for each $x \in \{x_1, x_2, \cdots, x_k\}$ and u satisfies

$$\lim_{s \to x-0} u(s) \geq (\leq) \lim_{s \to x+0} u(s) \quad \text{if} \quad v(x) = 1(-1). \tag{33}$$

4. Proof of Theorem 4:

We sketch the proof of Theorem 4. That of Theorem 5 is similar. Let H_1 be defined on $S^1 \times \mathbf{R} \times \mathbf{R}$ such that

$$H_1(x,u,v) = \begin{cases} H(x,u,v), & v \in [-1,1] \\ H(x,-u,2-v), & v \in [1,3] \end{cases}$$

and H_1 is 4-periodic function in v. Consider the solutions of the equation

$$-u' = H_{1,v}(x,u,v), \quad v' = H_{1,u}(x,u,v) \quad (34)$$

satisfying

$$u(0) = u(2\pi), \quad 4k + v(0) = v(2\pi). \quad (35)$$

Solutions: $w_k = (u_k, v_k) : [0, 2\pi] \to \mathbf{R}^2$ and a finite set $\{x_1, x_2, \cdots, x_{2k}\} \subset [0, 2\pi)$ with the following properties:

(1) (u_k, v_k) is C^1 on $[0, 2\pi] \setminus \{x_1, x_2, \cdots, x_{2k}\}$, satisfies (34) and (35);

(2) v_k is strictly increasing, continuous and for $j = 1, 2, \cdots, 2k$, satisfies

$$v_k(x_j) \quad \text{is an odd integer;} \tag{36}$$

(3) u_k is positive and

$$\lim_{x \to x_j - 0} u(x) > 0, \qquad \lim_{x \to x_j + 0} u(x) > 0; \tag{37}$$

(4) $H(x, u_k(x), v_k(x))$ is continuous.

Then

$$u(x) = \begin{cases} u_k(x), & 4j - 5 \le v_k(x) < 4j - 3 \\ -u_k(x), & 4j - 3 \le v_k(x) < 4j - 1, \end{cases}$$

is a solution of

$$-\left(\frac{u'}{\sqrt{1 + u'^2}}\right)' = f(x, u)$$

in $BV(S^1)$ and has the properties listed in Theorem 4.

Difficulties: the right hand side of (34) **is not continuous near** $v = \pm 1$ and we can not specify the set $\{x_1, x_2, \cdots, x_{2k}\}$ **a priori**.

For $0 < \delta << 1$, let χ_δ be a C^2 function such that

$$\chi_\delta(s) = \begin{cases} 1, & s \le -\delta \\ 0, & s \ge \delta \end{cases}$$

and $-\frac{4}{\delta} \le \chi'_\delta \le 0$;

$$F_\delta(x,u,v) = \begin{cases} \chi_\delta(v-1)F(x,u) & \\ +(1-\chi_\delta(v-1))F(x,-u), & v \in [-1+\delta, \\ \chi_\delta(v-3)F(x,-u) & \\ +(1-\chi_\delta(v-3))F(x,u), & v \in [1+\delta, 3 \end{cases}$$

G_δ be a C^2 and 4-periodic function satisfying

$$G_\delta(v) = \begin{cases} 1 - \sqrt{1-v^2}, & v \in [-1+\delta, 1-\delta] \\ 1 - \sqrt{1-(2-v)^2}, & v \in [1+\delta, 3-\delta], \end{cases}$$

$$0 \leq G'_\delta(v) \leq \frac{v}{\sqrt{1-v^2}}, \quad -1+\delta \leq v \leq 1,$$

$$\frac{(v-2)}{\sqrt{1-(2-v)^2}} \leq G'_\delta(v) \leq 0, \quad 1 \leq v \leq 3+\delta.$$

Then the function $H_\delta(x,u,v) = G_\delta(v) + F_\delta(x,u,v)$ is C^1 and 2π-periodic in x, 4-periodic in v. We fix a positive integer k and consider the solutions of the Hamiltonian system

$$-u' = H_{\delta,v}(x,u,v), \quad v' = H_{\delta,u}(x,u,v) \quad (38)$$

with the boundary condition

$$u(0) = u(2\pi), \quad 4k + v(0) = v(2\pi). \quad (39)$$

Proposition 8. *Let* $F(x,u) = \int_0^u f(x,s)ds$ *be a* C^1 *function,* 2π*-periodic in* x *such that*

(1) f *is 'superlinear',*

$$sgn(u)f(x,u) \to +\infty \quad \textit{as} \qquad |u| \to +\infty; \tag{$f4$}$$

(2) there exists a constant $C > 0$ *such that*

$$\left|\frac{\partial F(x,u)}{\partial x}\right| \leq C(1 + F(x,u)), (x,u) \in S^1 \times \mathbf{R}. \tag{$f5$}$$

Then for all $0 < \delta << 1$ *and integer* $k \geq 1$*, the equation (38) with the boundary condition (39) possesses 2 solutions* $w^i_{k,\delta} = (u^i_{k,\delta}, v^i_{k,\delta})$*,* $i = 1, 2$*. Moreover, there exist two sequences* $\{A(k)\}_1^\infty$ *and* $\{B(k)\}_1^\infty$*, independent of* δ *with* $\lim_{k\to\infty} B(k) = +\infty$ *such that*

$$A(k) \geq u^i_{k,\delta}(x) \geq B(k), \quad i = 1, 2, x \in S^1. \tag{40}$$

Using the estimate (40), we can take the limit $\delta \to 0$,

$$u_k = \lim_{\delta \to 0} u_{k,\delta}, \quad v_k = \lim_{\delta \to 0} v_{k,\delta},$$

then (u_k, v_k) is a solution of

$$-u' = H_{1,v}(x,u,v), \quad v' = H_{1,u}(x,u,v) \quad (41)$$

and

$$u(0) = u(2\pi), \quad 4k + v(0) = v(2\pi), \quad (42)$$

where $H(x,u,v) = 1 - \sqrt{1-v^2} + F(x,u)$ and

$$H_1(x,u,v) = \begin{cases} H(x,u,v), & v \in [-1,1] \\ H(x,-u,2-v), & v \in [1,3] \end{cases}$$

and H_1 is 4-periodic function in v. Then

$$u(x) = \begin{cases} u_k(x), & 4j-5 \le v_k(x) < 4j-3 \\ -u_k(x), & 4j-3 \le v_k(x) < 4j-1, \end{cases}$$

u is a solution of (5) in $BV(S^1)$ and has the properties listed in Theorem 4.

5. More Solutions

The solutions u_k given by Theorem 4 have the property: at each discontinuous point x_j of u_k, there holds

$$\lim_{x\to x_j-0} F(x, u_k(x)) = \lim_{x\to x_j+0} F(x, u_k(x)).$$

Slightly modify the approach, much more BV solutions can be obtained if this restriction is removed.

Theorem 9. *Let $F(x,u) = \int_0^x f(x,s)ds$ be a C^1 function, 2π-periodic in $x$satisfying conditions (f4) and (f5). Then for $\alpha_0 > 0$, there is a positive integer $k_0(\alpha_0)$ depending on α_0 such that for $k \geq k_0$ and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_k)$ with $\alpha_i \geq \alpha_0$, $i = 1, \cdots, k$, the equation*

$$-\left(\frac{u'}{\sqrt{1+u'^2}}\right)' = f(x,u) \tag{43}$$

possesses a 2π-periodic solution $u_k(\alpha; x) \in BV(S^1)$ having the properties:

(1) there is a finite set $\{x_1, x_2, \cdots, x_{2k}\} \subset S^1$ *such that* u_k *is* C^2 *and satisfies (43) on the set* $S^1 \setminus \{x_1, x_2, \cdots, x_{2k}\}$ *and* $\forall v \in C^1(S^1)$,

$$\int_0^{2\pi} \frac{u_k'}{\sqrt{1+u_k'^2}} v' dx = \int_0^{2\pi} f(x, u_k) v dx;$$

(2) for each x_i, *there hold:*

$$\begin{aligned} &\lim_{x \to x_{2i-1}-0} u_k(x) > 0 > \lim_{x \to x_{2i-1}+0} u_k(x), \\ &\lim_{x \to x_{2i-1} \pm 0} u_k'(x) = -\infty, \end{aligned} \tag{44}$$

$$\begin{aligned} &\lim_{s \to x_{2i}-0} u_k(x) < 0 < \lim_{x \to x_{2i}+0} u_k(x), \\ &\lim_{x \to x_{2i} \pm 0} u_k'(x) = +\infty, \end{aligned} \tag{45}$$

$$\alpha_i \lim_{x\to x_{2i}-0} F(x,u_k(x)) = \lim_{x\to x_{2i}+0} F(x,u_k(x)), \tag{46}$$

$$\lim_{x\to x_{2i+1}-0} F(x,u_k(x)) = \alpha_i \lim_{x\to x_{2i+1}+0} F(x,u_k(x)); \tag{47}$$

(3) for fixed k there hold

$$\max_{x\in S^1} |u_k(\alpha;x)| \to \infty \quad |\alpha| \to \infty \tag{48}$$

where $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_k$.

Theorem 10. *Under the same condition as that of Theorem 9, the same conclusion holds for*

$$-(\frac{u'}{|u'|})' = f(x,u). \tag{49}$$

6 Concluding Remarks:

We only consider the "superlinear" case

$$sgn(u)f(x,u) \to +\infty \quad \text{as} \quad |u| \to +\infty. \quad (f4)$$

Using the same methods, one can obtain some results for the "asymptotically linear" case

$$sgn(u)f(x,u) \to \lambda_{\pm} \quad \text{as} \quad u \to \pm\infty.$$

The existence is related to the eigenvalue problem of

$$-\left(\frac{u'}{|u'|}\right)' = \lambda \frac{u}{|u|}. \tag{50}$$

Theorems 4-5, Theorems 9-10 hold for other boundary value conditions, for instance, the Dirichlet and Neumann boundary conditions.

Conclusion: For equations

$$-(\frac{u'}{\sqrt{1+u'^2}})' = f(x,u) \tag{51}$$

and

$$-(\frac{u'}{|u'|})' = f(x,u), \tag{52}$$

there may be no solutions in $W^{1,1}$, but there are too many (**uncountable many**) solutions in BV space if f is superlinear.