A Bahri-Lions Theorem and a Brezis-Nirenberg Problem Revisited

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Theorem

Happy birthday to professor P. Rabinowitz !

Contents

1. A Bahri-Lions Theorem Revisited

1.1. Background

Consider

$$\begin{cases} -\Delta u = h(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbf{R}^n$ is an open bounded smooth domain.

Case 1. $h(x,u)$ is odd in u: it is relatively easy to prove the existence of infinitely many solutions.

Case 2. $h(x,u)$ is not odd in u: A long standing open question which even today is not adequately settled is whether the symmetry of the functional is crucial for the existence of infinitely many critical points. This open question and conjecture were also raised by Bahri, Berestycki, Lions, Rabinowitz and Struwe, etc. in their papers or books.

Several partial answers had been obtained in the past 30 years. Let us sketch the history on this direction. The special case

$$-\Delta u = |u|^{p-2}u + f(x), \quad u \in H_0^1(\Omega) \tag{1.1}$$

was first studied by Bahri-Brestycki (TAMS, 1981) and Struwe (Manus. Math. 1982) independently. In A. Bahri (JFA, 1981), the author considered (1.1) and proved that there is an open dense set of f in $W^{-1,2}(\Omega)$ such that (1.1) has infinitely many solutions if $p < 2N/(N-2)$.

Theorem

Rabinowitz Theorem: (TAMS, 1982) *considered the general problem*

$$-\Delta u = g(x,u) + f(x,u), \quad u \in H_0^1(\Omega), \tag{1.2}$$

where $g, f : \Omega \times \mathbb{R} \to \mathbb{R}$ *are Carathéodory functions satisfying*

(L_1) $g(x,s) \in \mathbb{C}(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$ *is odd in* s*;*

(L_2) $|g(x,s)| \le a_1 + a_2|s|^{p-1},\ 1 \le p < 2^*;\ a_1, a_2$ *are constants;*

(L_3) $g(x,s)s \ge \mu G(x,s) > 0$ *for all* $|s| \ge r$ *and* $x \in \Omega$*, where* $r > 0, \mu > 2$ *are constants and* $G(x,s) := \int_0^s g(x,\xi)d\xi$*;*

(L_4) $|f(x,s)| \le a_3 + a_4|s|^k$ *for all* $s \in \mathbb{R}, x \in \Omega$*;* $0 \le k < \mu - 1,\ a_3$ *and* a_4 *are constants.*

It was proved in that, if

$$\frac{(N+2)-(N-2)(p-1)}{N(p-2)} > \frac{\mu}{\mu-(1+k)}, \tag{1.3}$$

then (1.2) has infinitely many solutions.

Y. Long (TAMS, 1989) considered the perturbed superquadratic second order Hamiltonian systems. Tehrani (CPDE, 1996) considered the case of a sign-changing potential. P. Bolle-N. Ghoussoub-H. Tehrani (Manus. Math. 2000) got some existence results on the following perturbed elliptic equations:

$$-\Delta u = |u|^{p-2}u + f(x), \quad \text{in } \Omega; \qquad u = u_0, \quad \text{on } \partial\Omega, \tag{1.4}$$

where $u_0 \in \mathbf{C}^2(\bar{\Omega}, \mathbb{R})$ with $\Delta u_0 = 0$ and $2 < p < \frac{2N}{N-1}$. The papers mentioned above mainly concern the existence of infinitely many solutions.

Later, A. Bahri and P. L. Lions (CPAM, 1988) (see also K. Tanaka (1989, CPDE)) studied the following problem with perturbation from symmetry:

$$-\Delta u = |u|^{p-2}u + f(x,u), \qquad u \in H_0^1(\Omega), \tag{1.5}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N (N \geq 2)$;
$2 < p < 2N/(N-2)$ ($p < \infty$ if $N = 2$); $f(x,u)$ is not necessarily to be odd in u and satisfies

$(\mathbf{B_1})$ $|f(x,t)| \leq h(x) + C|t|^q$ a. e., for some $C \geq 0,\ h \in L_+^r(\Omega)$, where $q = (N+2)/(N-2)$ ($q < \infty$ if $N \leq 2$), $r = 2N/(N+2)$ ($r > 1$ if $N \leq 2$);

$(\mathbf{B_2})$ $|F(x,t)| \leq a(x) + b(x)|t|^\sigma$ a. e., for some $0 \leq \sigma < 2$, where $a \in L_+^1(\Omega),\ b \in L_+^\beta(\Omega)$ with $\beta > 1$,
$\beta' < 2N/(N-2)(1/\sigma)$ ($\beta > 1$ if $N \leq 2$), $F(x,t) = \int_0^t f(x,s)ds$.

Under the above assumptions, A. Bahri and P. L. Lions obtained the following existence result.

Theorem

Bahri-Lions Theorem (1988). *Let*

$$2 < p < \frac{2N - 2\sigma}{N - 2}. \tag{1.6}$$

Then equation (1.5) has infinitely many solutions.

K. Tanaka (1989, CPDE) studied (1.2) (with $g(x,u) = g(u), f(x,u) = f(x)$) and got similar result as the Bahri-Lions Theorem.

A natural question is: How far can we go? Two directions:

- weaken the restriction of p.
- more properties on the solutions.

under the the assumptions of the Bahri-Lions Theorem, what is the profile for the solutions? Whether equation (1.5) has infinitely many sign-changing solutions? —— In this paper, we give a positive answer to this question. Precisely, we shall obtain infinitely many sign-changing solutions to the problem

$$-\Delta u = g(x,u) + f(x,u), \quad u \in H_0^1(\Omega), \tag{1.7}$$

where $g, f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

$(\mathbf{H_1})$ $g(x,s)$ is odd in s and $\frac{g(x,s)}{s} \to 0$ as $s \to 0$ uniformly in x;

$(\mathbf{H_2})$ $0 \le g(x,s)s \le C(|s|^p + 1),\ 2 < p < 2^*$;

$(\mathbf{H_3})$ $g(x,s)s \ge \mu G(x,s) - C, C > 0, \mu > 2$, where $G(x,s) := \int_0^s g(x,\xi)d\xi$;

$(\mathbf{H_4})$ $f(x,s) = o(s)$ as $s \to 0$ uniformly in x, and $0 \le f(x,s)s \le C(|s|^\sigma + 1)\forall s$, for some $C > 0, 0 < \sigma < \mu$.

1.2. Main Results

Theorem

(Ramos-Tavares-Zou, 2008) Assume (H_1)-(H_4). *If moreover*

$$2 < p < \frac{2N\mu}{N\mu - 2\mu + 2\sigma}, \tag{1.8}$$

then the problem (1.7) admits a sequence of sign-changing solutions $(u_{k_n})_{n\in\mathbb{N}}$ *whose energy levels* $J(u_{k_n})$ *satisfy*

$$c_1 k_n^{\frac{2p}{N(p-2)}} \leq J(u_{k_n}) \leq c_2 k_n^{\frac{2\mu}{N(\mu-2)}}$$

for some $c_1, c_2 > 0$ *independent of* n, *where* J *is the energy functional*

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega G(x,u) - \int_\Omega F(x,u), \quad u \in H_0^1(\Omega).$$

In the above theorem, if we set $g(x,u) = |u|^{p-2}u$, then $\mu = p$ and (1.6) is exactly (1.8). So, the Bahri-Lions' existence theorem is a special case of our Theorem. Note that (1.3) is stronger than (1.8).

Ideas:

- We first study the corresponding even functional for the symmetric equation and we provide a precise estimate on the lower and upper growth of the Morse index of a sequence of sign-changing solutions. For that, we will introduce a suitable notion of linking.
- Based on this information together with a perturbation argument (on level subsets of the energy functional) and a very recent result on odd continuous extensions due to Castro and Clapp, we will construct a sequence of critical values with sign-changing critical points.

2. Perturbed Fourth order equations

Consider the fourth order problem:

$$\begin{cases} \Delta^2 u = g(x,u) + f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions satisfying (H_1)-(H_4) with $2 < p < 2N/(N-4)$ and $N \geq 5$. If further,

$(\mathbf{H_5})$ $g(x,s)$ and $f(x,s)$ are nondecreasing in s.

Theorem

(Ramos-Tavares-Zou, 2008) Under assumptions (H_1)-(H_5), if moreover $\dfrac{4p}{(N+4)(p-2)} > \dfrac{\mu}{\mu - \sigma}$, then (2.1) has an unbounded sequence of sign-changing solutions $u_n \in H^2(\Omega) \cap H_0^1(\Omega)$.

Theorem

(Mail Tool: Ramos-Tavares-Zou, 2008) For every pair of finite-dimensional subspaces $V \subset W$ of H with $\dim W = \dim V + 1$, every odd map $\varphi : V \to H$ and every $R > 0$ such that $\varphi(v) = v$ if $\|v\| \geq R$, there exist $\tilde{R} \geq R$ and an odd map $\tilde{\varphi} : W \to H$ which satisfy:

$$\tilde{\varphi}(v) = \varphi(v), \forall v \in V;$$

$$\tilde{\varphi}(w) = w, \forall w \in W : \|w\| \geq \tilde{R};$$

and

$$\sup_{\tilde{\varphi}(W)} (J + \lambda_0 I) \leq \alpha \sup_{\varphi(V)} (J + \lambda_0 I) + \beta,$$

where

$$I(u) := \frac{1}{2}\|u\|^2 - \frac{1}{p}\int_\Omega |u|^p,$$

$$J(u) = \frac{1}{2}\int_\Omega (\Delta u)^2 - \int_\Omega G(x,u) - \int_\Omega F(x,u).$$

Concerning the basic estimate on the Morse index, we have the following lemma.

Lemma

Given $2 < p < 2N/(N-4)$, let u be a solution of the problem

$$\Delta^2 u = |u|^{p-2}u, \quad u \in H,$$

with augmented Morse index greater than or equal to $k \in \mathbb{N}$. Then the corresponding energy level is greater than or equal to $Ck^{\frac{4p}{(N+4)(p-2)}}$, where C is a universal constant.

3. Strongly coupled elliptic systems

Theorem

(Ramos-Tavares-Zou, 2008) Assume $p, q > 2$ *and* $\frac{1}{p} + \frac{1}{q} > \frac{N-2}{N}$. *Then*

$$-\Delta u = |v|^{q-2}v, \quad -\Delta v = |u|^{p-2}u, \quad (u, v) \in H_0^1(\Omega) \times H_0^1(\Omega), \tag{3.1}$$

admits an unbounded sequence of sign-changing solutions (u_k, v_k), *in the sense that both* $(u_k + v_k)^+ \neq 0$ *and* $(u_k + v_k)^- \neq 0$ *for every* k.

For strongly coupled systems such as (3.1), we are merely aware of the following works:

- S. Angenent, R. van der Vorst: (Math. Z. 1999): obtained an unbounded sequence of solutions when
$$\frac{N}{2}(1-\frac{1}{p}-\frac{1}{q})<1. \tag{3.2}$$
- C. Tarsi (Adv. Nonl. Stud. 2007): Perturbation case:
$$-\Delta u=|v|^{q-2}v+k(x), \quad -\Delta v=|u|^{p-2}u+h(x), \tag{3.3}$$
where
$$\frac{1}{p}+\frac{1}{q}+\frac{p}{(p-1)q}>\frac{2N-2}{N}. \tag{3.4}$$
- Later, (3.4) was weakened by D. Bonheure and M. Ramos (2007, preprint):
$$\frac{N}{2}(1-\frac{1}{p}-\frac{1}{q})<\frac{p-1}{p}, \quad q\geq p>2. \tag{3.5}$$

These papers have no information on the property of sign-changingness.

We assume that $p \le q$, and we denote $f(s) = |s|^{p-2}s, g(s) = |s|^{q-2}s$.

The solutions of problem (3.1) are given by the critical points of the C^2 functional

$$I(u,v) = \langle u, v\rangle - \int_\Omega F(u) - \int_\Omega G(v), \quad (u,v) \in H_0^1(\Omega) \times H_0^1(\Omega),$$

where we have denoted $\langle u, v\rangle = \int_\Omega \langle \nabla u, \nabla v\rangle, F(u) = |u|^p/p, G(v) = |v|^q/q$.

We will use the even symmetric functional

$$J(u) := I(u + \psi_u, u - \psi_u), \quad u \in H_0^1(\Omega),$$

where ψ_u is defined by $I(u + \psi_u, u - \psi_u) = \max_{\psi \in H_0^1(\Omega)} I(u + \psi, u - \psi)$.

That is, ψ_u is the solution of the following equation in $H_0^1(\Omega)$,

$$-2\Delta\psi = g(u-\psi) - f(u+\psi), \quad \psi \in H_0^1(\Omega).$$

It turns out that $J \in C^2(H_0^1(\Omega);\mathbb{R})$ and

$$J'(u)\varphi = I'(u+\psi_u, u-\psi_u)(\varphi,\varphi), \quad \forall u,\varphi \in H_0^1(\Omega).$$

In particular, u is a critical point of J iff $(u+\psi_u, u-\psi_u)$ is a critical point of I, hence a solution of (3.1). In this way, to prove main Theorem, we are reduced ourselves in finding sign-changing critical points of the reduced functional J.

Combining Morse index, it yields sign-changing critical points u_k such that

$$c_1 k^{\frac{2pq}{(pq-p-q)N}} \leq J(u_k) \leq c_1 k^{\frac{2p}{N(p-2)}} + c_2,$$

with $c_1, c_2 > 0$.

- p or q may $\geq \frac{2N}{N-2}$.
- consider modified functional.

4. H. Brézis-L. Nirenberg Problem

4.1. Background

Consider the H. Brézis-L. Nirenberg problem (CPAM, 1983):

$$-\Delta u = \lambda u + |u|^{2^*-2}u, \qquad u \in H_0^1(\Omega), \tag{4.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N (N \geq 3)$, $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent and $\lambda > 0$.

(1) The pioneering paper on the equation (4.1) was due to H. Brézis-L. Nirenberg in 1983: The authors showed that for $N \geq 4$ and $\lambda \in (0, \lambda_1)$ problem (4.1) has at least one positive solution. In the sequel, λ_1 denotes the principal eigenvalue of $-\Delta$ on Ω. The same conclusion was proved in H. Brézis-L. Nirenberg for $N = 3$ when Ω is a ball and $\lambda \in (\lambda_1/4, \lambda_1)$. In this case, equation (4.1) has no radial solution when $\lambda \in (0, \lambda_1/4)$.

(2) The first multiplicity result was obtained in G. Cerami-D. Fortunato-M. Struwe (Ann. Inst. H. Poinvaré-Analyse, Nonlin., 1984), they proved that the number of solutions of (4.1) is bounded below by the number of eigenvalues of $(-\Delta, \Omega)$ lying in the open interval $(\lambda, \lambda + S|\Omega|^{-2/N})$, where S is the best constant for the Sobolev embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ and $|\Omega|$ is the measure of Ω.

(3) If $N \geq 4$ and Ω rotationally invariant, then for any $\lambda > 0$, problem (4.1) has infinitely many sign-changing solutions which are built using particular symmetry of the domain Ω (see Fortunato-Jannelli : Proc. Roy. Soc. Edinburgh, 1987). If $N = 3$, the number of solutions increases with λ.

(4) If $N \geq 7$ and Ω is a ball, then for each $\lambda > 0$, problem (4.1) has infinitely many sign-changing radial solutions (see S. Solimini: Ann. Inst. H. Poinvaré-Analyse, Nonlin., 1995). The symmetry of the ball$= \Omega$ plays an essential role and therefore, their methods are invalid for general domains.

(5) In G. Cerami-S. Solimini-M. Struwe (1986, JFA) it was proved for $N \geq 6$, (4.1) has two pairs of solutions on any smooth bounded domain for $\lambda \in (0, \lambda_1)$.

- Very recent results are:

(6) In F. V. Atkinson-H. Brézis-L. A. Peletier (JDE, 1990): Ω is a unit ball and $4 \leq N \leq 6$, this equation has no radial sign-changing solution for $\lambda \in (0, \lambda^*)$.

(7) In 2002 (Adv. Diff. Eq.), G. Devillanova-S. Solimini showed that, if $N \geq 7$, problem (4.1) has infinitely many solutions for each $\lambda > 0$.

(8) In 2005, M. Clapp-T. Weth (Adv. DE) got finitely many solutions to (4.1) for each $\lambda > 0$ and $N \geq 4$. These paper have no information on the sign-changingness of those solutions obtained there for $(\lambda < \lambda_1)$.

(9) G. Arioli, etc (NoDEA, 2008): $N = 4, \Omega = unit\ ball$. They study the bifurcation branch arising from the second radial eigenvalue of $-\Delta$. They also proved this equation does not admit a nontrivial radial solution when $\lambda = \lambda_1$.

4. 2. Main Theorem

A natural open question is whether (4.1) has infinitely many sign-changing solutions on each bounded domain Ω and for any $\lambda > 0$. The results of F. V. Atkinson-H. Brézis-L. A. Peletier (1990) suggest that the hypothesis $N > 6$ should be imposed since it is needed in the radial sign-changing case.

Theorem

(Schechter-Zou, 2008) If $N \geq 7$, the Brézis-Nirenberg problem has infinitely many sign-changing solutions.

- G. Devillanova-S. Solimini' Uniformly bounded theorem.

5. Perturbed H. Brézis-L. Nirenberg Problem

5.1. Background

A perturbed H. Brézis-L. Nirenberg problem:

$$-\Delta u = \lambda u + |u|^{2^*-2}u + \beta f(x,u), \quad u \in H_0^1(\Omega). \tag{5.1}$$

If $\beta = 0$, the H. Brézis-L. Nirenberg problem has infinitely many sign-changing solutions. However, what happens if $\beta \neq 0$ and $f \not\equiv 0$ (f is not odd)?

5.2. Main Theorems

Assume

(C) $f(x,t)t \geq 0$ for all $x \in \Omega$ and $t \in \mathbb{R}$; $f(x,t) = o(t)$ as $t \to 0$ uniformly in $x \in \Omega$; $|f(x,t)| \leq T_1$, $|f(x,t)t| \leq T_2$ for all $x \in \Omega$ and $t \in \mathbb{R}$, T_i are constants, $i = 1, 2$.

We shall prove the following theorem.

Theorem

(N. Hirano-Zou, 2008) Assume (C) and $N \geq 7, \lambda \in (0, \lambda_1)$. For any $m \in \mathbb{N}$, there is a $\beta_m > 0$ such that for each $\beta \in (0, \beta_m)$, the perturbed H. Brézis-L. Nirenberg Problem

$$-\Delta u = \lambda u + |u|^{2^*-2}u + \beta f(x,u), \qquad u \in H_0^1(\Omega), \tag{5.2}$$

has m distinct sign-changing solutions.

5.3. A Perturbation Method

Let $(E, \|\cdot\|)$ be a Banach space and $I \in \mathbb{C}^1(E, \mathbb{R})$. Denote $I^a := \{u \in E : I(u) \leq a\}$.

Definition

(N. Hirano-Zou, 2008) Let $a, b \in \mathbb{R}$ with $a \leq b$. The pair (I^b, I^a) is called a S-trivial pair if for any $\delta \in (0, \delta_0)$ and $\bar{\varepsilon} > 0$, there exist $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in (0, \bar{\varepsilon}), \varepsilon_3 > \varepsilon_2$, and a map $\eta_\delta : [0,1] \times E \to E$ satisfying:

(1) $\eta_\delta(0, u) = u, \quad \forall u \in E$;

(2) $\eta_\delta(t, \mathcal{D}_\delta^*) \subset \mathcal{D}_\delta^*$ for all $t \in [0,1]$;

(3) ($\eta_\delta(t, \cdot)|_{I^{a-\varepsilon_3}} = \mathbf{id}$ for all $t \in [0,1]$;

(4) $\eta_\delta(1, I^{b+\varepsilon_1}) \subset I^{a-\varepsilon_2} \cup D_\delta^*$;

(5) $\eta_\delta(t, u) = u$ for all $t \in [0,1]$ and $u \in \mathcal{K}$,

where $\mathcal{D}_\delta^*$ is an open convex set of E.

Theorem

(N. Hirano-Zou, 2008) Suppose that I satisfies the $(PS)_c$ condition for some $c \in \mathbb{R}$. Then c is a critical value of a sign-changing critical point of I if and only if (I^c, I^c) is not a S-trivial pair.

Definition

(N. Hirano-Zou, 2008) We call $c \in \mathbb{R}$ an S-essential value if (I^c, I^c) is not S-trivial.

Theorem

(N. Hirano-Zou, 2008) Let $c \in \mathbb{R}$ be an S-essential value of I. Then for every $\varepsilon > 0$, there is a $\beta > 0$ such that every $\mathbb{C}^1$-functional $J : E \to \mathbb{R}$ with $\sup\{|I(u) - J(u)| : u \in E\} < \beta$ has an S-essential value in $[c - \varepsilon, c + \varepsilon]$.

Remark: I don't know whether or not such a perturbation problem has infinitely many (nodal) solutions.

6. Pure Critical Exponent Problem

6.1. Background

We consider the pure critical exponent problem:

$$\begin{cases} -\Delta u = |u|^{2^*-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N, N \geq 3$ and 2^* is the critical Sobolev exponent. Equations of this type arise in fundamental questions in Differential Geometry like Yamabe's problem or the prescribed scalar curvature problem. It is well known that the existence of solutions depends on the topology of the domain Ω.

- Pohožaev (Soviet Math. Dokl.,1965) showed that (6.1) has no nontrivial solution if Ω is strictly star-shaped.

- If the domain Ω is an annulus

$$\Omega(a,b) := \{x \in \mathbf{R}^N : 0 < a < |x| < b\},$$

then it was shown there are infinitely many radial solutions to (6.1) (see Kazdan and Warner: CPAM, 1975).

- The first existence result for general domain is due to J. M. Coron (C. R. Avadamy of Sciences, Paries, Ser., 1984). He proved that (6.1) has one positive solution if $0 \notin \Omega$ and $\Omega(a,b) \subset \Omega$ with a/b small enough. Later A. Bahri and J. M. Coron (CPAM, 1988) obtained the same result for $b = 1$ and any $a > 0$. In fact they got at least one positive solution in every domain Ω having assumptions on the topology of Ω. See also M. Clapp-T. Weth (Calc. Var., 2004); E. Dancer (Bull. Lond. Math. Soc., 1988); W. Y. Ding (J. Partial Diff. Eq.,1989); D. Passaseo (Manus. Math., 1989 and Top. Meth. Nonl. Anal., 1994).

- M. Clapp and T. Weth (Comm. Contem. Math., in press) extended J. M. Coron' s results by showing that (6.1) has two solutions if Ω has a small enough hole. But the sign of the second solution have not been determined.

- If the domain has a certain symmetry and a hole whose boundary is a sphere, then the number of nodal solutions becomes arbitrarily large as the radius of the sphere goes to zero (see M. Musso and A. Pistoia: J. Math. Pure Appl., 2006).

- Very recently, M. Clapp and F. Pacella (Math. Z, 2008) studied (6.1) again, where they assumed that the domain Ω is annular-shaped:

$$0 \notin \Omega, \quad \{x \in \mathbf{R}^N : a < |x| < b\} \subset \Omega. \tag{6.2}$$

Moreover, Ω is invariant under the action of a closed subgroup of orthogonal linear transformations in $\mathbb{R}^N$. They establish the existence of multiple nodal solutions if either the subgroup is arbitrary and a/b is small enough or a/b is arbitrary and the minimal orbit of the group in Ω is large enough.

Summing up, to the best of our knowledge, the following problems about the multiple sign changing solutions to (6.1) have not been solved before:

- Estimates on the nodal domains,
- Precise estimates of the energies,
- Estimates on the Morse indices.

The present paper is devoted to solving these problems. Let $O(N)$ be the group of orthogonal linear transformations in $\mathbb{R}^N$ and $\mathcal{Z}$ is a closed subgroup. We denote by $\mathcal{Z}x := \{gx : g \in \mathcal{Z}\}$ the $\mathcal{Z}$-orbit of $x \in \Omega$ and by $\sharp\mathcal{Z}x$ its cardinality. Let $\Xi = \Xi(\mathcal{Z}) := \min\{\sharp\mathcal{Z}x : x \in \Omega\}$ be the minimal $\mathcal{Z}$-orbit of Ω. Recall that Ω is said to be $\mathcal{Z}$-invariant if $\mathcal{Z}x \in \Omega$ for any $x \in \Omega$, a function $u : \Omega \to \mathbb{R}$ is $\mathcal{Z}$-invariant if $u(gx) = u(x)$ for any $g \in \mathcal{Z}$ and $x \in \Omega$. Define

$$G(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{2^*}\int_\Omega |u|^{2^*} dx.$$

5.2. Main Theorems

Theorem

Assume (6.2). Then for any $k_0 \in \mathbb{N}$, there is an m_0 such that if $\Xi > m_0$, (6.1) has k_0 pairs of sign-changing solutions $\{\pm u_k\}$ satisfying

(1) $c_1 k \leq G(\pm u_k), k = 1, 2, ..., k_0$; *where c_1 is a constant independent of k;*

(2) *the augmented Morse index of u_k is $\geq k$;*

(3) *the number of $\mathcal{Z}$-symmetric nodal domain of u_k is $\leq k$.*

Theorem

Assume (6.2). If $\Xi = \infty$, then (6.1) has infinitely many sign-changing solutions $\{\pm u_k\}$ satisfying

(1) $c_1 k \leq G(u_k) \leq c_2 k^{\frac{N}{N-m}}, k = 1, 2, ...,$ *where c_i $(i = 1, 2)$ is a constant independent of k;*

(2) *the augmented Morse index of u_k is $\geq k$;*

(3) *the number of $\mathcal{Z}$-symmetric nodal domain of u_k is $\leq k$.*

6.3. A Variational Principle

Let $(E, \langle\cdot,\cdot\rangle)$ be a Hilbert space with the corresponding norm $\|\cdot\|$, $\mathcal{P} \subset E$ be a closed convex (positive) cone.
Let Z be a subspace of E with $E = Z^{\perp} \oplus Z$, $\dim Z^{\perp} = k-1$. The nontrivial elements of Z are sign-changing. We assume that $\mathcal{P}$ is weakly closed in the sense that $\mathcal{P} \ni u_k \rightharpoonup u$ weakly in $(E, \|\cdot\|)$ implies $u \in \mathcal{P}$.
Suppose that there is another norm $\|\cdot\|_*$ of E such that $\|u\|_* \leq C_* \|u\|$ for all $u \in E$, here $C_* > 0$ is a constant. Moreover, we assume that $\|u_n - u^*\|_* \to 0$ whenever $u_n \rightharpoonup u^*$ weakly in $(E, \|\cdot\|)$. Let $G \in \mathbf{C}^1(E, \mathbb{R})$ be an even functional and the gradient G' be of the form $G'(u) = u - \mathcal{G}(u)$, where $\mathcal{G} : E \to E$ is a continuous operator, $\mathcal{D}$ is an open convex neighborhood of $\mathcal{P}$, set $\mathcal{S} = E \backslash \mathcal{D}$. We make the following assumptions.

(A_1) $\mathcal{G}(\pm\mathcal{D}) \subset \pm\mathcal{D}$; K$\cap\big(E\backslash\{-\mathcal{P}\cup\mathcal{P}\}\big) \subset \mathcal{S}$.

(A_2) For any $a, b > 0$, $G^a \cap \{u \in E : \|u\|_* = b\}$ is bounded in $(E, \|\cdot\|)$.

(A_3) There exists a $\rho > 0$ such that $\inf_{u\in Z, \|u\|_*=\rho} G > -\infty$.

(A_4) $\lim_{u\in Y,\ \|u\|\to\infty} G(u) = -\infty$, for any subspace $Y \subset E$ with $\dim Y < \infty$.

In applications, conditions (A_1)-(A_4) are satisfied readily.

Definition

The functional G is said to satisfy the (w^*-PS) condition on $[a, b]$ if for any sequence $\{u_n\}$ such that $G(u_n) \to c \in [a, b]$ and $G'(u_n) \to 0$, we have either $\{u_n\}$ is bounded and has a convergent subsequence or $\|G'(u_n)\|\|u_n\| \to \infty$.

Theorem

Assume (A_1)-(A_4) hold and $G \in \mathbf{C}^1(E, \mathbb{R})$ is even. If there is a $\lambda_0 > 0$ such that G satisfies a weak compactness condition on $(-\infty, \beta + \lambda_0]$, then G has a sign-changing critical points u^* with

$$G(u^*) \le \beta := \inf_{\substack{Y \subset E, \\ k \le \dim Y < \infty}} \sup_Y G.$$

If further G is of $\mathbf{C}^2$, then the augmented Morse index of u^* is $\ge k$.

Remark: In the paper by T. Bartsch-K. C. Chang-Z. Wang (Math. Z, 2000), where two different estimates for the Morse indices of sign-changing solutions were obtained: one concerns a sign-changing solution of mountain pass type with Morse index less than or equal to 1; another one concerns a possibly degenerate critical point having Morse index 2.

Thank You!