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Understanding iterates of closed geodesics

Dedicated to Professor Paul Rabinowitz on the occasion of his 70th birthday

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Definition M - a manifold, $dim M < +\infty$. If $F : TM \to [0, +\infty)$ satisfies

(F1) F is C^∞ on $TM \setminus \{0\}$,

(F2) $F(x, \lambda y) = \lambda F(x, y)$, $\forall\, y \in T_xM$, $x \in M$, $\lambda > 0$, ($\Longrightarrow$ *length*(c) is well-defined).

(F3) $\forall y \in T_xM \setminus \{0\}$, the following quadratic form is positive definite,

$$g_{x,y}(u, v) \equiv \frac{1}{2}\frac{\partial^2}{\partial s \partial t} F^2(x, y + su + tv)|_{t=s=0}, \qquad \forall u, v \in T_xM,$$

($\Longrightarrow$ Local existence and uniqueness of geodesics connecting nearby points)

then $F : TM \to [0, +\infty)$ is a *Finsler metric*, (M, F) is a *Finsler manifold*.

F is *reversible*, if $F(x, -y) = F(x, y)$, $\forall y \in T_xM, x \in M$.

F is *Riemannian*, if $F(x, y)^2 = \frac{1}{2} g(x) y \cdot y$, for some symmetric pos. def. matrix function $g(x) \in GL(T_xM)$ depending on $x \in M$ smoothly.

$c : \mathbf{R} \to (M, F)$ is a closed geodesic, if

(i) it is a closed curve, and

(ii) it is always locally the shortest curve.

i.e., $length(c|_{p,q})$ =distance from p to q for nearby p and q on c.

Closed geodesics—Global questions on Existence, Multiplicity, Stability.

Let $c : S^1 = \mathbf{R}/\mathbf{Z} \to (M, F)$ be a closed geodesic.

$$c^m(t) = c(mt), \quad \forall \theta \in \mathbf{R},\ t \in \mathbf{R}.$$

c is prime, if $c \neq d^m$ for any CG d and $m \geq 2$.

F is an irrev. Finsler metric on M.

Prime c and d on (M, F) are distinct, if $c(t) \neq d(t + \theta)$, $\forall t$ and $\theta \in [0, 1]$.

F is an rev. Finsler (Riemannian) metric on M.

Closed geodesics c and d on (M, g) are geometrically distinct, if $c(\mathbf{R}) \neq d(\mathbf{R})$.

$\mathrm{CG}(M, F) = \{$distinct/geom. distinct CGs on (M, F) when F is irrev./rev. Finsler$\}$

Existence of at least one closed geodesic ($\mathrm{CG}(M, F)$):

1898, Hadamard, 1905 Poincaré.

1917-1927, Birkhoff: ${}^{\#}\mathrm{CG}(S^d, g) \geq 1$, $\forall$ Riemannian g on S^d.

1951, Lyusternik-Fet: ${}^{\#}\mathrm{CG}(M, g) \geq 1$, $\forall$ Riemannian g on a compact M.

Variational method $\Longrightarrow$

${}^{\#}\mathrm{CG}(M, F) \geq 1$, $\forall$ Finsler metric F on a compact manifold M.

Question: Estimate ${}^{\#}\mathrm{CG}(M, F)$ or ${}^{\#}\mathrm{CG}(M, g)$?

For a compact manifold M, define its Betti numbers via its free loop space $\Lambda M = \{c \in W^{1,2}(S^1, M) \mid c \text{ is abs. contin.}\}$:

$$b_k(M) = \dim H_k(\Lambda M; \mathbf{Q}), \qquad \forall\, k \in \mathbf{Z}.$$

1969, Gromoll-Meyer: (M, g) compact, $\dim M \geq 2$, g is Riemannian. Then

$\{b_j(M)\}_{j\in\mathbf{N}}$ is unbounded $\Longrightarrow$ ${}^{\#}\mathrm{CG}(M, g) = +\infty$.

1976, Vigué-Poirrier and Sullivan: (M, g) is a cpt. simply conn. Riem. mfd. Then

$\{b_k(M)\}_{k\in\mathbf{N}}$ is bounded $\Longleftrightarrow$ $H^*(M; \mathbf{Q})$ has only one generator.

$\Longleftrightarrow$ $H^*(M; \mathbf{Q}) \cong T_{d,h+1}(x) = \mathbf{Q}[x]/(x^{h+1} = 0)$

with a generator x of degree $d \geq 2$ and hight $h + 1 \geq 2$.

1980, Matthias: These two theorems work for (rev. or irrev.) Finsler manifolds too.

Most interesting manifolds $--$ spheres !

Known multiplicity results for Riemannian S^d:

1968, Klingenberg; 1982, Ballmann-Thorbergsson-Ziller:

$$1/4 \le K_g \le 1 \Longrightarrow {}^{\#}\mathrm{CG}(S^d, g) \ge d.$$

1965, Fet: (M, F)-cpt, bumpy, revers. Finsler manifold, $\Longrightarrow {}^{\#}\mathrm{CG}(M, F) \ge 2$.

Bumpy, i.e., all the closed geodesics (with their iterates) are non-degenerate.

1990, Bangert, Franks: ${}^{\#}\mathrm{CG}(S^2, g) = +\infty$, $\forall$ Riemannian metric g on S^2.

Others: Bangert, Hingston, Ballmann, Wang-Long, Wang, ...

Conjecture: ${}^{\#}\mathrm{CG}(M, g) = +\infty$ $\forall$ Riemannian metric g on every compact manifold M.

1973, Katok's metric on S^d:

$$^{\#}\mathrm{CG}(S^d, F_{Katok}) = 2[\frac{d+1}{2}], \qquad ^{\#}\mathrm{CG}(S^2, F_{Katok}) = 2,$$
$$^{\#}\mathrm{CG}(S^3, F_{Katok}) \quad = \quad ^{\#}\mathrm{CG}(S^4, F_{Katok}) = 4.$$

where $[a] = \max\{k \in \mathbf{Z} \,|\, k \leq a\}$ for any real number a.

2003, Hofer-Wysocki-Zehnder on S^2: $\quad ^{\#}\mathrm{CG}(S^2, F) = 2 \quad \text{or} \quad +\infty$,

provided the irrev. F is bumpy, and all stable and unstable mfds intersect transversally at every hyperbolic closed geodesics.

2005, Bangert-Long on S^2: $\quad ^{\#}\mathrm{CG}(S^2, F) \geq 2 \qquad \forall$ irrev. Finsler F.

Others: Rademacher, Wang, Duan, Long, ...

Multiplicity results

Theorem (Duan-Long, JDE 2007, Rademacher TAMS, 2008)

(S^d, F)-compact irreversible Finsler, F is bumpy. $\Longrightarrow$ $^{\#}\mathrm{CG}(S^d, F) \geq 2$.

Theorem (Rademacher, 2008)

$(\mathbf{C}P^2, F)$-irreversible Finsler, F is bumpy. $\Longrightarrow$ $^{\#}\mathrm{CG}(\mathbf{C}P^2, F) \geq 2$.

A natural conjecture for compact manifold M with $\dim M = n$:

There exist integers $0 < p_n < q_n < +\infty$ such that $p_n \to +\infty$ as $n \to +\infty$,

$$^{\#}\mathrm{CG}(M, F) \in [p_n, q_n] \cup \{+\infty\}, \qquad \forall \text{ irrev. Finsler metric } F \text{ on } M.$$

Known: $p_2 = 2$. $(p_3 \geq 2, \quad p_4 \geq 2)$.

Conjecture: $q_2 = 2$? $q_n(S^n) = 2[\frac{n}{2}]$?

New multiplicity results:

Theorem (Long-Duan, AIM 2009): M is cpt, simply conn. $\dim M = 3$,

(i) ${}^{\#}\mathrm{CG}(M, F) \geq 2$, $\forall$ irrev. Finsler F on M.

(ii) ${}^{\#}\mathrm{CG}(M, g) \geq 2$, $\forall$ rev. Finsler (Riemannian) g on M.

Theorem (Duan-Long, 2009): M is cpt, simply conn. $\dim M = 4$,

(i) ${}^{\#}\mathrm{CG}(M, F) \geq 2$, $\forall$ irrev. Finsler F on M.

(ii) ${}^{\#}\mathrm{CG}(M, g) \geq 2$, $\forall$ rev. Finsler (Riemannian) g on M.

Ideas of our study

Let (M, F) be a compact simply conn. irrev./rev. Finsler (Riemannian) manifold.

Gromoll-Meyer + Vigue-Sullivan + $^{\#}\mathrm{CG}(M, F) < +\infty \Longrightarrow$

$$H^*(M; \mathbf{Q}) \cong T_{d,h+1}(x) = \mathbf{Q}[x]/(x^{h+1} = 0)$$

with a generator x of degree $d \geq 2$ and hight $h + 1 \geq 2$.

Main (not all) examples: the compact rank one symmetric spaces, i.e.,

Spheres S^d of dimension d with $h = 1$,

Complex projective spaces $\mathbf{C}P^h$ of dimension $2h$ with $d = 2$,

Quaternionic projective spaces $\mathbf{H}P^h$ of dimension $4h$ with $d = 4$,

Cayley plane $\mathbf{C}aP^2$ of dimension 16 with $d = 8$ and $h = 2$.

Suggestions from Morse's works on ellipsoids:

Theorem (Morse 1934). *Let E_d be a d-dim. ellipsoid in $\mathbf{R}^{d+1}$. For any given $N \in \mathbf{N}$, every closed geodesic c which is not an iterate of some main ellipse must have Morse index satisfying $i(c) \geq N$, provided all the semi-axis of E_d are close to 1 enough.*

Consequently, all the global homologies of the free loop space on M at dimensions less than N are generated by iterates of the main ellipses only.

Let $b_j(M) = \dim H_j(\overline{\Lambda}M, \overline{\Lambda}^0 M; \mathbf{Q}) = \dim H_j(\Lambda M/S^1, \Lambda^0 M/S^1; \mathbf{Q})$ for all $j \geq 0$.

S^4,	$j:$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	$\cdots$,
	$b_j:$	0	0	0	1	0	1	0	1	0	2	0	1	0	1	0	2	0	$\cdots$.

Need to understand properties of higher iterations of each single CG.

Need to understand properties of higher iterations of each single CG.

Need to try the homological level.

First things first:

Understand Morse indices and local homologies of iterates of each single prime CG.

By product: The existence of at least two CGs on a certain manifold.

The variational structure for CGs. Define

$$
\begin{aligned}
&E(\gamma) = \int_0^1 F(\dot{\gamma}(t))^2 dt, \qquad \forall\, \gamma \in \Lambda \equiv H^1(S^1, M),\\
&\Lambda(c^m) = \{\gamma \in \Lambda \mid E(\gamma) < E(c^m)\}, \quad \Lambda^\kappa = \{\gamma \in \Lambda \mid E(\gamma) \le \kappa\},\\
&\overline{C}_q(E, c^m) \equiv H_q\left((\Lambda(c^m) \cup S^1 \cdot c^m)/S^1, \Lambda(c^m)/S^1\right)\\
&\qquad = \left(H_{i(c^m)}(U^-_{c^m} \cup \{c^m\}, U^-_{c^m}) \otimes H_{q-i(c^m)}(N^-_{c^m} \cup \{c^m\}, N^-_{c^m})\right)^{+\mathbf{Z}_m}\\
&\qquad = H_{q-i(c^m)}(N^-_{c^m} \cup \{c^m\}, N^-_{c^m})^{\epsilon(c^m)\mathbf{Z}_m},
\end{aligned}
$$

where $\epsilon(c^m) = (-1)^{i(c^m)-i(c)}$. Let $\kappa_m = E(c^m) = m^2 E(c) > 0$ for all $m \ge 1$. Then

$$
\begin{aligned}
&\kappa_0 \equiv 0 < \kappa_1 < \kappa_2 < \cdots < \kappa_m < \kappa_{m+1} < \cdots,\\
&\kappa_m \to +\infty \ \text{ as } \ m \to +\infty,\\
&\hat{i}(c) > 0, \quad i(c^m) \to +\infty \ \text{ as } \ m \to +\infty.
\end{aligned}
$$

We write $\overline{\Lambda}^m = \overline{\Lambda}^{\kappa_m} = \Lambda^{\kappa_m}/S^1 = \{\gamma \in \Lambda \mid E(\gamma) \le \kappa_m\}/S^1$.

Ideas to study iterates of only one closed geodesic:

Global homologies $H_j = H_j(\overline{\Lambda} M, \overline{\Lambda}^0 M; \mathbf{Q})$

with $j \geq 0$ of the free loop space pair $(\overline{\Lambda} M, \overline{\Lambda}^0 M)$ (Example: $d = 4$ and $h = 1$):

$+\infty$																						
.																						
.																						
$\dim H_j$	0	0	0	1	0	1	0	1	0	2	0	1	0	1	0	2	0	1	0	1	0	$\cdots$
.																						
.																						
0																						
j in H_j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	$\cdots$

Local homologies of only one CG c **Distribution diagram** of $\dim \overline{C}_j(E, c^m)$.

κ_{2n}													$*$
κ_{2n-1}										$*$	$\cdots$	$\cdots$	$\cdots$
$\cdots$									$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
κ_{n+1}								$*$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$*$
κ_n						$*$	$*$	$\cdots$	$*$	$\cdots$	$*$	$*$	
κ_{n-1}			$*$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$*$					
$\cdot$		$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$						
κ_1	$*$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$*$							
m in c^m	$\overline{C}_0$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\overline{C}_{i(c^n)}$	$\cdots$	$\overline{C}_{i(c^n)+\mu}$	$\overline{C}_{i(c^n)+\mu+1}$	$\overline{C}_{i(c^n)+\mu+2}$	$\cdots.$		

j in $\overline{C}_j$

Question: Can local homologies generate global homologies ?

Ideas to study iterates of only one closed geodesic:

1. Classify CGs into two classes: rational and irrational.

   Assume $^{\#}(M, F) = 1$ with $\dim M \geq 2$ and the prime CG c is rational (or rational):

2. Morse indices of iterations of rational or irrational CGs.

3. Local homological properties of only one CG c.

4. Rademacher identity $\Longrightarrow$ local and global relations.

5. Morse theory $\Longrightarrow$ local and global relations.

4 & 5 $\Longrightarrow$ Contradiction !

Classify CGs: Rational and irrational CGs

Definition(Long 1999) For $M \in \mathrm{Sp}(2n)$,

$$\Omega(M) = \{N \in \mathrm{Sp}(2n) \quad | \quad \sigma(N) \cap \mathbf{U} = \sigma(M) \cap \mathbf{U}, \text{ and}$$
$$\nu_\lambda(N) = \nu_\lambda(M) \; \forall \, \lambda \in \sigma(M) \cap \mathbf{U}\},$$

where $\nu_\lambda(M) = \dim_{\mathbf{C}} \dim_{\mathbf{C}}(M - \lambda I)$. We call the path conn. component $\Omega^0(M)$ of $\Omega(M)$ containing M the *homotopy component* of M in $\mathrm{Sp}(2n)$. Denote by $N \approx M$, if $N \in \Omega^0(M)$.

$[M] \;\equiv\; \{P^{-1}MP \, |; P \in \mathrm{Sp}(2n)\} \subset \Omega^0(M)$!

Purpose: $\gamma \sim \beta$ and $\gamma(\tau) \approx \beta(\tau) \Longrightarrow i_1(\gamma^m) = i_1(\beta^m)$ for all $m \geq 1$.

Theorem.(Long 1999) For any $M \in \mathrm{Sp}(2n)$, $\exists$ basic normal form decomposition

$$M \approx M_1 \diamond M_2 \diamond \cdots \diamond M_k.$$

Basic normal forms:

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \ \begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}, \ R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \ H(a) = \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix},$$

where $b = 0, \pm 1, \ a \in \mathbf{R} \setminus \{0, \pm 1\}; \ \theta \in \mathbf{R}$,

$$N(\theta, B) = \begin{pmatrix} R(\theta) & B \\ 0 & R(\theta) \end{pmatrix}, \text{ where } B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}.$$

$N(\theta, B)$ is trivial, if $(b_2 - b_3)\sin\theta > 0$; is non-trivial, otherwise.

Notation: the direct sum of two symplectic matrices:

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \diamond \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}.$$

Let c be a CG on a Finsler (M, F) with $\dim M = d$.

The linearized Poincare map of c: $P_c \in \mathrm{Sp}(2d-2)$ s.t.

$$P_c \approx M_1 \diamond \cdots \diamond M_k.$$

c is **irrational**, if $\exists \ \geq 1 \ \ M_j = R(\theta) \equiv \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ with $\theta/\pi \in \mathbf{R} \setminus \mathbf{Q}$,

c is **rational**, otherwise.

The **analytical period** $n(c)$ of a closed geodesic c is defined by

$$n(c) = \min\{k \in \mathbf{N} \,|\, \nu(c^k) = \max_{m \geq 1} \nu(c^m), \ \ i(c^{m+k}) - i(c^m) \in 2\mathbf{Z}, \ \ \forall\, m \in \mathbf{N}\}.$$

Theorem (Long, 2000) *Let c be a CG on (M,F) with linearized Poincaré map P_c. Then*

$$\begin{aligned}
P_c \ \approx\ & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{\diamond p_-} \diamond I_{2p_0} \diamond \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{\diamond p_+} \\
& \diamond \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}^{\diamond q_-} \diamond I_{2q_0} \diamond \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}^{\diamond q_+} \\
& \diamond R(\theta_1) \diamond \cdots \diamond R(\theta_r) \\
& \diamond \begin{pmatrix} R(\alpha_1) & A_1 \\ 0 & R(\alpha_1) \end{pmatrix} \diamond \cdots \diamond \begin{pmatrix} R(\alpha_{r_*}) & A_{r_*} \\ 0 & R(\alpha_{r_*}) \end{pmatrix} \quad \{\text{non} - \text{trivial ones}\} \\
& \diamond \begin{pmatrix} R(\beta_1) & B_1 \\ 0 & R(\beta_1) \end{pmatrix} \diamond \cdots \diamond \begin{pmatrix} R(\beta_{r_0}) & B_{r_0} \\ 0 & R(\beta_{r_0}) \end{pmatrix} \quad \{\text{trivial ones}\} \\
& \diamond \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}^{\diamond h_+} \diamond \begin{pmatrix} -2 & 0 \\ 0 & -1/2 \end{pmatrix}^{\diamond h_-},
\end{aligned}$$

and there hold

$$\begin{aligned} i(c^m) &= m(i(c)+p_-+p_0-r)+2\sum_{j=1}^{r}E(\frac{m\theta_j}{2\pi})-r \\ &\quad -p_- - p_0 - \frac{1+(-1)^m}{2}(q_0+q_+) \\ &\quad +2(\sum_{j=1}^{r_*}\varphi(\frac{m\alpha_j}{2\pi})-r_*), \\ \nu(c^m) &= \nu(c)+\frac{1+(-1)^m}{2}(q_-+2q_0+q_+)+2\zeta(m,P_c), \\ \hat{i}(c) &= \lim_{m\to+\infty}\frac{i(c^m)}{m} = i(c)+p_-+p_0-r+\sum_{j=1}^{r}\frac{\theta_j}{\pi}, \end{aligned}$$

where we denote by

$$\zeta(m,P_c)=r-\sum_{j=1}^{r}\varphi(\frac{m\theta_j}{2\pi})+r_*-\sum_{j=1}^{r_*}\varphi(\frac{m\alpha_j}{2\pi})+r_0-\sum_{j=1}^{r_0}\varphi(\frac{m\beta_j}{2\pi}),$$

$E(a)=\min\{k\in\mathbf{Z}\,|\,k\ge a\}$, $[a]=\max\{k\in\mathbf{Z}\,|\,k\le a\}$, $\varphi(a)=E(a)-[a]$ *for* $a\in\mathbf{R}$.

Morse indices of iterates of rational CGs

Theorem. *Let c be a rational prime closed geodesic on a d-dimensional Finsler manifold (M, F). Let $n = n(c)$ be the analytical period of c. Then the following conclusions hold:*

(A) (Periodicity) For all $m \in \mathbf{N}$,

$$i(c^{m+n}) = i(c^m) + i(c^n) + p(c), \qquad \nu(c^{m+n}) = \nu(c^m),$$

where $p(c) = p_0(P_c) + p_-(P_c) + q_0(P_c) + q_+(P_c) + r(P_c) + 2r_*(P_c)$.

(B) (Boundedness) For all $1 \le m < n$,

$$i(c^m) + \nu(c^m) \le i(c^n) + p(c) + d - 3, \qquad \nu(c^m) = \nu(c^{n-m}).$$

(C) (Period-mean index) $n\,\hat{i}(c) = i(c^n) + p(c)$, *where* $\hat{i}(c) = \lim_{m\to\infty} i(c^m)/m$.

(D) (Relative parity) $i(c^n) = p(c) \mod 2$.

(E) (Nullity-periodicity) $\nu(c^n) \le p(c) + d - 1$.

Homologies of level sets related to only one rational CG c

Distribution diagram of $\dim \overline{C}_j(E, c^m)$. $(\beta = i(c^n))$

κ_{2n}													0	$*$
κ_{2n-1}										$*$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$									$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
κ_{n+1}								$*$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$*$	
κ_n						0	$*$	$\cdots$	$*$	$\cdots$	$*$	0		
κ_{n-1}			$*$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$*$						
$\cdot$		$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$							
κ_1	$*$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$*$								
m in c^m	$\overline{C}_0$	$\cdots$	$\overline{C}_{\beta-1}$	$\overline{C}_\beta$	$\cdots$	$\overline{C}_{\beta+p(c)}$	$\cdots$	$\overline{C}_{\beta+\mu}$	$\overline{C}_{\beta+\mu+1}$	$\overline{C}_{\beta+\mu+2}$	$\cdots$.			

New Identity Theorem. *(M, F) is a Finsler mfd with $H^*(M;\mathbf{Q}) \cong T_{d,h+1}(x) = \mathbf{Q}[x]/(x^{h+1}=0)$ and a generator x of degree $d \geq 2$ and hight $h+1 \geq 2$* and ${}^{\#}\mathrm{CG}(M,F) = 1$. The only prime CG c is rational with $n = n(c)$ being the analytical period.

Suppose that there exists an integer $\mu \geq -1$ ($\mu = p(c) + dh - 3$ in appl.)satisfying:

$$i(c^{m+n}) = i(c^n) + i(c^m) + p(c), \qquad \forall\, m \geq 1,$$

$$i(c^m) + \nu(c^m) \leq i(c^n) + \mu, \qquad \forall\, 1 \leq m < n,$$

$$H_j(N^- \cup \{c^n\}, N^-)^{\epsilon(c^n)\mathbf{Z}_{m(c)}} = 0, \qquad \forall\, j \geq i(c^n) + \mu + 2,$$

$$H_j(\overline{\Lambda}, \overline{\Lambda}^{\kappa_n}) = 0, \qquad \forall\, j = i(c^n) + \mu + 1.$$

$\Rightarrow$ $\exists$ an integer $\kappa \geq 0$ such that

$$B(d,h)(i(c^n) + p(c)) + (-1)^{i(c^n)+\mu}\kappa = \sum_{j=\mu-p(c)+1}^{i(c^n)+\mu} (-1)^j b_j,$$

where $B(d,h) = -\frac{h(h+1)d}{2d(h+1)-4}$ if $d \in 2\mathbf{N}$, $B(d,h) = \frac{d+1}{2(d-1)}$ if $d \in 2\mathbf{N} - 1$.

Idea of the proof of the new identity:

(1) **Rademacher identity, 1992, special case** *(M, F) is a Finsler mfd with $H^*(M;\mathbf{Q}) \cong T_{d,h+1}(x)$. There exists only one CG c with $n = n(c)$ and $\hat{i}(c_j) > 0$. Then*

$$\begin{aligned} n\hat{i}(c)B(d,h) &= \sum_{\substack{1\le m\le n(c)\\ 0\le l\le 2d-2}} (-1)^{i(c^m)+l} k_l^{\epsilon(c^m)}(c^m) \\ &= \sum_{\substack{1\le m\le n(c)\\ 0\le l\le 2d-2}} (-1)^{i(c^m)+l} \dim H_l(N_{c^m}^- \cup \{c^m\}, N_{c^m}^-)^{\epsilon(c^m)\mathbf{Z}_m} \\ &= \sum_{j=0}^{i(c^n)+p(c)+d-3} (-1)^j u_j, \end{aligned}$$

where $\epsilon(c^m) = (-1)^{i(c^m)-i(c)}$ and $H_j(\overline{\Lambda}^{\kappa_n}, \overline{\Lambda}^0) = \mathbf{Q}^{u_j}$.

(2) For $j \in \mathbf{Z}$, denote by

$$B_j = H_j(\overline{\Lambda}, \overline{\Lambda}^0) = \mathbf{Q}^{\hat{b}_j}, \quad C_j = H_j(\overline{\Lambda}, \overline{\Lambda}^{\kappa_n}) = \mathbf{Q}^{c_j}.$$

Then the long exact sequence of $(\overline{\Lambda}, \overline{\Lambda}^{\kappa_n}, \overline{\Lambda}^0)$ yields the following: $(\mu = p(c) + dh - 3)$

$$0 = \sum_{j=0}^{i(c^n)+\mu} (-1)^j (u_j - b_j + c_j).$$

Go through the long exact sequences for all the triples $(\overline{\Lambda}^{\kappa_m}, \overline{\Lambda}^{\kappa_{m-1}}, \overline{\Lambda}^0)$ with $m = n, n-1, \ldots, 2$ in the diagram for the distributions of $\dim \overline{C}_q(E, c^m)$.

Thus we obtain

$$n\hat{i}(c)B(d,h) = (-1)^{i(c^n)+\mu} u_{i(c^n)+\mu} - \sum_{j=0}^{i(c^n)+\mu} (-1)^j (b_j - c_j).$$

Claim: $c_j = b_{j-i(c^n)-p(c)} \qquad \forall\, j \in \mathbf{Z}.$ (∗)

$\Longrightarrow \quad B(d,h)(i(c^n) + p(c)) + (-1)^{i(c^n)+\mu}\kappa = \sum_{j=\mu-p(c)+1}^{i(c^n)+\mu} (-1)^j b_j.$

Proof of $(*)$

Isomorphism Theorem. *$M = (M, F)$ is a Finsler manifold with ${}^{\#}\mathrm{CG}(M, F) = 1$,*

the prime CG c is rational, with $n = n(c)$ being the analytical period of c.

Then for any non-negative integers $b > a$ and $h \in \mathbf{Z}$,

$\exists$ a chain map f on singular chains which induces an isomorphism

$$f_* : H_h(\overline{\Lambda}^{\kappa_b}, \overline{\Lambda}^{\kappa_a}) \xrightarrow{\cong} H_{h+i(c^n)+p(c)}(\overline{\Lambda}^{\kappa_{n+b}}, \overline{\Lambda}^{\kappa_{n+a}}).$$

Difficulty: $\kappa_a < \kappa_{a+1} < \cdots < \kappa_{b-1} < \kappa_b$ for $b - a > 1$!

Idea of the proof of the Isomorphism Theorem. For any $h \in \mathbf{Z}$ let

$$A_h = H_h(\overline{\Lambda}^{\kappa_b}, \overline{\Lambda}^{\kappa_{b-1}}), \qquad B_h = H_h(\overline{\Lambda}^{\kappa_{b-1}}, \overline{\Lambda}^{\kappa_a}),$$

$$A'_h = H_h(\overline{\Lambda}^{\kappa_{n+b}}, \overline{\Lambda}^{\kappa_{n+b-1}}), \qquad B'_h = H_h(\overline{\Lambda}^{\kappa_{n+b-1}}, \overline{\Lambda}^{\kappa_{n+a}}).$$

For any $h \in \mathbf{Z}$, letting $\bar{h} = h + i(c^n) + p(c)$, we consider the following diagram:

$$\begin{array}{ccccccccc}
A_{h+1} & \xrightarrow{\partial_{h+1*}} & B_h & \xrightarrow{i_{h*}} & H_h(\overline{\Lambda}^{\kappa_b}, \overline{\Lambda}^{\kappa_a}) & \xrightarrow{j_{h*}} & A_h & \xrightarrow{\partial_{h*}} & B_{h-1} \\
\downarrow f^{b,b-1}_{h+1*} & & \downarrow f^{b-1,a}_{h*} & & \downarrow f^{b,a}_{h*} & & \downarrow f^{b,b-1}_{h*} & & \downarrow f^{b-1,a}_{h-1*} \\
A'_{\bar{h}+1} & \xrightarrow{\partial'_{\bar{h}+1*}} & B'_{\bar{h}} & \xrightarrow{i'_{\bar{h}*}} & H_{\bar{h}}(\overline{\Lambda}^{\kappa_{n+b}}, \overline{\Lambda}^{\kappa_{n+a}}) & \xrightarrow{j'_{\bar{h}*}} & A'_{\bar{h}} & \xrightarrow{\partial'_{\bar{h}*}} & B'_{\bar{h}-1}.
\end{array}$$

Case of $b - a = 1$

+ 5-lemma (Comm. at far squares) $\Longrightarrow$ Middle iso. $\Longrightarrow$ Comm. at mid. squares !

+ Induction arguments

Homologies of level sets related to only one rational CG c

Corollary. (M, F) *is a Finsler mfd with* ${}^{\#}\mathrm{CG}(M, F) = 1$,

c is the unique rational prime CG with $n = n(c)$ being the analytical period.

Then for all $j \in \mathbf{Z}$ there holds

$$c_j = \dim H_j(\overline{\Lambda}, \overline{\Lambda}^{\kappa_n}) = \dim H_j(\overline{\Lambda}, \overline{\Lambda}^0) = b_{j-i(c^n)-p(c)}. \qquad (*)$$

On cpt simply conn. irr/re Finsler mfd (M,F) with with $H^*(M;\mathbf{Q})\cong T_{d,h+1}(x)$ and ${}^{\#}\mathrm{CG}(M,F)=1$. Denote this CG by c.

Let $\mu=p(c)+\dim M-3$, Identity Theorem $\Longrightarrow$

$$B(d,h)(i(c^n)+p(c))+(-1)^{i(c^n)+\mu}\kappa=\sum_{j=\dim M-2}^{i(c^n)+\mu}(-1)^j b_j. \qquad (Identity)$$

$\Longrightarrow$

$$i(c^n)+p(c)\le\frac{1}{|B(d,h)|}\sum_{j=\dim M-2}^{i(c^n)+\mu}(-1)^j b_j<i(c^n)+p(c).$$

$\Longrightarrow$ Contradiction ! i.e., c can not be rational!

Theorem Cpt simply conn. irr/re Finsler mfd (M,F), ${}^{\#}\mathrm{CG}(M,F)=1$

$\Longrightarrow$ c can be neither rational nor complete non-degenerate !

For every integer $k \geq d-1+(h-1)d = hd-1$, letting $D = d(h+1)-2$, we have

$$\sum_{q=0}^{k} b_q = \frac{h(h+1)d}{2D}(k-(d-1)) - \frac{h(h-1)d}{4} + 1 + \epsilon_{d,h}(k),$$

where

$$\begin{aligned}\epsilon_{d,h}(k) &= \{\frac{D}{hd}\{\frac{k-(d-1)}{D}\}\} - (\frac{2}{d} + \frac{d-2}{hd})\{\frac{k-(d-1)}{D}\} \\ &\quad -h\{\frac{D}{2}\{\frac{k-(d-1)}{D}\}\} - \{\frac{D}{d}\{\frac{k-(d-1)}{D}\}\},\end{aligned}$$

and there hold $\epsilon_{d,h}(k) \in (-(h+2), 1)$ and $\epsilon_{d,1}(k) \in (-2, 0]$ for all integer $k \geq d-1$.

Understanding irrational CGs, Quasi-monotonicity of index growth.

Rademacher identity $^{\#}\mathrm{CG}(M,F)=1$ and c is irrational $\Longrightarrow$ c has at least 2 rotation matrices $R(\theta_1)\diamond R(\theta_2)$ in its basic normal form decomposition with $\theta_j/\pi\in\mathbf{R}\setminus\mathbf{Q}$ for $j=1,2$.

Theorem. *Let c be a closed geodesic with mean index $\hat{i}(c)>0$ on a Finsler manifold (M,F) of dimension $d\geq 2$. Then for any integer $A\in[0,k]$, there exist infinitely many integers $T\in\mathbf{N}$ such that*

$$\begin{aligned} i(c^m)-i(c^T) &\geq K_1\equiv\lambda+(q_0+q_+)+2(r-k)+2(r_*-k_*)+2A, \quad \forall\, m\geq T+1, \\ i(c^T)-i(c^m) &\geq K_2\equiv\lambda-(q_0+q_+)+2k-2(r_*-k_*)-2A, \quad \forall\, 1\leq m\leq T-1, \end{aligned}$$

where $\lambda=i(c)+p_-+p_0-r$.

Definition. *For a prime closed geodesic c with mean index* $\hat{i}(c) > 0$ *on a Finsler manifold* (M, F) *of dimension* $d \geq 2$. *Using* $\chi_c(m) = m(i(c) + p_- + p_0 - r) + \sum_{j=1}^{r} [\frac{m\theta_j}{2\pi}]$, *we define*

$$m_0(c) = \max\{m \in \mathbf{N} \mid \chi_c(m) < 0\}, \quad I_0(c) = \{m \in \mathbf{N} \mid \chi_c(m) < 0\},$$
$$\sigma_0(c) = \max\{0, \ -\chi_c(m) \mid 1 \leq m \leq m_0(c)\}.$$

Theorem. *Let c be a closed geodesic with mean index* $\hat{i}(c) > 0$ *on a Finsler manifold* (M, F) *of dimension* $d \geq 2$. *Then for every integer* $A \in [0, k]$, *there exist infinitely many integers* $T > m_0(c)$ *such that*

$$i(c^m) - i(c^T) \geq K_3 \equiv 2A - \sigma_0(c), \qquad \forall\, m \geq T + 1,$$
$$i(c^T) - i(c^m) \geq K_4 \equiv 2(k - A - (r_* - k_*)), \qquad \forall\, 1 \leq m \leq T - 1, \ m \notin I_0(c),$$
$$i(c^T) - i(c^m) \geq K_5 \equiv 2(k - A - (r_* - k_*)) - \sigma_0(c), \qquad \forall\, m \in I_0(c).$$

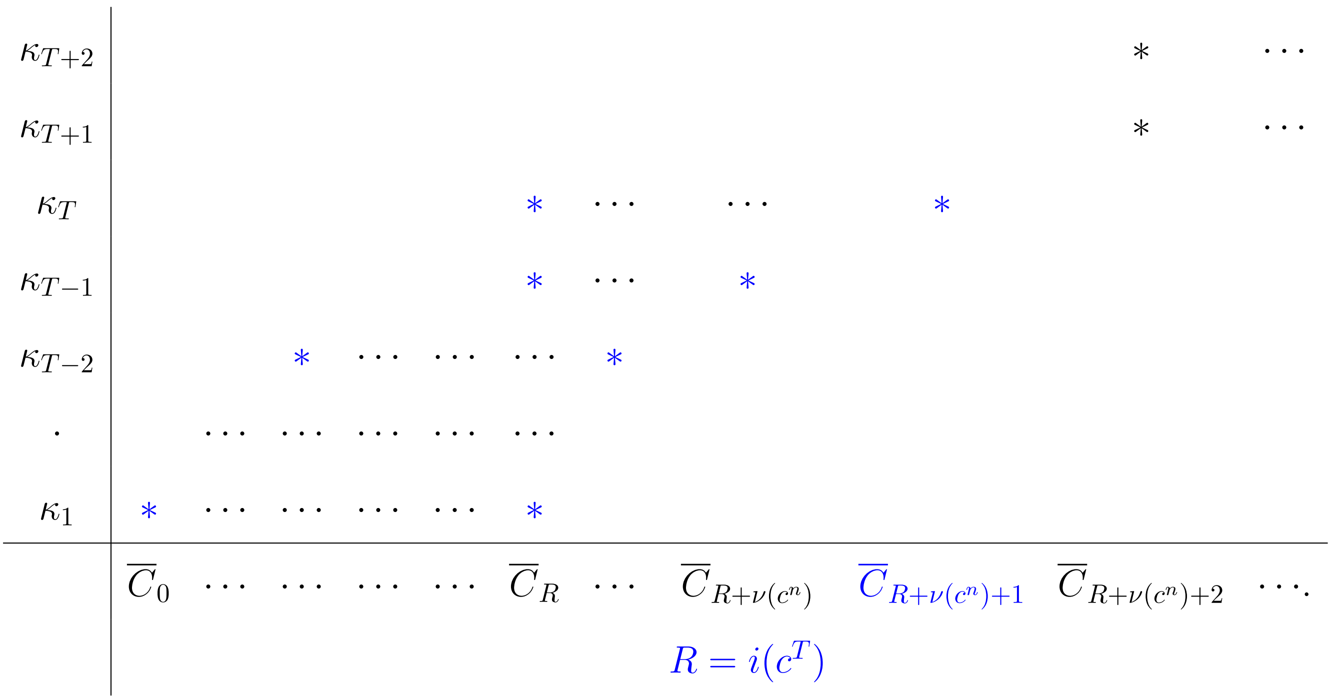
Exclusive Theorem. *Finsler manifold (M,F) with $H^*(M;\mathbf{Q})\cong T_{d,h+1}(x)$ with $h\geq 1$ and $d\geq 2$. c is a prime CG with $\hat{i}(c)>0$. Suppose that there exists an even $T\in n(c)\mathbf{N}$ and an integer $A\in[0,k]$ such that*

$$
\begin{aligned}
&i(c^T)\geq d-1,\\
&i(c^m)\geq i(c^T)+\nu(c^n)+1+\tau^{\pm}(i(c^T)+\nu(c^n)), \qquad \forall\, m\geq T+1,\\
&i(c^m)\leq i(c^T), \qquad \forall\, 1\leq m\leq T-1,
\end{aligned}
$$

$s(c)>\nu(c^n)+\tau^-(i(c^T)+\nu(c^n))+2(k-A)-d-\frac{\beta}{4}+\frac{2D}{h(h+1)d}(1+\epsilon_{d,h}(\mu)-\frac{h(h-1)d}{4})$, *where d is even, $\mu=i(c^T)+\nu(c^n)+\tau^-(i(c^T)+\nu(c^n))-1$;*

or $s(c)>\nu(c^n)+\tau^+(i(c^T)+\nu(c^n))+2(k-A)-\frac{\beta}{4}-d+2-\frac{4}{d+1}$, where d is odd. Here $s(c)=r+p_-+p_0+q_++q_0+2(r_*-k_*)$.

Then c can not be the only prime closed geodesic on (M,F).


κ_{T+2}
κ_{T+1}
κ_T
κ_{T-1}
κ_{T-2}
κ_1
$\overline{C}_0$
$\overline{C}_R$
$\overline{C}_{R+\nu(c^n)}$
$\overline{C}_{R+\nu(c^n)+1}$
$\overline{C}_{R+\nu(c^n)+2}$
$R = i(c^T)$

Definition. (M, F), c–*a prime closed geodesic on* (M, F). *For integers* T *and* $k \in \mathbf{N}$, *define the* **compensated sum** *of* c *respective to* T *and* k *by*

$$\Delta_c(T, k) = \sum_{\substack{1 \le m \le T,\ 0 \le l_m \le \nu(c^m) \\ i(c^m) + l_m > k}} (-1)^{i(c^m)+l_m} k_{l_m}^{\epsilon(c^m)}(c^m). \tag{1}$$

Compensated Exclusive Theorem. (M, F) *cpt. simply conn. mfd.* c *is a prime closed geodesic with* $\hat{i}(c) > 0$. $H^*(M; \mathbf{Q}) \cong T_{d,h+1}(x)$ *with* $h \ge 1$ *and* $d \ge 2$. *Suppose for some* $A \in [0, k]$ *and an even* $T \in n(c)\mathbf{N}$ *the following hold:*

$2A \ge \sigma_c(0) + \nu(c^n) + 1 + \tau^{\pm}(i(c^T) + \nu(c^n))$ *and* $k - A \ge r_* - k_*$;

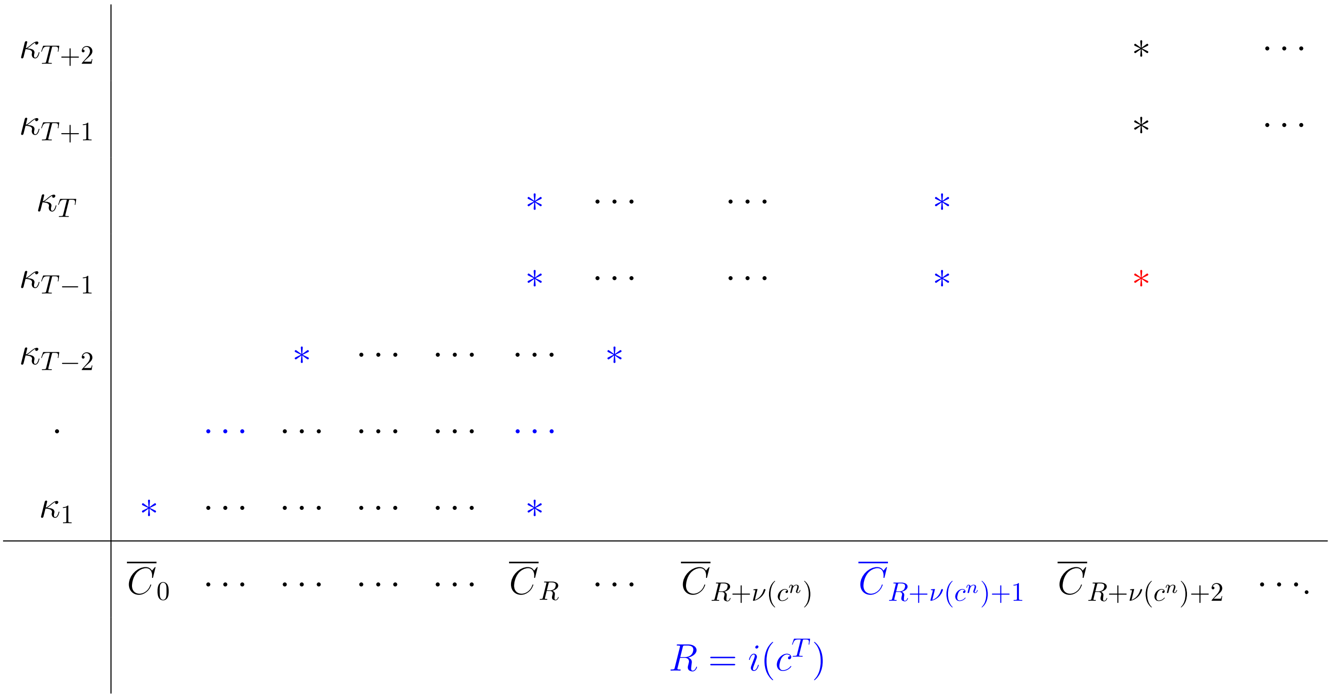
$s(c) > \nu(c^n) + \tau^-(i(c^T) + \nu(c^n)) - d + \frac{2D}{h(h+1)d}\Big(1 + \epsilon_{d,h}(\mu) - \frac{h(h-1)d}{4} - \Delta_c(T, R)\Big) + 2(k - A)$,

where d *is even,* $R = i(c^T) + \nu(c^n) + \tau^-(i(c^T) + \nu(c^n))$ *and* $\mu = R - 1$;

or $s(c) > \nu(c^n) + \tau^+(i(c^T) + \nu(c^n)) - d - \frac{4}{d+1} + 2 - \frac{2(d-1)}{d+1}\Delta_c(T, R) + 2(k - A)$,

where d *is odd,* $R = i(c^T) + \nu(c^n) + \tau^+(i(c^T) + \nu(c^n))$.

Then c *can not be the only prime closed geodesic on* (M, F).


κ_{T+2}
κ_{T+1}
κ_T
κ_{T-1}
κ_{T-2}
κ_1
$\overline{C}_0$
$\overline{C}_R$
$\overline{C}_{R+\nu(c^n)}$
$\overline{C}_{R+\nu(c^n)+1}$
$\overline{C}_{R+\nu(c^n)+2}$
$R = i(c^T)$

Proof of the existence of at least two distinct closed geodesics

on every cpt. simply conn. 4-dim M with $d = 4$ and $h = 1$, or $d = h = 2$.

Case 1, $i(c) \geq \dim M - 2$.

$\Longrightarrow i(c^m) \leq i(c^{m+1}) \qquad \forall\, m \geq 1$.

+ the above Exclusive Theorem $\Longrightarrow$ $^{\#}\mathrm{CG}(M, F) \geq 2$.

Case 2, $i(c) = 1$.

+ the above Exclusive Theorem $\Longrightarrow$ $^{\#}\mathrm{CG}(M, F) \geq 2$.

Case 3, $i(c) = 0$.

Case 3, $i(c) = 0$.

Basic Normal form decomp. $\Longrightarrow$ $P_c \approx R(\theta_1) \diamond R(\theta_2) \diamond \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ & $i(c^m)$ is even $\forall\, m$.

Rademacher identity $\Longrightarrow$ $-\frac{1}{|B(d,h)|}(k_0(c) - k_1^+(c)) = \hat{i}(c) = -2 + \frac{\theta_1+\theta_2}{\pi} > 0$

$\Longrightarrow$ $k_1^+(c^m) = k_1^+(c) = 1, \quad k_0(c^m) = 0, \quad \forall\, m \geq 1,$ $\Longrightarrow$ $d = h = 2$, (by Morse ineq.)

$\Longrightarrow$ $\frac{\theta_1+\theta_2}{\pi} = 2 + \frac{1}{|B(d,h)|} \geq \frac{8}{3}, \qquad \hat{i}(c) = \frac{1}{|B(d,h)|} \geq \frac{2}{3},$

$\Longrightarrow$ $\frac{\theta_1}{2\pi} \in (\frac{2}{3}, 1)$.

Compensated Exclusive Theorem $\Longrightarrow$

$$T\hat{i}(c)B(d,h) - \Delta_c(T,R) = \sum_{q=0}^{R}(-1)^q M_q \geq \sum_{q=0}^{R}(-1)^q b_q$$

where $-\Delta_c(T,R) \leq 2$, $B(2,2) = -3/2$, $R = i(c^T) + \nu(c^T) + * \in 2\mathbf{N}$.

$\Longrightarrow$ (2 = number of irr. rotations)

$$\begin{aligned} i(c^T) + 2 - \epsilon &= T\hat{i}(c) \leq \frac{1}{-B(2,2)}\left(\sum_{q=0}^{R}(-1)^q b_q - \Delta_c(T,R)\right) \\ &\leq i(c^T) + \nu(c^T) + \tau^-(i(c^T) + \nu(c^T)) - d + \frac{1}{|B(2,2)|}(-\Delta_c(T,R)) \\ &\leq i(c^T) + 1 + 1 - 2 + \frac{2}{3} \cdot 2 = i(c^T) + \frac{4}{3}. \end{aligned}$$

$\epsilon > 0$ small enough $\Longrightarrow$ Contradiction !

$\Longrightarrow$ ${}^{\#}\mathrm{CG}(M,F) \geq 2.$

Open problems

Understand irrational closed geodesics.

Conjecture 1: ${}^{\#}\mathrm{CG}(S^2, F) = 2 \quad or \quad +\infty, \quad \forall F \in \mathcal{F}(S^2)$.

Conjecture 2: $\forall\, n \geq 2$, $\exists\; 2 \leq p_n \leq q_n$ such that $p_n \to +\infty$ and

$$\{\,{}^{\#}\mathrm{CG}(S^n, F) \,|\, F \in \mathcal{F}(S^n)\} = \{k \in \mathbf{N} \,|\, p_n \leq k \leq q_n\} \cup \{+\infty\}.$$

[Long-Duan], [Duan-Long] $\Longrightarrow$ $p_3 \geq 2$, $p_4 \geq 2$.

It is very likely that $q_n \leq 2[(n+1)/2]$ holds.

Conjecture 3: ${}^{\#}\mathrm{CG}(M, g) = +\infty$, $\forall$ Riemannian metric g on M^n with $n \geq 3$.

${}^{\#}\mathrm{CG}(M, g) \geq 2$, $\forall$ Riemannian metric g on cpt. simply conn. M.

Conjecture 4: $\exists \geq 1$ elliptic closed geodesic on (S^n, F), $\forall\, F \in \mathcal{F}(S^n)$.

Conjecture 5: $^{\#}\mathrm{CG}(S^n, F) < +\infty \Longrightarrow$ all closed geodesics are irrational elliptic.

Hyperbolic $\Longrightarrow$ $^{\#}\mathrm{CG}(S^n, F) = +\infty$.

More identity ?

Problem: How are closed geodesics distributed on (S^n, F) and (S^n, g), in terms of minimal period, or geographical location ? What factors determine the situation ?

Thank You !