Making connections to the big ideas in mathematics: Promoting proportional reasoning

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Abstract

The focus of this paper is on proportional reasoning, emphasising its pervasiveness throughout the mathematics curriculum, but also highlighting its elusiveness. Proportional reasoning is required for students to operate successfully in many rational number topics (fractions, decimals, percentages), but also other topics (scale drawing, probability, trigonometry). Proportional reasoning is also required in many other school curriculum topics (for example, drawing timelines in history; interpreting density, molarity, speed calculations in science). In this paper, an overview of mathematics education research on proportional reasoning will be presented, highlighting the complex nature of the development of proportional reasoning and implications for learning and instruction. Through presentation of results of a current research project on proportional reasoning in the middle years, teaching approaches that have captured and engaged students' interest in exploring proportion-related situations will be shared.

Background

Proportional reasoning is a fundamental cornerstone of mathematics knowledge (Lesh, Post, & Behr, 1988). Proportional reasoning is the ability to understand situations of comparison. Examples of everyday tasks that require proportional reasoning include estimating the better buy, interpreting scales and maps, determining chances associated with gambling and risk-taking. Proportional reasoning has been described as one of the most commonly applied mathematics concepts in the real world (Lanius & Williams, 2003). Underdeveloped proportional reasoning potentially impacts real-world situations, sometimes with life-threatening or disastrous consequences, for example, incorrect doses in medicine (Preston, 2004). Proportional reasoning therefore is a major aspect of numeracy, yet it is implicit in school curricula and often limited to the study of rate and ratio in mathematics only.

The development of proportional reasoning is a complex operation, and

> ... [it] requires firm grasp of various rational number concepts such as order and equivalence, the relationship between the unit and its parts, the meaning and interpretation of ratio, and issues dealing with division, especially as this relates to dividing smaller numbers by larger ones. A proportional reasoner has the mental flexibility to approach problems from multiple perspectives and at the same time has understandings that are stable enough not to be radically affected by large or 'awkward' numbers, or the context within which a problem is posed. (Post, Behr & Lesh, 1988, p. 80)

Proportional reasoning is intertwined with many mathematical concepts. For example, English and Halford (1995) stated that: 'Fractions are the building blocks of proportion' (p. 254). Similarly, Behr et al. (1992) stated that 'the concept of fraction order and equivalence and proportionality are one component of this very significant and global mathematical concept' (p. 316). Also, Streefland (1985) suggested that 'Learning to view something 'in proportion', or 'in proportion with ...' precedes the acquisition of the proper concept of ratio' (p. 83). Developing students' understanding of ratio and proportion is difficult because the concepts of multiplication, division, fractions and decimals are the building blocks of proportional reasoning, and students' knowledge of such topics is generally poor (Lo & Watanabe, 1997).

The development of proportional reasoning is a gradual process, underpinned by increasingly sophisticated multiplicative thinking and the ability to compare two quantities in relative (multiplicative), rather than absolute (additive) terms (Lamon, 2005). The essence of proportional reasoning is on understanding the multiplicative structures inherent in proportion situations (Behr, Harel, Post & Lesh, 1992). Children's intuitive strategies for solving proportion problems are typically additive (Hart, 1981). The teacher's role, therefore, is to build on students' intuitive additive strategies and guide them towards building multiplicative structures. Strong multiplicative structures develop as early as the second grade for some children, but are also seen to take time to develop to a level of conceptual stability, often beyond fifth grade (Clark & Kamii, 1996). Behr et al. (1992) suggested that exploring change will help students develop multiplicative understanding. For example, students can be encouraged to discuss the change to 4 which will result in 8. From an additive view, 4 can change to 8 by adding 4. From a multiplicative view, 4 can change to 8 by multiplying by 2. The difference between the additive and multiplicative view can be seen by looking at other numbers. The additive rule holds for 13 changing to 17, but not the multiplicative rule. According to Behr et al. (1992), 'the ability to represent change (or difference) in both additive and multiplicative terms and to understand their behaviour under transformation is fundamental to understanding fraction and ratio equivalence' (p. 316). Moving students towards formal ratio and proportion principles and procedures is termed by Streefland (1985) as 'anticipating ratio', where the teacher capitalises on students' informal intuitive problem solving procedures, guiding students to 'formulae and algorithmisation' (p. 84). Such an approach was taken in a teaching experiment conducted by Lo and Watanabe (1997) where a Year 5 child was exposed to proportional reasoning tasks to promote intuitive multiplicative reasoning skills and hence develop proportional reasoning.

Research has indicated that students' (and teachers') understanding of proportion is generally poor (e.g., Behr et al., 1992; Fisher, 1988; Hart, 1981). Streefland (1985) stated that 'Ratio is introduced too late to be connected with mathematically related ideas such as equivalence of fractions, scale, percentage' (p. 78). English and Halford (1995) suggested that proportional reasoning is taught in isolation and thus remains unrelated to other topics. Behr et al. (1992) stated, 'We believe that the elementary school curriculum is deficient by failing to include the basic concepts and principles relating to multiplicative structures necessary for later learning in intermediate grades'(p. 300). Behr et al. also added, 'There is a great deal of agreement that learning rational number concepts remains a serious obstacle in the mathematical development of children ... In contrast there is no clear argument about how to facilitate learning of rational number concepts' (p. 300).

As the proportion concept is intertwined with many mathematical concepts, this has implications for instruction. The development of a rich concept of rational number, and thus proportional relationships, takes a long time (Streefland, 1985). The proportional nature of various rational number topics must be the focus of instruction as these topics are revisited continually throughout the curriculum, in order to build and link students' proportional understanding (Behr et al., 1992). Building proportional reasoning must be through multiple perspectives (Post et al., 1988). The literature provides various suggestions for activities and strategies for promoting the proportion concept. The use of ratio tables has been suggested as one means for building students' ratio understanding (English & Halford, 1995; Middleton & Van den Heuvel-Panhuizen, 1994; Robinson, 1981; Streefland, 1985). English and Halford (1995) provided the following example of a ratio table, which assists in the comparison of the number of soup cubes per person:

soup cubes	2	4	6	8
people	4	8	12	16

English and Halford stated, 'A table of this nature provides an effective means of organising the problem data and enables children to detect more readily all the relations displayed, both within and between the series ... it serves as a permanent record of proportion as an equivalence relation' (p. 254).

The MC SAM project

Promoting proportional reasoning has been the focus of a large research project undertaken by The University of Queensland (2007–2010). Not only did this project target proportional reasoning in mathematics but in science as well, as proportional reasoning is fundamental to many topics in both mathematics and science (Lamon, 2005). The MC SAM project, an acronym for Making Connections: Science and Mathematics, brought together middle years' mathematics and science teachers around this important topic, providing an opportunity for teachers to explore the proportional reasoning linkages between topics in both mathematics and science, and to create, implement and evaluate innovative and engaging learning experiences to assist students to promote and connect essential mathematics and science knowledge. The project had two major aims. First, it aimed to develop an instrument to assess middle years students' proportional reasoning knowledge. Second, it aimed to use this data to develop and trial specific learning experiences in both mathematics and science that may support students' access to particular topics in those subjects and promote proportional reasoning skills.

There is a large corpus of existing research that has provided analysis of strategies applied by students to various proportional reasoning tasks (e.g., Misailidou & Williams, 2003; Hart, 1981), Such research has highlighted issues associated with the impact of 'awkward' numbers (that is, common fractions and decimals as opposed to whole numbers), the common application of an incorrect additive strategy, and the blind application of rules and formulae to proportion problems. Prior research has also emphasised the complexity of the development of proportional reasoning and the need for further and continued work in the field to support students' development of proportional reasoning. In fact, it is estimated that approximately only 50 per cent adults can reason proportionately (Lamon, 2005). In our study, we wanted to take a snapshot of a large group of students' proportional reasoning on tasks that relate to mathematics and science curriculum in the middle years of schooling. This component of the project was concerned with the development of an instrument that would provide a 'broad brush' measure of students' proportional reasoning and their thinking strategies, and that would have some degree of diagnostic power. This challenge was undertaken with full awareness of both the pervasiveness and the elusiveness of proportional reasoning throughout the curriculum and that its development is dependent upon many other knowledge foundations in mathematics and science.

Developing the instrument was guided by literature and especially the American Association for the Advancement of Science (AAAS) (2001) Atlas of Science Literacy. The Atlas identifies two key components of proportional reasoning: Ratios and Proportion (parts and wholes, descriptions and comparisons and computation) and Describing Change (related changes, kinds of change, and invariance). The AAAS provided the framework for the development of the proportional reasoning assessment instrument. The test included items on direct proportion (whole number and fractional ratios), rate and inverse proportion items, as well as fractions, probability, speed and density items. Guided by the words of Lamon (2005), who suggested that students must be provided with many different contexts, 'to analyse quantitative relationships in context, and to represent those relationships in symbols, tables, and graphs' (p. 3), the items included contexts of shopping, cooking, mixing cordial, painting fences, graphing stories, saving money, school excursions anddual measurement scales. For each item on the test, students were required to provide the answer and explain the thinking they applied to solve the problem.

Approximately 700 students in the middle years of schooling (Years 4–9) participated in this assessment. Initially, project teachers had mixed feelings about the test's capacity to assess their students' proportional reasoning. The ninth grade teachers stated that they thought the test would be too easy for their students; the fourth grade teachers stated that the test was too hard. The highest average score however, for the ninth-graders on one item was just 75 per cent, with the fourth-graders averaging 15 per cent for that item. On several other items, the eighth and ninth graders scored less than 50 per cent. On one particular item, the ninth graders averaged just 21 per cent and the fourth graders averaged 5 per cent for the same item. The results were a wake-up call to all teachers in the project: the fourth and fifth grade teachers realised that there were some very good proportional reasoners in their grades, and the eighth and ninth grade teachers realised that they were taking for granted the proportional reasoning skills of their students. Item analysis and students' results provided direction for targeted teaching. Collectively, results of the whole test suggested that a much greater focus on proportional reasoning must occur in all classes at every opportunity.

Throughout the project, a series of integrated mathematics and science tasks has been developed, shared and adapted by the teachers. One of the simplest, and one that has been taken up most widely by all fourth grade to ninth grade teachers, is an exploration into why penguins huddle, incorporating the surface area to volume ratio. By using three 2-cm cubic blocks, penguins can be created. Focusing on one penguin, the surface area of the penguin can be found by counting the faces of the cubes (14) and the volume can be counted by counting the number of cubes (3). A huddle is formed by putting 9 penguins into a cubic arrangement. A data table is constructed and students can analyse the results to consider how the surface area to volume ratio changes as the huddle gets bigger.

One of the capstone elements of the project has been the development of a unit of work on density. Although density is typically regarded as a topic within the middle years science curriculum, conceptual understanding of density requires understanding of mathematics topics including mass and volume, as well as number sense and mental computation. It also requires data gathering, data analysis, interpretation of data, graphing, measuring, using measuring instruments, problem solving, problem posing, conducting experiments and controlling variables, which are components of both mathematics and science curricula. The integrated unit on density was developed and trialled in a number of middle years mathematics and/or science classrooms. It was implemented to varying degrees in most classes by project teachers, but was specifically implemented by the project team in a fifth and seventh grade classroom. At the beginning of the unit, the students' had limited knowledge of density, with developing understanding of mass and volume. At the end of the unit, students could describe how an object might sink or float in water by simultaneously considering both its volume and mass. All students could verbalise the concept of density and showed greater conceptualisation of units of measure for volume. Results of this study provide evidence of the capacity of targeted, integrated mathematics and science units for the development of connected mathematics and science knowledge and promotion of proportional reasoning skills.

Concluding comments

The development of proportional reasoning is a slow process exacerbated by its nebulous nature and lack of specific prominence in school syllabus documents. Our project teachers have revisited their traditional work program and its two-week mathematics unit on ratio and proportion. They have put greater emphasis on proportional reasoning and multiplicative thinking in the study of scale drawing, linear equations, trigonometry, percentages, number study, mapping, ratio and rate situations. Science teachers in the project a greater awareness of the mathematical foundations of proportional reasoning and how science topics and presentations of equations (e.g., density equation and force equation) may be based on assumptions of students' proportional reasoning that are not stable. The significance of this project has been that it brought together mathematics and science teachers to explore the synergies between mathematics and science curriculum through proportional reasoning.

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