Deviation property of Periodic measures in the Symbolic systems.

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Abstract

Fixing a continuous observable and using thermodynamic formalism and the method of convex analysis, we obtain upper and lower bounds for the exponential decay rate of periodic measures far from a given invariant measure in the two-side symbolic system.

1 Preliminaries

One branch of large deviation theory concerns the exponential decay rate of the periodic measures keeping away from some given measure. For a continuous self map $f : M \to M$ of some domain M, let m be an f-invariant measure on M and $\varphi : M \to \mathbb{R}$ be a observable function, given $\delta > 0$, let $B_n(\delta, f) := \{x \in Fix(f^n) \,|\, |\frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i x) - \int \varphi dm| \geq \delta\}$ and $C_n(\delta, f) := \{x \in Fix(f^n) \,|\, |\frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i x) - \int \varphi dm| > \delta\}$, where $Fix(f^n)$ denotes the set of periodic points with period n. People are interested in how to describe the exponential decay rate of $\sharp B_n$ or $\sharp C_n$, where $\sharp A$ denotes the cardinality of set A, by certain characteristics of dynamical systems. The research in this branch could be traced back to the work of Kifer [3], which is recovered by Pollicott in [4]. In their work, the system is assumed to be a uniformly hyperbolic flow $\phi_t : \Lambda \to \Lambda$ on a flow-invariant set Λ, and the invariant measures they concerned are those supported on the periodic orbits. They gave an upper bound of the exponential decay rate with a given weight function G of the meausures contained in a closed subset of the flow-invariant measures; and gave a lower bound of the exponential decay rate with the weight function G of the measures contained in an open subset.

Here we consider a two-side symbolic system $T : X \to X$. By a classical result of Sigmund [6], any T-invariant measure could be approximated by periodic measures. Inspired by this, in the "opposite researching direction", we consider the deviation property of the periodic measures far from a given T-invariant measure. We get two results in this direction for symbolic systems. The first one, Theorem 1.1, states that in such system the exponential decay rate of $\sharp B_n(\delta, T)$ could be controlled **from top** by the supremum of the measure theoretic entropy on a **closed subset** of the T-invariant measures. The second one Theorem 1.2, states that the exponential decay rate of $\sharp C_n(\delta, T)$ could be controlled **from bottom** by the supremum of measure theoretic entropy on an **open subset** of the T-invariant measures.

We employ two main tools in our approaches. The first is the general entropy in the measure sense introduced by Gelfert and Wolf in their paper♮, see Definition 2.1. The second is the method of convex analysis, see Lemma 3.3, which plays an important role in the present paper. Kifer [3] first introduced this method to the large deviation field, and it is also referred in paper of Gelfert and Wolf ♮. In the present paper, we give a more concise version of this method for our case.

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♮ Gelfert, K., Wolf, C., On the distribution of periodic orbits, Preprint, 2009.

Let $Y = \{1, 2, \ldots, k\}$ and $X = \prod_{-\infty}^{+\infty} Y$ and consider $T : X \to X$, $T : (x_i)_{-\infty}^{+\infty} \mapsto (x_{i+1})_{-\infty}^{+\infty}$. As usual, we call (X, T) two-side symbolic system. The metric $d(\cdot,\cdot) : X \times X \to \mathbb{R}$ is given by $d(x, y) := \sum_{-\infty}^{+\infty} \frac{|x_i - y_i|}{2^{|n|}}$, where $x = (x_i)_{-\infty}^{+\infty}$, $y = (y_i)_{-\infty}^{+\infty} \in X$. T is expansive, that is, there exists a constant ξ, expansive constant, such that $d(T^n x, T^n y) \le \xi$ for any $n \in \mathbb{Z}$ implies $x = y$. We denote by $\mathcal{M}_{inv}(X, T)$ the set of all the T-invariant probability measures on X, and denote $\omega_x := \frac{1}{l} \sum_{i=0}^{l-1} \delta_{f^i x}$ for $x \in Fix(f^l)$.

Theorem 1.1. *Let $T : X \to X$ be a two-side symbolic system preserving a probability measure μ. Then given $\varphi \in C(X)$ and $\delta > 0$, we have*

$$\limsup_{l\to+\infty} \frac{1}{l} \log \sharp\{x \in Fix(T^l) \mid \int \varphi d\omega_x - \int \varphi d\mu| \ge \delta\} \le \sup\{h_\nu(T) \mid |\int \varphi d\nu - \int \varphi d\mu| \ge \delta\}.$$

Theorem 1.2. *Let $T : X \to X$ be a two-side symbolic system preserving a probability measure μ. Then given $\varphi \in C(X)$ and $\delta > 0$, we have*

$$\liminf_{l\to+\infty} \frac{1}{l} \log \sharp\{x \in Fix(T^l) \mid \int \varphi d\omega_x - \int \varphi d\mu| > \delta\} \ge \sup\{h_\nu(T) \mid |\int \varphi d\nu - \int \varphi d\mu| > \delta\}.$$

We prove Theorem 1.1 and Theorem 1.2 in Section 2 and Section 3 respectively.

2 Proof of Theorem 1.1

In this section, we start from introducing the generalized measure theoretic entropy.

Definition 2.1. *Let $f : M \to M$ be a homomorphism on the compact manifold. Given $\nu \in \mathcal{M}_{inv}(M, f)$, we call*

$$\hat{h}_\nu(f) := \inf_{\psi\in C(M)} (P(\psi) - \int \psi d\nu)$$

the generalized entropy of f with respect to ν, where $P(\psi)$ denotes the topological pressure of ψ.

Remark 2.2. For our case, it is easy to see that $h_\nu(T) \le \hat{h}_\nu(T) \le h_{top}(T) < +\infty$, $\forall \nu \in \mathcal{M}_{inv}(X, T)$, where $h_{top}(T)$ denotes the topological entropy of T.

Remark 2.3. From the definition, it is standard to check that the function $\hat{h}_\cdot(f) : \mathcal{M}_{inv}(M, f) \to \mathbb{R} \cup \{+\infty\}$ is concave, i.e., for any non-negetive a_1, a_2 with $a_1 + a_2 = 1$ and $\nu_1, \nu_2 \in \mathcal{M}_{inv}(M, f)$, it holds that $\hat{h}_{a_1\nu_1+a_2\nu_2}(f) \ge a_1 \hat{h}_{\nu_1}(f) + a_2 \hat{h}_{\nu_2}(f)$.

Remark 2.4. The generalized measure theoretic entropy satisfies the variation principle, i.e., $P(\psi) = \sup_{\nu\in\mathcal{M}_{inv}(M,f)}(\hat{h}_\nu(M) + \int \psi d\nu)$. Indeed, by Definition 2.1, $\hat{h}_\nu(T) + \int \psi d\nu \le P(\psi)$ for any $\psi \in C(X)$ and $\nu \in \mathcal{M}_{inv}(M, f)$, which means that $\sup_{\nu\in\mathcal{M}_{inv}(M,f)}(\hat{h}_\nu(M) + \int \psi d\nu) \le P(\psi)$. And the opposite direction of this equality follows from the fact that $h_\nu(T) \le \hat{h}_\nu(T)$.

Remark 2.5. $\hat{h}_\nu(f) = h_\nu(f)$ if and only if the entropy map $h_\cdot(f) : \mathcal{M}_{inv}(M, f) \to \mathbb{R} \cup \{+\infty\}$ is upper semi-continuous at ν, see Theorem 9.12 in[7]. In our case, when T is expansive, the upper semi-continuity property follows by, see, Theorem 8.2 in [7]. Thus, $\hat{h}_\nu(T)$ and $h_\nu(T)$ coincide.

Remark 2.6. Recall that for a system $f : M \to M$ and a continuous function $\varphi \in C(M)$, we say $\nu \in \mathcal{M}_{inv}(M, f)$ is an equilibrium state of φ, if $h_\nu(f) + \int \varphi d\nu = P(\varphi)$. Thus, for our case, suppose $\phi \in C(X)$, then by Remark 2.5, $\nu \in \mathcal{M}_{inv}(X, T)$ is said to be an equilibrium state of ϕ, whenever ν satisfies $\hat{h}_\nu(T) + \int \phi d\nu = P(\varphi)$.

To get the first main result,Theorem 1.1, we prove a more general proposition.

Proposition 2.7. *Let $T : X \to X$ be a two-side symbolic system, let $\mathcal{V}$ be a closed subset of $\mathcal{M}_{inv}(X, T)$ and $\psi \in C(X)$. Then*

$$\limsup_{l\to+\infty} \frac{1}{l} \log \sum_{x\in Fix(T^l),\omega_x\in\mathcal{V}} \exp(S_l\psi(x)) \le \sup_{\nu\in\mathcal{V}} (\hat{h}_\nu(T) + \int \psi d\nu).$$

Proof. By Definition 2.1 together with Remark 2.2, $\hat{h}_\nu(T) = \inf_{\varphi\in C(X)}(P(\varphi) - \int \varphi d\nu) < +\infty$. Then for each $\nu \in \mathcal{M}_{inv}(X,T)$ and $\epsilon > 0$, there is a $\phi_\nu \in C(X)$ such that

$$P(\psi + \phi_\nu) - \int (\psi + \phi_\nu) d\nu < \hat{h}_\nu(T) + \epsilon. \tag{2.1}$$

Notice that $\psi, \psi + \phi_\nu \in C(X)$, which means that the maps $\nu \mapsto \int \psi d\nu$ and $\nu \mapsto \int (\psi + \phi_\nu) d\nu$ are continuous in $\mathcal{M}_{inv}(X,T)$, then there exists an open neighborhood $\mathcal{V}_\nu$ of ν, such that

$$|\int \psi d\tau - \int \psi d\nu| \le \epsilon \text{ and } |\int (\psi + \phi_\nu) d\tau - \int (\psi + \phi_\nu) d\nu| \le \epsilon, \text{ for any } \tau \in \mathcal{V}_\nu. \tag{2.2}$$

Combine (2.1) and (2.2), it follows that

$$\hat{h}_\nu(T) + \int (\psi + \phi_\nu) d\tau - P(\psi + \phi_\nu) + 2\epsilon > 0, \text{ for any } \tau \in \mathcal{V}_\nu. \tag{2.3}$$

Until now, we have shown that for each $\nu \in \mathcal{M}_{inv}(M,f)$, there exist $\phi_\nu \in C(X)$ and a open neighborhood $\mathcal{V}_\nu$ satisfying (2.3). Clearly, the union $\bigcup_{\nu\in\mathcal{V}} \mathcal{V}_\nu$ forms an open cover of $\mathcal{V}$. By the compactness of $\mathcal{V}$, we could choose a finite open subcover $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_r$ such that $\bigcup_{i=1}^r \mathcal{V}_i \supseteq \mathcal{V}$. And for each $\mathcal{V}_i$ there exist $\nu_i \in \mathcal{V}_i$ and $\phi_i \in C(X)$ such that $|\int \psi d\tau - \int \psi d\nu_i| \le \epsilon$, $|\int (\psi + \phi_i) d\tau - \int (\psi + \phi_i) d\nu_i| \le \epsilon$ and $\hat{h}_{\nu_i}(T) + \int (\psi + \phi_i) d\tau - P_{top}(\psi + \phi_i) + 2\epsilon > 0$ hold for any $\tau \in \mathcal{V}_i$, which means that

$$\exp(l \cdot (\hat{h}_{\nu_i}(T) + \int (\psi + \phi_i) d\tau - P_{top}(\psi + \phi_i) + 2\epsilon)) > 1, \text{ holds for any } l \in \mathbb{N}. \tag{2.4}$$

According to above argument, it follows that

$$\begin{aligned}
& \limsup_{l\to+\infty} \frac{1}{l} \log \sum_{x\in Fix(T^l), \omega_x\in\mathcal{V}} \exp(S_l\psi(x)) \\
\le\ & \limsup_{l\to+\infty} \frac{1}{l} \log \sum_{i=1}^r \sum_{x\in Fix(T^l), \omega_x\in\mathcal{V}_{\nu_i}} \exp(S_l\psi(x) \exp(l \cdot (\int (\psi + \phi_i) d\omega_x + \hat{h}_{\nu_i}(T) - P(\psi + \phi_i) + 2\epsilon)) \\
=\ & \limsup_{l\to+\infty} \frac{1}{l} \log \sum_{i=1}^r \sum_{x\in Fix(T^l), \omega_x\in\mathcal{V}_{\nu_i}} \exp(S_l(\psi + \phi_i)(x)) \exp(l \cdot (\int \psi d\omega_x + \hat{h}_{\nu_i}(T) - P(\psi + \phi_i) + 2\epsilon)) \\
\le\ & \limsup_{l\to+\infty} \frac{1}{l} \log \sum_{i=1}^r \sum_{x\in Fix(T^l), \omega_x\in\mathcal{V}_{\nu_i}} \exp(S_l(\psi + \phi_i)(x)) \exp(l \cdot (\int \psi d\nu_i + \hat{h}_{\nu_i}(T) - P(\psi + \phi_i) + 3\epsilon)) \\
\le\ & \limsup_{l\to+\infty} \frac{1}{l} \log \sum_{i=1}^r \sum_{x\in Fix(T^l), \omega_x\in\mathcal{V}_{\nu_i}} \exp(S_l(\psi + \phi_i)(x) - l \cdot P(\psi + \phi_i)) \exp l \cdot (\int \psi d\nu_i + \hat{h}_{\nu_i}(T) + 3\epsilon) \\
\le\ & \sup_{\nu\in\mathcal{V}} (\hat{h}_\nu(T) + \int \psi d\nu) + 3\epsilon + \limsup_{l\to+\infty} \frac{1}{l} \log \sum_{i=1}^r \sum_{x\in Fix(T^l), \omega_x\in\mathcal{V}_{\nu_i}} \exp(S_l(\psi + \phi_i)(x) - l \cdot P(\psi + \phi_i)).
\end{aligned}$$

Now quoting a classical result of Ruelle [5] saying that

$$\lim_{l\to+\infty} \frac{1}{l} \log \sum_{x\in Fix(T^l)} \exp(S_l(\varphi)(x)) = P(\varphi)$$

for any $\varphi \in C(X)$, we get

$$\limsup_{l\to+\infty} \frac{1}{l} \log \sum_{x\in Fix(T^l), \omega_x\in\mathcal{V}_{\nu_i}} \exp(S_l(\psi + \phi_i)(x) - l \cdot P(\psi + \phi_i)) \le 0, \text{ for any } 1 \le i \le r,$$

and thus

$$\limsup_{l\to+\infty}\frac{1}{l}\log\sum_{i=1}^{r}\sum_{x\in Fix(T^l),\omega_x\in\mathcal{V}_{\nu_i}}\exp(S_l(\psi+\phi_i)(x)-l\cdot P(\psi+\phi_i))\leq 0. \tag{2.5}$$

Applying (2.5) to the above argument, we have that

$$\limsup_{l\to+\infty}\frac{1}{l}\log\sum_{x\in Fix(T^l),\omega_x\in\mathcal{V}}\exp(S_l\psi(x))\leq\sup_{\nu\in\mathcal{V}}(\hat{h}_\nu(T)+\int\psi d\nu)+3\epsilon.$$

Notice that ϵ is taken arbitrarily, let $\epsilon\to 0$, then it holds that

$$\limsup_{l\to+\infty}\frac{1}{l}\log\sum_{x\in Fix(T^l),\omega_x\in\mathcal{V}}\exp(S_l\psi(x))\leq\sup_{\nu\in\mathcal{V}}(\hat{h}_\nu(T)+\int\psi d\nu),$$

which finishes our proof. □

Proof of Theorem 1.1 For given $\varphi\in C(X)$, $\delta>0$ and $\mu\in\mathcal{M}_{inv}(X,T)$, let $\mathcal{V}:=\{\nu\in\mathcal{M}_{inv}(X,T)\,|\,|\int\varphi d\nu-\int\varphi d\mu|\geq\delta\}$ and take $\psi\equiv 0$, then by Proposition 2.7, we get that

$$\limsup_{l\to+\infty}\frac{1}{l}\log\sharp\{x\in Fix(T^l)\,|\,|\int\varphi d\omega_x-\int\varphi d\mu|\geq\delta\}\leq\sup_{\nu\in\mathcal{V}}\hat{h}_\nu(T). \tag{2.6}$$

By Remark 2.5, $\hat{h}_\nu(T)=h_\nu(T)$, which together with (2.6) gives rise to Theorem 1.1 □

3 Proof of Theorem 1.2

Suppose $\alpha>0$, a function $\varphi\in C(X)$ is said to be α-hölder continuous if there exists $C>0$ such that $|\varphi(x)-\varphi(y)|\leq Cd(x,y)^\alpha$, $\forall\, x,y\in X$. In this section, we start from the subset $C^\alpha(X)$ consisting of α-hölder functions on X. It is a well known result for the symbolic system that each α-hölder continuous function has a unique equilibrium state, see [2], which is a critical property in our proof. Since $C^\alpha(X)$ is dense in $C(X)$, we could find a countable subfamily of α-hölder continuous functions dense in $C(X)$, denoted by $\{\varphi_i\}_{i=1}^{\infty}\subseteq C^\alpha(X)$. Recall that $\{\varphi_i\}_{i=1}^{+\infty}$ could induce a metric $\rho:\mathcal{M}_{inv}(X,T)\times\mathcal{M}_{inv}(X,T)\to\mathbb{R}$ as following:

$$\rho(\mu,\nu):=\sum_{i=1}^{+\infty}\frac{|\int\varphi_i d\mu-\int\varphi_i d\nu|}{2^i\|\varphi_i\|},$$

for any $\mu,\nu\in\mathcal{M}_{inv}(X,T)$, where the norm $\|\cdot\|$ is given by $\|\varphi\|=\sup_{x\in X}|\varphi(x)|$. This metric is compatible with the weak* topology of $\mathcal{M}_{inv}(X,T)$.

In the following, we fix $n\in\mathbb{N}$ and consider the subspace $\{\sum_{i=1}^n a_i\varphi_i\,|\,a_i\in\mathbb{R},1\leq i\leq n\}$, denoted by $C_n^0(X)$. We will give some new denotations. We say ν is equivalent to ν', denoted by $\nu\sim_n\nu'$, if and only if $\int\phi d\nu=\int\phi d\nu'$ for any $\phi\in C_n^0(X)$. This equivalent relation induces a quotient space $\mathcal{M}_{inv}(X,T)/\sim_n$, denoted by $\mathcal{M}_{inv}^n(X,T)$. Thus, the sequence $\{\varphi_i\}_{i=1}^n$ could also induce a metric $\rho_n:\mathcal{M}_{inv}^n(X,T)\times\mathcal{M}_{inv}^n(X,T)\to\mathbb{R}$ as following:

$$\rho_n(\mu,\nu):=\sum_{i=1}^{n}\frac{|\int\varphi_i d\mu-\int\varphi_i d\nu|}{2^i\,\|\varphi_i\|}.$$

Obviously, $\rho_n(\mu,\nu)\leq\rho(\mu,\nu)\leq\rho_n(\mu,\nu)+2^{-(n-1)}$, $\forall\,\mu,\nu\in\mathcal{M}_{inv}^n(X,T)$. The new metric ρ_n induces a topology of $\mathcal{M}_{inv}^n(X,T)$, we call it the ρ_n-topology in the following. Here, we point out that for $\nu\in\mathcal{M}_{inv}(X,T)$, the set $\{\nu'\in\mathcal{M}_{inv}(X,T)\,|\,\nu'\sim_n\nu\}$ is closed in the weak*-topology. Then, we define a function $\hat{h}^n_\cdot(T):\mathcal{M}_{inv}(X,T)\to\mathbb{R}$ as $\hat{h}^n_\nu(T):=\sup_{\nu\sim_n\nu'}\hat{h}_{\nu'}(T)$. Clearly, $\hat{h}^n_\nu(T)=\hat{h}^n_{\nu'}(T)$ holds for any $\nu\sim_n\nu'$. Moreover, we note that for each ν, there exists some $\tilde{\nu}\sim_n\nu$ satisfying $\hat{h}_{\tilde{\nu}}(T)=\hat{h}^n_\nu(T)$, because $\hat{h}_\cdot(T):\mathcal{M}_{inv}(X,T)\to\mathbb{R}$ is upper semi-continuous and $\{\nu'\in\mathcal{M}_{inv}(X,T)\,|\,\nu'\sim_n\nu\}$ is closed. Then we conclude that:

Claim 3.1. *Given $n \in \mathbb{N}$, then for each $\nu \in \mathcal{M}_{inv}(X,T)$ and $\phi \in C_n^0(X)$, it holds that*

$$P(\phi) = \sup_{\nu \in \mathcal{M}_{inv}(X,T)} (\int \phi d\nu + \hat{h}_\nu^n(T)).$$

Proof. By Remark 2.4, it holds that

$$\begin{aligned} P(\phi) &= \sup_{\nu \in \mathcal{M}_{inv}(X,T)} (\hat{h}_\nu(T) + \int \phi d\nu) \\ &= \sup_{\nu \in \mathcal{M}_{inv}(X,T)} \sup_{\nu' \sim_n \nu} (\hat{h}_{\nu'}(T) + \int \phi d\nu'). \end{aligned}$$

Notice that $\phi \in C_n^0(X)$ implies $\int \phi d\nu = \int \phi d\nu'$ for any $\nu' \sim_n \nu$, then

$$\begin{aligned} P(\phi) &= \sup_{\nu \in \mathcal{M}_{inv}(X,T)} \sup_{\nu' \sim_n \nu} (\hat{h}_{\nu'}(T) + \int \phi d\nu') \\ &= \sup_{\nu \in \mathcal{M}_{inv}(X,T)} (\sup_{\nu' \sim_n \nu} (\hat{h}_{\nu'}(T)) + \int \phi d\nu) \\ &= \sup_{\nu \in \mathcal{M}_{inv}(X,T)} (\hat{h}_\nu^n(T) + \int \phi d\nu). \end{aligned}$$

□

Claim 3.2. *The function $-\hat{h}_\cdot^n(T) : \mathcal{M}_{inv}(X,T) \to \mathbb{R}$ is convex and lower semi-continuous in the ρ_n-topology.*

Proof. By the definition of $\hat{h}_\nu(T)$, for any non-negative real numbers a_1, a_2 with $a_1 + a_2 = 1$, it holds that

$$\begin{aligned} \hat{h}^n_{a_1\nu_1 + a_2\nu_2}(T) &= \sup_{\nu' \sim_n (a_1\nu_1 + a_2\nu_2)} \hat{h}_{\nu'}(T) \\ &\geq \sup_{\nu_1' \sim_n \nu_1, \nu_2' \sim_n \nu_2} \hat{h}_{a_1\nu_1' + a_2\nu_2'}(T) \\ &\geq \sup_{\nu_1' \sim_n \nu_1, \nu_2' \sim_n \nu_2} (a_1 \hat{h}_{\nu_1'}(T) + a_2 \hat{h}_{\nu_2'}(T)) \quad \text{(by Remark 2.3)} \\ &= a_1 \sup_{\nu_1' \sim_n \nu_1} \hat{h}_{\nu_1'}(T) + a_2 \sup_{\nu_2' \sim_n \nu_2} \hat{h}_{\nu_2'}(T) \\ &= a_1 \hat{h}^n_{\nu_1}(T) + a_2 \hat{h}^n_{\nu_2}(T). \end{aligned}$$

Then the convexity of $-\hat{h}_\cdot^n(T)$ follows. To get the lower semi-continuity of $-\hat{h}_\cdot^n(T)$ in the ρ_n-topology, we show that the function $\hat{h}_\cdot^n(T) : \mathcal{M}_{inv}(X,T) \to \mathbb{R}$ is upper semi-continuous, i.e., for arbitrarily $\epsilon > 0$ and $\nu \in \mathcal{M}_{inv}(X,T)$, there exists $\delta = \delta(\epsilon, \nu) > 0$ such that $\hat{h}_\mu^n(T) < \hat{h}_\nu^n(T) + \epsilon$ holds for any μ with $\rho_n(\nu, \mu) < \delta$. Otherwise, there exists $\nu \in \mathcal{M}_{inv}(X,T)$ and $\epsilon > 0$ such that for each $k \in \mathbb{N}$ there is a μ_k satisfying $\rho_n(\mu_k, \nu) < \frac{1}{k}$ and $\hat{h}^n_{\mu_k}(T) \geq \hat{h}^n_\nu(T) + \epsilon$. Recall that for each μ_k, there exists $\tilde{\mu}_k \sim_n \mu_k$ satisfying that $\hat{h}^n_{\mu_k}(T) = \hat{h}_{\tilde{\mu}_k}(T)$, thus, let μ be a accumulation point of the sequence $\{\tilde{\mu}_k\}_{k=1}^{+\infty}$ in the weak*-topology, it is clear that $\rho_n(\mu, \nu) = 0$, i.e., $\mu \sim_n \nu$. By the the upper semi-continuity of $\hat{h}_\cdot(T) : \mathcal{M}_{inv}(X,T) \to \mathbb{R}$, it follows that

$$\hat{h}^n_\nu(T) \geq \hat{h}_\mu(T) \geq \limsup_{k \to +\infty} \hat{h}_{\tilde{\mu}_k}(T) = \limsup_{k \to +\infty} \hat{h}^n_{\mu_k}(T) \geq \hat{h}^n_\nu(T) + \epsilon,$$

a contradiction. This gives rise to our claim. □

Now we give the following lemma that

Lemma 3.3. *Given $n \in \mathbb{N}$, then for each interior point ν of $\mathcal{M}_{inv}(X,T)$, there exists a $\phi \in C_n^0(X)$ and $\nu' \sim_n \nu$ such that ν' is the unique equilibrium state of ϕ.*

To get Lemma 3.3, we need some basic concepts from convex analysis and an element result Lemma 3.4 below. Let X be a Bananch space and X^* be the dual space of X. We say $V : X \to \mathbb{R} \cup \{+\infty\}$ is a proper function if there exists $x \in X$ such that $V(x) < +\infty$. Let $V : X \to \mathbb{R} \cup \{+\infty\}$ be a proper function. The function $V^* : X^* \to \mathbb{R} \cup \{+\infty\}$ defined by $V^*(p) := \sup_{x \in X}(\langle p, x\rangle - V(x))$ is called the conjugate function of V. Similarly, we define $V^{**} : X \to \mathbb{R} \cup \{+\infty\}$ as $V^{**}(x) := \sup_{p \in X^*}(\langle p, x\rangle - V^*(x))$.

Lemma 3.4. *Let $V : X \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semi-continuous function. If x is an interior point of $\{x \,|\, V(x) < +\infty\}$, then the set $\{p \in X^* \,|\, \langle p, x\rangle = V(x) + V^*(p)\}$ is nonempty.*

The proof could be found in [1], see Theorem 17, p199 and Proposition 3, p202.

Proof of Lemma 3.3. Define $F_n : \mathcal{M}_{inv}(X,T) \to \mathbb{R}^n$ as $F_n(\nu) := (\int \varphi_1 d\nu, \int \varphi_2 d\nu, \cdots, \int \varphi_n d\nu)$. It is easy to see that for $\alpha = (a_1, a_2, \cdots, a_n)$ and $\phi = \sum_{i=1}^n a_i\varphi_i$, it holds that $\int \phi d\nu = \sum_{i=1}^n a_i \int \varphi_i d\nu = \langle \alpha, F_n(\nu)\rangle$, where $\langle \cdot, \cdot \rangle$ denotes the general inner product on $\mathbb{R}^n$. Moreover, note that $\mathcal{M}_{inv}(X,T)$ is closed and convex, the map F_n is a linear isomorphism from $\mathcal{M}_{inv}(X,T)$ (with respect to the ρ_n-topology) to its image, then $F_n(\mathcal{M}_{inv}(X,T))$ is also closed and convex in $\mathbb{R}^n$. We define function $A : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ as

$$A(\alpha) := \begin{cases} -\hat{h}_\nu^n(T), & \alpha = F_n(\nu) \text{ for some } \nu \in \mathcal{M}_{inv}(X,T) \\ +\infty, & \alpha \in \mathbb{R}^n \setminus F_n(\mathcal{M}_{inv}(X,T)) \end{cases}$$

By the definition of $\hat{h}_\nu^n(T)$, $\hat{h}_\nu^n(T) = \hat{h}_{\nu'}^n(T)$ for any $\nu \sim_n \nu'$, which implies that the function $A : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is well defined. We define function $B : \mathbb{R}^n \to \mathbb{R}$ as following. For each $\alpha = (a_1, a_2, \cdots, a_n) \in \mathbb{R}^n$, set $B(\alpha) := P(\sum_{i=1}^n a_i\varphi_i)$. Denote $\phi := \sum_{i=1}^n a_i\varphi_i$, then by Claim 3.1, it holds that

$$\begin{aligned} B(\alpha) &= P(\phi) = \sup_{\nu \in \mathcal{M}_{inv}(X,T)} (\int \phi d\nu + \hat{h}_\nu^n(T)) \\ &= \sup_{\nu \in \mathcal{M}_{inv}(X,T)} (\langle \alpha, F_n(\nu)\rangle - A(F_n(\nu))). \end{aligned}$$

Note that $A(\alpha) = +\infty$ for any $\alpha \in \mathbb{R}^n \setminus F_n(\mathcal{M}_{inv}(X,T))$, then

$$B(\alpha) = \sup_{\beta \in \mathbb{R}^n} (\langle \alpha, \beta\rangle - A(\beta)), \tag{3.7}$$

which means that $B = A^*$. To apply Lemma 3.4, it suffices to show that $A : \mathbb{R}^n \to \mathbb{R}$ is convex and lower semi-continuous. By Claim 3.2, the map $-\hat{h}_\cdot^n(T) : \mathcal{M}_{inv}(X,T) \to \mathbb{R}$ is convex and lower semi-continuous. And recall that F_n is an isomorphism from $\mathcal{M}_{inv}(X,T)$ (with respect to the ρ_n topology) to its image, then it is standard to check that $A|_{F_n(\mathcal{M}_{inv}(X,T))} = -\hat{h}_\cdot(T) \circ F_n^{-1} : F_n(\mathcal{M}_{inv}(X,T)) \to \mathbb{R}$ is also convex and lower semi-continuous. The function $A : \mathbb{R}^n \to \mathbb{R}$ could be seen as an extension of $A|_{F_n(\mathcal{M}_{inv}(X,T))}$ by setting $A(\alpha) = +\infty$ for $\alpha \in \mathbb{R}^n \setminus F_n(\mathcal{M}_{inv}(X,T))$, then the convexity and lower semi-continuity of A follows from that of $A|_{F_n(\mathcal{M}_{inv}(X,T))}$.

Recall that $F_n(\mathcal{M}_{inv}(X,T)) = \{\alpha \in \mathbb{R}^n \,|\, A(\alpha) < +\infty\}$, then by Lemma 3.4, for each interior point α of $F_n(\mathcal{M}_{inv}(X,T))$ there exists a $\beta \in \mathbb{R}^n$, satisfying $\langle \alpha, \beta\rangle = A(\alpha) + B(\beta)$. Consequently, we have that for each interior point $\nu \in \mathcal{M}_{inv}(X,T)$, there exists $\phi_\nu = \sum_{i=1}^n b_i\varphi_i \in C^\alpha(X)$ such that

$$P(\phi_\nu) = \int \phi_\nu d\nu + \hat{h}_\nu^n(T). \tag{3.8}$$

Note that the function $\hat{h}_\cdot^n(T) : \mathcal{M}_{inv}(X,T) \to \mathbb{R}$ is upper semi-continuous and the subset $\{\nu' \in \mathcal{M}_{inv}(X,T) \,|\, \nu' \sim_n \nu\}$ is closed in the weak* topology, then there exists a $\nu' \sim_n \nu$ satisfying that

$$\hat{h}_{\nu'}(T) = \sup_{\nu' \sim_n \nu} \hat{h}_{\nu'}(T) = \hat{h}_\nu^n(T). \tag{3.9}$$

And notice that the ϕ_ν in (3.8) belongs to $C_n^0(X)$ and $\nu' \sim_n \nu$, then $\int \phi_\nu d\nu = \int \phi_\nu d\nu'$. This together with (3.8) (3.9) gives that ν' is an equilibrium state of ϕ_ν. Moreover, the fact $\phi_\nu \in C^\alpha(X)$ implies that ν' is the unique equilibrium state of ϕ_ν. □

To get Proposition 3.6 below (which is a key proposition for Theorem 1.2), we also need the following fact that

Lemma 3.5. *Let $\{a_l\}_{l=1}^{+\infty}$ and $\{b_l\}_{l=1}^{+\infty}$ be two sequences of positive real numbers. Suppose that*

$$\lim_{l\to+\infty} \frac{1}{l}\log(a_l+b_l) = c \quad and \quad \limsup_{l\to+\infty}\frac{1}{l}\log b_l < c. \tag{3.10}$$

Then it holds that

$$\liminf_{l\to+\infty}\frac{1}{l}\log a_l = c.$$

This is a salient fact. For convenience of readers, we give its proof in below.

Proof. Otherwise, notice $b_l > 0$, it follows that $\liminf_{l\to+\infty}\frac{1}{l}\log a_l < c$. Without loss of generality, we set

$$a := \liminf_{l\to+\infty}\frac{1}{l}\log a_l \quad \text{and} \quad b := \limsup_{l\to+\infty}\frac{1}{l}\log b_l,$$

and according to our assumptions, $a, b < c$. Thus, for any $\epsilon > 0$, there exist $N = N(\epsilon)$ and an increasing subsequence of natural numbers $\{l_i\}_{i=1}^{+\infty}$ such that as long as $i \geq N$, we have that

$$a_{l_i} \leq e^{l_i(a+\epsilon)} \quad \text{and} \quad b_{l_i} \leq e^{l_i(b+\epsilon)}.$$

Consequently,

$$\limsup_{i\to+\infty}\frac{1}{l_i}\log(a_{l_i}+b_{l_i}) \leq \limsup_{i\to+\infty}\frac{1}{l_i}\log(e^{l_i(a+\epsilon)}+e^{l_i(b+\epsilon)}) \leq \lim_{i\to+\infty}\frac{1}{l_i}\log 2e^{l_i(\max\{a,b\}+\epsilon)} \leq \max\{a,b\}+\epsilon. \tag{3.11}$$

Take $\epsilon > 0$ small enough to satisfy that $\max\{a,b\}+\epsilon < c$, then

$$\lim_{i\to+\infty}\frac{1}{l_i}\log(a_{l_i}+b_{l_i}) < c,$$

which contradicts to our assumption (3.10). This argument gives rise to our lemma. □

Proposition 3.6. *Let $T : X \to X$ be a two-side symbolic system, let $\mathcal{U}$ be an open subset of $\mathcal{M}_{inv}(X,T)$ and $\psi \in C(X)$. Then*

$$\liminf_{l\to+\infty}\frac{1}{l}\log\sum_{x\in Fix(T^l),\omega_x\in\mathcal{U}}\exp(S_l\psi(x)) \geq \sup_{\nu\in\mathcal{U}}(\hat{h}_\nu(T)+\int\psi d\nu).$$

Proof. Given $\epsilon > 0$, there exists $\nu_\epsilon \in \mathcal{U}$ satisfying

$$\sup_{\nu\in\mathcal{U}}(\hat{h}_\nu(T)+\int\psi d\nu)-\epsilon \leq \hat{h}_{\nu_\epsilon}(T)+\int\psi d\nu_\epsilon \leq \sup_{\nu\in\mathcal{U}}(\hat{h}_\nu(T)+\int\psi d\nu).$$

For any $\eta > 0$, we denote $B(\nu_\epsilon,\eta) := \{\tau \in \mathcal{M}_{inv}(X,T)\,|\,\rho(\tau,\nu_\epsilon) < \eta\}$. Since $\mathcal{U}$ is open, there exists an $\eta_0 > 0$ such that $B(\nu_\epsilon,\eta_0) \subset \mathcal{U}$ and

$$|\int\psi d\tau - \int\psi d\nu_\epsilon| \leq \epsilon$$

holds for each $\tau \in B(\nu_\epsilon,\eta_0)$. Moreover, for any $\eta > 0$ and $n \in \mathbb{N}$, we denote $B_n(\nu_\epsilon,\eta) := \{\tau \in \mathcal{M}_{inv}(X,T)\,|\,\rho_n(\tau,\nu_\epsilon) < \eta\}$. It is standard to check that $B_n(\nu_\epsilon,\eta)$ is open in the weak* topology. Here, recall that $\rho_n(\mu,\nu) \leq \rho(\mu,\nu) \leq \rho_n(\mu,\nu)+2^{-(n-1)}$, for any $\mu,\nu \in \mathcal{M}_{inv}(X,T)$. Thus, let the natural number n satisfy $2^{-(n-1)} \leq \eta_0/4$ and let $\eta_1 \leq \eta_0/4$, then it is easy to check that $B_n(\nu_\epsilon,\eta_1) \subset B(\nu_\epsilon,\eta_0)$.

Notice that ν_ϵ is clearly an interior point of $\mathcal{M}_{inv}(X,T)$, applying Lemma 3.3, it follows that there exist $\nu'_\epsilon \sim_n \nu_\epsilon$ and $\psi+\phi_\epsilon \in C^0_n(X)$ such that ν'_ϵ is the unique equilibrium state of $\psi+\phi_\epsilon$. Obviously, $\nu'_\epsilon \in B_n(\nu_\epsilon,\eta_1)$. Moreover, since $\psi+\phi_\epsilon \in C^0_n(X)$ has been fixed, we could modify η_1, if necessary, to ensure that

$$|\int(\psi+\phi_\epsilon)d\tau - \int(\psi+\phi_\epsilon)d\nu_\epsilon| \le \epsilon$$

holds for any $\tau \in B_n(\nu_\epsilon,\eta_1)$.

Then

$$\begin{aligned}
&\liminf_{l\to+\infty}\frac{1}{l}\log\sum_{x\in Fix(T^l),\omega_x\in\mathcal{U}}\exp(S_l\psi(x))\\
\ge\ &\liminf_{l\to+\infty}\frac{1}{l}\log\sum_{x\in Fix(T^l),\omega_x\in B_n(\nu_\epsilon,\eta_1)}\exp(S_l\psi(x))\\
\ge\ &\liminf_{l\to+\infty}\frac{1}{l}\log\sum_{x\in Fix(T^l),\omega_x\in B_n(\nu_\epsilon,\eta_1)}\exp(S_l\psi(x))\exp(l\cdot(\hat{h}_{\nu_\epsilon}(T)+\int(\psi+\phi_\epsilon)d\nu_\epsilon-P(\psi+\phi_\epsilon)))\\
\ge\ &\liminf_{l\to+\infty}\frac{1}{l}\log\sum_{x\in Fix(T^l),\omega_x\in B_n(\nu_\epsilon,\eta_1)}\exp(S_l\psi(x))\exp(l\cdot(\hat{h}_{\nu_\epsilon}(T)+\int(\psi+\phi_\epsilon)d\omega_x-\epsilon-P(\psi+\phi_\epsilon)))\\
\ge\ &\liminf_{l\to+\infty}\frac{1}{l}\log\sum_{x\in Fix(T^l),\omega_x\in B_n(\nu_\epsilon,\eta_1)}\exp(S_l\psi(x))\exp(l\cdot(\hat{h}_{\nu_\epsilon}(T)+\int\psi d\nu_\epsilon-\epsilon+\int\phi_\epsilon d\omega_x-\epsilon-P(\psi+\phi_\epsilon)))\\
=\ &\liminf_{l\to+\infty}\frac{1}{l}\log\sum_{x\in Fix(T^l),\omega_x\in B_n(\nu_\epsilon,\eta_1)}\exp(l\cdot(\hat{h}_{\nu_\epsilon}(T)+\int\psi d\nu_\epsilon-2\epsilon))\exp(S_l\psi(x)+l\cdot(\int\phi_\epsilon d\omega_x-P(\psi+\phi_\epsilon)))\\
\ge\ &\liminf_{l\to+\infty}\frac{1}{l}\log\sum_{x\in Fix(T^l),\omega_x\in B_n(\nu_\epsilon,\eta_1)}\exp(l\cdot(\hat{h}_{\nu_\epsilon}(T)+\int\psi d\nu_\epsilon-2\epsilon))\exp(l\cdot(\int(\psi+\phi_\epsilon)d\omega_x-P(\psi+\phi_\epsilon)))\\
=\ &\hat{h}_{\nu_\epsilon}(T)+\int\psi d\nu_\epsilon-2\epsilon+\liminf_{l\to+\infty}\frac{1}{l}\log\sum_{x\in Fix(T^l),\omega_x\in B_n(\nu_\epsilon,\eta_1)}\exp(S_l(\psi+\phi_\epsilon)(x)-l\cdot P(\psi+\phi_\epsilon))\\
\ge\ &\sup_{\nu\in\mathcal{U}}(\hat{h}_\nu(T)+\int\psi d\nu)-3\epsilon+\liminf_{l\to+\infty}\frac{1}{l}\log\sum_{x\in Fix(T^l),\omega_x\in B_n(\nu_\epsilon,\eta_1)}\exp(S_l(\psi+\phi_\epsilon)(x)-l\cdot P(\psi+\phi_\epsilon))
\end{aligned}$$

It remains to show that

$$\liminf_{l\to+\infty}\frac{1}{l}\log\sum_{x\in Fix(T^l),\omega_x\in B_n(\nu_\epsilon,\eta_1)}\exp(S_l(\psi+\phi_\epsilon)(x)-l\cdot P(\psi+\phi_\epsilon))=0. \tag{3.12}$$

For the sake of simplicity, we denote $c := P(\psi+\phi_\epsilon)$,

$$a_l := \sum_{x\in Fix(T^l),\omega_x\in B_n(\nu_\epsilon,\eta_1)}\exp(S_l(\psi+\phi_\epsilon)(x) \text{ and } b_l := \sum_{x\in Fix(T^l),\omega_x\in\mathcal{M}(X,T)\setminus B_n(\nu_\epsilon,\eta_1)}\exp(S_l(\psi+\phi_\epsilon)(x).$$

Consequently,

$$\lim_{l\to+\infty}\frac{1}{l}\log(a_l+b_l)=\lim_{l\to+\infty}\frac{1}{l}\log\sum_{x\in Fix(T^l)}\exp(S_l(\psi+\phi_\epsilon)(x)=P(\psi+\phi_\epsilon)=c. \tag{3.13}$$

To apply Lemma 3.5, it suffices to show $\limsup_{l\to+\infty}\frac{1}{l}\log b_l < c$, i.e.,

$$\limsup_{l\to+\infty}\frac{1}{l}\log\sum_{x\in Fix(T^l),\omega_x\in\mathcal{M}(X,T)\setminus B_n(\nu_\epsilon,\eta_1)}\exp(S_l(\psi+\phi_\epsilon)(x)<P(\psi+\phi_\epsilon). \tag{3.14}$$

If (3.14) does not hold, then it follows that

$$\limsup_{l\to+\infty}\frac{1}{l}\log\sum_{x\in Fix(T^l),\omega_x\in\mathcal{M}(X,T)\setminus B_n(\nu_\epsilon,\eta_1)}\exp(S_l(\psi+\phi_\epsilon)(x)=P(\psi+\phi_\epsilon). \tag{3.15}$$

By Proposition 2.7, we get that

$$\limsup_{l\to+\infty}\frac{1}{l}\log\sum_{x\in Fix(T^l),\omega_x\in\mathcal{M}_{inv}(X,T)\setminus B_n(\nu_\epsilon,\eta_1)}\exp(S_l(\psi+\phi_\epsilon)(x)$$
$$\leq \sup_{\nu\in\mathcal{M}(X,T)\setminus B_n(\nu_\epsilon,\eta_1)}(\hat{h}_\nu(T)+\int(\psi+\phi_\epsilon)d\nu).$$

This together with (3.15) gives that

$$\sup_{\nu\in\mathcal{M}(X,T)\setminus B_n(\nu_\epsilon,\eta_1)}(\hat{h}_\nu(T)+\int(\psi+\phi_\epsilon)d\nu)=P(\psi+\phi_\epsilon).$$

Recall that $\hat{h}_\cdot(T):\mathcal{M}_{inv}(X,T)\to\mathbb{R}$ is upper semi-continuous, then it could take its supremum on the closed subset $\mathcal{M}_{inv}(X,T)\setminus B_n(\nu_\epsilon,\eta_1)$, i.e., there exists $\tau\in\mathcal{M}_{inv}(X,T)\setminus B_n(\nu_\epsilon,\eta_1)$ satisfying

$$\hat{h}_\tau(T)+\int(\psi+\phi_\epsilon)d\tau=P(\psi+\phi_\epsilon).$$

Notice that $\hat{h}_\tau(T)=h_\tau(T)$ for our case, then τ is also an equilibrium state of $\psi+\phi_\epsilon$, which contradicts to the uniqueness of ν'_ϵ. This contradiction gives rise to (3.14). Combining (3.13)(3.14) and applying Lemma 3.5, it follows

$$\liminf_{l\to+\infty}\frac{1}{l}\log\sum_{x\in Fix(T^l),\omega_x\in B_n(\nu_\epsilon,\eta')}\exp(S_l(\psi+\phi_\epsilon)(x))=\liminf_{l\to+\infty}\frac{1}{l}\log a_l=c=P(\psi+\phi_\epsilon).$$

Thus, (3.12) holds. Now, we have that

$$\liminf_{l\to+\infty}\frac{1}{l}\log\sum_{x\in Fix(T^l),\omega_x\in\mathcal{U}}\exp(S_l\psi(x))\geq\sup_{\nu\in\mathcal{U}}(\hat{h}_\nu(T)+\int\psi d\nu)-3\epsilon.$$

Recall that $\epsilon>0$ is taken arbitrarily, then let $\epsilon\to0$, it follows that

$$\liminf_{l\to+\infty}\frac{1}{l}\log\sum_{x\in Fix(T^l),\omega_x\in\mathcal{U}}\exp(S_l\psi(x))\geq\sup_{\nu\in\mathcal{U}}(\hat{h}_\nu(T)+\int\psi d\nu),$$

which finishes our proof. □

Proof of Theorem 1.2 For given $\varphi\in C(X)$, $\delta>0$ and $\mu\in\mathcal{M}_{inv}(X,T)$, let $\mathcal{U}:=\{\nu\in\mathcal{M}_{inv}(X,T)\,|\,|\int\varphi d\nu-\int\varphi d\mu|>\delta\}$ and take $\psi\equiv0$, then by Proposition 3.6, we get that

$$\limsup_{l\to+\infty}\frac{1}{l}\log\sharp\{x\in Fix(T^l)\,|\,|\int\varphi d\omega_x-\int\varphi d\mu|\,\delta\}\leq\sup_{\nu\in\mathcal{U}}\hat{h}_\nu(T).\tag{3.16}$$

By Remark 2.5, $\hat{h}_\nu(T)=h_\nu(T)$, which together with (3.16) gives rise to Theorem 1.2 □

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