RENORMALIZED AND ENTROPY SOLUTIONS FOR NONLINEAR PARABOLIC EQUATIONS WITH VARIABLE EXPONENTS AND L^1 DATA

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ABSTRACT. In this paper we prove the existence and uniqueness of both renormalized solutions and entropy solutions for nonlinear parabolic equations with variable exponents and L^1 data. And moreover, we obtain the equivalence of renormalized solutions and entropy solutions.

1. INTRODUCTION

Suppose that Ω is a bounded open domain of $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$, T is a positive number. In this paper we study the following nonlinear parabolic problem

$$
\begin{cases}
\dfrac{\partial u}{\partial t} - \operatorname{div}\big(|\nabla u|^{p(x)-2}\nabla u\big) = f & \text{in} \quad Q \equiv \Omega \times (0,T), \\
u = 0 & \text{on} \quad \Gamma \equiv \partial\Omega \times (0,T), \\
u(x,0) = u_0(x) & \text{on} \quad \Omega,
\end{cases}
\tag{1.1}
$$

where the variable exponent $p : \bar{\Omega} \to (1,+\infty)$ is a continuous function, $f \in L^1(Q)$ and $u_0 \in L^1(\Omega)$.

The study of differential equations and variational problems with nonstandard growth conditions arouses much interest with the development of elastic mechanics, electro-rheological fluid dynamics and image processing, etc. We refer the readers to [32, 33, 37, 15] and references therein. $p(x)$-growth conditions can be regarded as a very important class of nonstandard (p,q)-growth conditions. There are already numerous results for such kind of problems (see [1, 2, 3, 19, 20, 18, 5]). The functional spaces to deal with these problems are the generalized Lebesgue spaces $L^{p(x)}(\Omega)$ and the generalized Lebesgue-Sobolev spaces $W^{k,p(x)}(\Omega)$.

Under our assumptions, it is reasonable to work with entropy solutions or renormalized solutions, which need less regularity than the usual weak solutions. The notion of renormalized solutions was first introduced by DiPerna and Lions [17] for the study of Boltzmann equation. It was then adapted to the study of some nonlinear elliptic or parabolic problems and evolution problems in fluid mechanics. We refer to [14, 16, 8, 10, 9, 27] for details. At the same time the notion of entropy solutions has been proposed by Bénilan et al. in [7]

for the nonlinear elliptic problems. This framework was extended to related problems with constant p in [13, 31, 11, 4, 29].

Recently, Sanchón and Urbano in [34] studied a Dirichlet problem of $p(x)$-Laplace equation and obtained the existence and uniqueness of entropy solutions for L^1 data, as well as integrability results for the solution and its gradient. The proofs rely crucially on *a priori* estimates in Marcinkiewicz spaces with variable exponents. Besides, Bendahmane and Wittbold in [6] proved the existence and uniqueness of renormalized solutions to nonlinear elliptic equations with variable exponents and L^1 data.

The aim of this paper is to extend the results in [34, 6] to the case of parabolic equations. As far as we know, there are no papers concerned with the nonlinear parabolic equations involving variable exponents and L^1 data. Inspired by [30] and [31], we develop a refined method. The advantage of our method is that we can not only obtain the existence and uniqueness of renormalized solutions for problem (1.1), but also find that the renormalized solution is equivalent to the entropy solution for problem (1.1). We first employ the difference and variation methods to prove the existence and uniqueness of weak solutions for the approximate problem of (1.1) under appropriate assumptions. Then we construct an approximate solution sequence and establish some *a priori* estimates. Next, we draw a subsequence to obtain a limit function, and prove this function is a renormalized solution. Based on the strong convergence of the truncations of approximate solutions, we obtain that the renormalized solution of problem (1.1) is also an entropy solution, which leads to an equality in the entropy formulation. By choosing suitable test functions, we prove the uniqueness of renormalized solutions and entropy solutions, and thus the equivalence of renormalized solutions and entropy solutions.

For the convenience of the readers, we recall some definitions and basic properties of the generalized Lebesgue spaces $L^{p(x)}(\Omega)$ and generalized Lebesgue-Sobolev spaces $W^{k,p(x)}(\Omega)$.

Set $C_+(\bar{\Omega}) = \{h \in C(\bar{\Omega}) : \min_{x \in \bar{\Omega}} h(x) > 1\}$. For any $h \in C_+(\bar{\Omega})$ we define

$$h_+ = \sup_{x \in \Omega} h(x) \quad \text{and} \quad h_- = \inf_{x \in \Omega} h(x).$$

For any $p \in C_+(\bar{\Omega})$, we introduce the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ to consist of all measurable functions such that

$$\int_\Omega |u(x)|^{p(x)} \, dx < \infty,$$

endowed with the Luxemburg norm

$$|u|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} \, dx \le 1 \right\},$$

which is a separable and reflexive Banach space. The dual space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$. If $p(x)$ is a constant function, then the variable exponent Lebesgue space coincides with the classical Lebesgue space. The variable exponent Lebesgue spaces is a special case of Orlicz-Musielak spaces treated by Musielak in [28].

For any positive integer k, denote

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \le k\},$$

where the norm is defined as

$$\|u\|_{W^{k,p(x)}} = \sum_{|\alpha| \le k} |D^\alpha u|_{p(x)}.$$

$W^{k,p(x)}(\Omega)$ is called generalized Lebesgue-Sobolev space, which is a special generalized Orlicz-Sobolev space. An interesting feature of a generalized Lebesgue-Sobolev space is that smooth functions are not dense in it without additional assumptions on the exponent $p(x)$. This was observed by Zhikov [36] in connection with Lavrentiev phenomenon. However, when the exponent $p(x)$ is log-Hölder continuous, i.e., there is a constant C such that

$$|p(x) - p(y)| \leq \frac{C}{-\log|x - y|} \tag{1.2}$$

for every $x, y \in \Omega$ with $|x - y| \leq \frac{1}{2}$, then smooth functions are dense in variable exponent Sobolev spaces and there is no confusion in defining the Sobolev space with zero boundary values, $W_0^{1,p(\cdot)}(\Omega)$, as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{W^{1,p(\cdot)}}$ (see [21]).

Throughout this paper we assume that $p(x) \in C_+(\bar{\Omega})$ satisfies the log-Hölder continuity condition (1.2). Let T_k denote the truncation function at height $k \geq 0$:

$$T_k(r) = \min\{k, \max\{r, -k\}\} = \begin{cases} k & \text{if } r \geq k, \\ r & \text{if } |r| < k, \\ -k & \text{if } r \leq -k, \end{cases}$$

and its primitive $\Theta_k : \mathbb{R} \to \mathbb{R}^+$ by

$$\Theta_k(r) = \int_0^r T_k(s)\, ds = \begin{cases} \frac{r^2}{2} & \text{if } |r| \leq k, \\ k|r| - \frac{k^2}{2} & \text{if } |r| \geq k. \end{cases}$$

It is obvious that $\Theta_k(r) \geq 0$ and $\Theta_k(r) \leq k|r|$.

We denote

$$\mathcal{T}_0^{1,p(\cdot)}(Q) = \{u : \bar{\Omega} \times (0, T] \to \mathbb{R} \text{ is measurable} \,|\, T_k(u) \in L^{p_-}\left(0, T; W_0^{1,p(\cdot)}(\Omega)\right)$$
$$\text{with } \nabla T_k(u) \in (L^{p(\cdot)}(Q))^N, \text{ for every } k > 0\}.$$

Next we define the very weak gradient for every measurable function $u \in \mathcal{T}_0^{1,p(\cdot)}(Q)$. The proof follows from Lemma 2.1 of [7] due to the fact that $W_0^{1,p(\cdot)}(\Omega) \subset W_0^{1,p_-}(\Omega)$.

Proposition 1.1. *For a measurable function $u \in \mathcal{T}_0^{1,p(\cdot)}(Q)$, there exists a unique measurable function $v : Q \to \mathbb{R}^N$, which we call the very weak gradient of u and denote $v = \nabla u$, such that*

$$\nabla T_k(u) = v\chi_{\{|u| < k\}}, \qquad \text{almost everywhere in } Q \text{ and for every } k > 0,$$

where χ_E denotes the characteristic function of a measurable set E. Moreover, if u belongs to $L^1(0, T; W_0^{1,1}(\Omega))$, then v coincides with the weak gradient of u.

The notion of the very weak gradient allows us to give the following definitions of renormalized solutions and entropy solutions for problem (1.1).

Definition 1.1. A function $u \in \mathcal{T}_0^{1,p(\cdot)}(Q) \cap C([0, T]; L^1(\Omega))$ is a renormalized solution to problem (1.1) if the following conditions are satisfied:

(i) $\displaystyle \lim_{n \to \infty} \int_{\{(x,t) \in Q : n \leq |u(x,t)| \leq n+1\}} |\nabla u|^{p(x)}\, dxdt = 0$;

(ii) For every function $\varphi \in C^1(\bar{Q})$ with $\varphi(\cdot, T) = 0$ and $S \in W^{2,\infty}(\mathbb{R})$ satisfying that S' has a compact support,

$$-\int_\Omega S(u_0)\varphi(x, 0)\, dx - \int_0^T \int_\Omega S(u)\frac{\partial\varphi}{\partial t}\, dxdt$$

$$+ \int_0^T \int_\Omega [S'(u)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi + S''(u)|\nabla u|^{p(x)}\varphi]\, dxdt$$

$$= \int_0^T \int_\Omega fS'(u)\varphi\, dxdt \tag{1.3}$$

holds.

Definition 1.2. A function $u \in \mathcal{T}_0^{1,p(\cdot)}(Q) \cap C([0, T]; L^1(\Omega))$ is an entropy solution to problem (1.1) if

$$\int_\Omega \Theta_k(u - \phi)(T)\, dx - \int_\Omega \Theta_k(u_0 - \phi(0))\, dx + \int_0^T \langle \phi_t, T_k(u - \phi) \rangle\, dt$$

$$+ \int_Q |\nabla u|^{p(x)-2}\nabla u \cdot \nabla T_k(u - \phi)\, dxdt = \int_Q fT_k(u - \phi)\, dxdt, \tag{1.4}$$

for all $k > 0$ and $\phi \in C^1(\bar{Q})$ with $\phi|_\Gamma = 0$.

Now we state our main results.

Theorem 1.1. *Assume that condition* (1.2) *holds. Then there exists a unique renormalized solution for problem* (1.1).

Theorem 1.2. *Assume that condition* (1.2) *holds. Then the renormalized solution u in Theorem* 1.1 *is also an entropy solution for problem* (1.1). *And the entropy solution is unique.*

Remark 1.1. The renormalized solution for problem (1.1) is equivalent to the entropy solution for problem (1.1).

The rest of this paper is organized as follows. In Section 2, we state some basic results that will be used later. We will prove the main results in Section 3. In the following sections C will represent a generic constant that may change from line to line even if in the same inequality.

2. Preliminaries

In this section, we first state some elementary results for the generalized Lebesgue spaces $L^{p(x)}(\Omega)$ and the generalized Lebesgue-Sobolev spaces $W^{k,p(x)}(\Omega)$. The basic properties of these spaces can be found from [23], and many of these properties were independently established in [20].

Lemma 2.1. ([20, 23]) (1) *The space $L^{p(\cdot)}(\Omega)$ is a separable, uniform convex Banach space, and its conjugate space is $L^{p'(\cdot)}(\Omega)$ where $1/p(x) + 1/p'(x) = 1$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, we have*

$$\Big| \int_\Omega uv\, dx \Big| \leq \Big(\frac{1}{p_-} + \frac{1}{(p_-)'} \Big)|u|_{p(x)}|v|_{p'(x)} \leq 2|u|_{p(x)}|v|_{p'(x)};$$

(2) *If $p_1, p_2 \in C_+(\bar{\Omega}), p_1(x) \le p_2(x)$ for any $x \in \Omega$, then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$, whose norm does not exceed $|\Omega| + 1$.*

Lemma 2.2. ([20]) *If we denote*

$$\rho(u) = \int_{\Omega} |u|^{p(x)} \, dx, \quad \forall u \in L^{p(x)}(\Omega),$$

then

$$\min\{|u|_{p(x)}^{p_-}, |u|_{p(x)}^{p_+}\} \le \rho(u) \le \max\{|u|_{p(x)}^{p_-}, |u|_{p(x)}^{p_+}\}.$$

Lemma 2.3. ([20]) $W^{k,p(x)}(\Omega)$ *is a separable and reflexive Banach space.*

Lemma 2.4. ([22, 23]) *Let $p \in C_+(\bar{\Omega})$ satisfy the log-Hölder continuity condition (1.2). Then, for $u \in W_0^{1,p(\cdot)}(\Omega)$, the $p(\cdot)$-Poincaré inequality*

$$|u|_{p(x)} \le C|\nabla u|_{p(x)}$$

holds, where the positive constant C depends on p and Ω.

Lemma 2.5. *Assume that $u_0 \in L^2(\Omega)$ and $f \in L^{(p_-)'}(0,T; L^{p'(x)}(\Omega))$. Then the following problem*

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = f & \text{in} \quad Q, \\ u = 0 & \text{on} \quad \Gamma, \\ u(x,0) = u_0 & \text{on} \quad \Omega, \end{cases}$$

admits a unique weak solution $u \in L^{p_-}(0,T; W_0^{1,p(\cdot)}(\Omega)) \cap C([0,T]; L^2(\Omega))$ with $\nabla u \in (L^{p(\cdot)}(Q))^N$ such that for any $\varphi \in C^1(\bar{Q})$ with $\varphi(\cdot, T) = 0$,

$$-\int_{\Omega} u_0(x)\varphi(x,0) \, dx + \int_0^T \int_{\Omega} [-u\varphi_t + |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi] \, dxdt$$

$$= \int_0^T \int_{\Omega} f\varphi \, dxdt$$

holds.

Proof. By employing the difference and variation methods (see [35]), we give a sketched proof.

Let n be a positive integer. Denote $h = T/n$. We first consider the following time-discrete problem

$$\begin{cases} \dfrac{u_k - u_{k-1}}{h} - \operatorname{div}(|\nabla u_k|^{p(x)-2}\nabla u_k) = [f]_h((k-1)h)), \\ u_k|_{\partial\Omega} = 0, \quad k = 1, 2, \ldots, n, \end{cases} \tag{2.1}$$

where $[f]_h$ denotes the Steklov average of f defined by

$$[f]_h(x,t) = \frac{1}{h} \int_t^{t+h} f(x,\tau) \, d\tau.$$

It is easy to see that $[f]_h(\cdot) \in L^{p'(\cdot)}(\Omega)$.

For $k = 1$, we introduce the variational problem

$$\min\{J(u) | u \in W\},$$

where

$$W = \left\{ u \in W_0^{1,p(x)}(\Omega) \cap L^2(\Omega) \right\}$$

and functional J is

$$J(u) = \frac{1}{2h} \int_\Omega u^2 \, dx + \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx - \frac{1}{h} \int_\Omega u_0 u \, dx - \int_\Omega [f]_h(0) u \, dx.$$

We will establish that $J(u)$ has a minimizer $u_1(x)$ in W.

By Lemma 2.1, Lemma 2.4, Young's inequality and Lemma 2.2, we have

$$\begin{aligned}
\left| \int_\Omega [f]_h(0) u \, dx \right| &\leq 2|[f]_h(0)|_{p'(x)} |u|_{p(x)} \\
&\leq C|[f]_h(0)|_{p'(x)} |\nabla u|_{p(x)} \\
&\leq \varepsilon |\nabla u|_{p(x)}^{p_-} + C(\varepsilon) |[f]_h(0)|_{p'(x)}^{(p_-)'} \\
&\leq \varepsilon \left(\int_\Omega |\nabla u|^{p(x)} \, dx \right)^{\beta p_-} + C(\varepsilon) |[f]_h(0)|_{p'(x)}^{(p_-)'} \\
&\leq \varepsilon \left(\int_\Omega |\nabla u|^{p(x)} \, dx + 1 \right) + C(\varepsilon) |[f]_h(0)|_{p'(x)}^{(p_-)'},
\end{aligned}$$

where ε is a small positive number and

$$\beta = \begin{cases} \frac{1}{p_-} & \text{if } |\nabla u|_{p(\cdot)} \geq 1, \\ \frac{1}{p_+} & \text{if } |\nabla u|_{p(\cdot)} \leq 1. \end{cases}$$

Choosing ε sufficiently small and using Young's inequality, we obtain

$$J(u) \geq \frac{1}{2p_+} \int_\Omega |\nabla u|^{p(x)} \, dx + \frac{1}{4h} \int_\Omega u^2 \, dx - C \left(\int_\Omega u_0^2 \, dx + |[f]_h(0)|_{p'(x)}^{(p_-)'} + 1 \right),$$

and thus $J(u)$ is lower bounded and coercive on W. On the other hand, $J(u)$ is weakly lower semicontinuous on W. Therefore, there exists a function $u_1 \in W$ such that

$$J(u_1) = \inf_{u \in W} J(u).$$

Thus the function u_1 is a weak solution of the corresponding Euler-Lagrange equation of $J(u)$, which is (2.1) in the case $k = 1$. And it is unique.

Following the same procedures, we find weak solutions u_k of (2.1) for $k = 2, \ldots, n$. It follows that, for every $\varphi \in W$,

$$\int_\Omega \frac{u_k - u_{k-1}}{h} \varphi \, dx + \int_\Omega |\nabla u_k|^{p(x)-2} \nabla u_k \cdot \nabla \varphi \, dx = \int_\Omega [f]_h((k-1)h) \varphi \, dx. \qquad (2.2)$$

For every $h = T/n$, we define the approximate solutions

$$u_h(x,t) = \begin{cases} u_0(x), & t = 0, \\ u_1(x), & 0 < t \leq h, \\ \cdots\cdots, & \cdots\cdots, \\ u_j(x), & (j-1)h < t \leq jh, \\ \cdots\cdots, & \cdots\cdots, \\ u_n(x), & (n-1)h < t \leq nh = T. \end{cases}$$

Taking $\varphi = u_k$ in (2.2), we can obtain an *a priori* estimate

$$\int_\Omega u_h^2(x,t) \, dx + \int_0^T \int_\Omega |\nabla u_h(x,t)|^{p(x)} \, dx dt \leq \int_\Omega u_0^2 \, dx + C \int_0^T |f|_{p'(x)}^{(p_-)'} \, dt,$$

which implies from Lemma 2.2 that

$$\int_0^T \min\{|\nabla u_h|_{p(x)}^{p_+}, |\nabla u_h|_{p(x)}^{p_-}\}\, dt \le \int_0^T \int_\Omega |\nabla u_h|^{p(x)}\, dxdt \le C$$

and

$$\|u_h\|_{L^\infty(0,T;L^2(\Omega))} + |\nabla u_h|_{p(x),Q} + \|u_h\|_{L^{p_-}(0,T;W_0^{1,p(x)}(\Omega))} \le C.$$

Thus we may choose a subsequence (we also denote it by the original sequence for simplicity) such that

$$u_h \rightharpoonup u, \quad \text{weakly-* in} \quad L^\infty(0,T;L^2(\Omega)),$$

$$u_h \rightharpoonup u, \quad \text{weakly in} \quad L^{p_-}\big(0,T;W_0^{1,p(x)}(\Omega)\big),$$

$$|\nabla u_h|^{p(x)-2}\nabla u_h \rightharpoonup \xi, \quad \text{weakly in} \quad (L^{p'(x)}(Q))^N.$$

Following the arguments in [35] with necessary changes in detail, we use the monotonicity method to show that $\xi = |\nabla u|^{p(x)-2}\nabla u$ a.e. in Q. Recalling the fact that $u \in L^{p_-}(0,T;W_0^{1,p(x)}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ and $u_t \in L^{(p_-)'}(0,T;W^{-1,p'(x)}(\Omega))$ from the equation, we conclude that u belongs to $C([0,T];L^2(\Omega))$. Therefore, we obtain the existence of weak solutions.

For uniqueness, suppose there exist two weak solutions u and v of problem (1.1). Then $w = u - v$ satisfies the following problem

$$\begin{cases} \dfrac{\partial w}{\partial t} - \text{div}\big(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\big) = 0 & \text{in} \quad Q, \\ w = 0 & \text{on} \quad \Gamma, \\ w(x,0) = 0 & \text{on} \quad \Omega. \end{cases}$$

Choosing w as a test function in the above problem, we have, for almost every $t \in (0,T)$,

$$\frac{1}{2}\int_\Omega w^2(t)\, dx + \int_0^t \int_\Omega [|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v] \cdot \nabla(u-v)\, dxds = 0.$$

Since the two terms on the left-hand side are nonnegative, we have $u = v$ a.e. in Q. This finishes the proof. $\square$

3. The proofs of main results

Now we are ready to prove the main results. Some of the reasoning is based on the ideas developed in [30] and [31] for the constant exponent case. First we prove the existence and uniqueness of renormalized solutions for problem (1.1).

Proof of Theorem 1.1.

(1) Existence of renormalized solutions.

We first introduce the approximate problems. Find two sequences of functions $\{f_n\} \subset C_0^\infty(Q)$ and $\{u_{0n}\} \subset C_0^\infty(\Omega)$ strongly converging respectively to f in $L^1(Q)$ and to u_0 in $L^1(\Omega)$ such that

$$\|f_n\|_{L^1(Q)} \le \|f\|_{L^1(Q)}, \|u_{0n}\|_{L^1(\Omega)} \le \|u_0\|_{L^1(\Omega)}. \tag{3.1}$$

Then we consider the approximate problem of (1.1)

$$
\begin{cases}
\dfrac{\partial u_n}{\partial t} - \operatorname{div}\big(|\nabla u_n|^{p(x)-2}\nabla u_n\big) = f_n & \text{in} \quad Q, \\[2mm]
u_n = 0 & \text{on} \quad \Gamma, \\[2mm]
u_n(x,0) = u_{0n} & \text{on} \quad \Omega.
\end{cases}
\tag{3.2}
$$

By Lemma 2.5, we can find a weak solution $u_n \in L^{p^-}\big(0,T;W_0^{1,p(\cdot)}(\Omega)\big)$ with $\nabla u_n \in (L^{p(\cdot)}(Q))^N$ for problem (3.2). Our aim is to prove that a subsequence of these approximate solutions $\{u_n\}$ converges to a measurable function u, which is a renormalized solution of problem (1.1). We will divide the proof into several steps. Although some of the arguments are not new, we present a self-contained proof for the sake of clarity and readability.

Step 1. Prove the convergence of $\{u_n\}$ in $C([0,T];L^1(\Omega))$ and find its subsequence which is almost everywhere convergent in Q.

Let m and n be two integers, then from (3.2) we can write the weak form as

$$
\int_0^T \langle (u_n - u_m)_t, \phi \rangle \, dt
$$
$$
+ \int_0^T \int_\Omega [|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u_m|^{p(x)-2}\nabla u_m] \cdot \nabla \phi \, dx dt
$$
$$
= \int_0^T \int_\Omega (f_n - f_m)\phi \, dx dt,
$$

for all $\phi \in L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega)) \cap L^\infty(Q)$ with $\nabla \phi \in (L^{p(\cdot)}(Q))^N$. Choosing $\phi = T_1(u_n - u_m)\chi_{(0,t)}$ with $t \le T$ and discarding the positive term, we get

$$
\begin{aligned}
\int_\Omega \Theta_1(u_n - u_m)(t)\,dx &\le \int_\Omega \Theta_1(u_{0n} - u_{0m})\,dx + \|f_n - f_m\|_{L^1(Q)} \\
&\le \|u_{0n} - u_{0m}\|_{L^1(\Omega)} + \|f_n - f_m\|_{L^1(Q)} := a_{n,m}.
\end{aligned}
$$

Therefore, we conclude that

$$
\int_{\{|u_n-u_m|<1\}} \frac{|u_n - u_m|^2(t)}{2}\,dx + \int_{\{|u_n-u_m|\ge 1\}} \frac{|u_n - u_m|(t)}{2}\,dx
$$
$$
\le \int_\Omega [\Theta_1(u_n - u_m)](t)\,dx \le a_{n,m}.
$$

It follows that

$$
\begin{aligned}
\int_\Omega |u_n - u_m|(t)\,dx &= \int_{\{|u_n-u_m|<1\}} |u_n - u_m|(t)\,dx \\
&\quad + \int_{\{|u_n-u_m|\ge 1\}} |u_n - u_m|(t)\,dx \\
&\le \Big(\int_{\{|u_n-u_m|<1\}} |u_n - u_m|^2(t)\,dx\Big)^{\frac{1}{2}} \operatorname{meas}(\Omega)^{\frac{1}{2}} + 2a_{n,m} \\
&\le (2\operatorname{meas}(\Omega))^{\frac{1}{2}} a_{n,m}^{\frac{1}{2}} + 2a_{n,m}.
\end{aligned}
$$

Since $\{f_n\}$ and $\{u_{0n}\}$ are convergent in L^1, we have $a_{n,m} \to 0$ for $n,m \to +\infty$. Thus $\{u_n\}$ is a Cauchy sequence in $C([0,T];L^1(\Omega))$ and u_n converges to u in $C([0,T];L^1(\Omega))$. Then

we find an a.e. convergent subsequence (still denoted by $\{u_n\}$) in Q such that

$$u_n \to u \quad \text{a.e. in } Q. \tag{3.3}$$

Step 2. Prove $\nabla T_k(u_n)$ strongly converges to $\nabla T_k(u)$ in $(L^{p(\cdot)}(Q))^N$, for every $k > 0$. Choosing $T_k(u_n)$ as a test function in (3.2), we have

$$\int_\Omega \Theta_k(u_n)(T)\,dx - \int_\Omega \Theta_k(u_{0n})\,dx$$

$$+ \int_0^T \int_\Omega |\nabla T_k(u_n)|^{p(x)}\,dxdt = \int_0^T \int_\Omega f_n T_k(u_n)\,dxdt.$$

It follows from the definition of $\Theta_k(r)$ and (3.1) that

$$\int_0^T \int_\Omega |\nabla T_k(u_n)|^{p(x)}\,dxdt \quad \leq \quad k\big(\|f_n\|_{L^1(Q)} + \|u_{0n}\|_{L^1(\Omega)}\big)$$

$$\leq \quad k\big(\|f\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}\big). \tag{3.4}$$

Combining (3.4) with Lemma 2.2, we deduce that

$$\int_0^T \min\{|\nabla T_k(u_n)|^{p+}_{p(x)}, |\nabla T_k(u_n)|^{p-}_{p(x)}\}\,dt \leq \int_0^T \rho(\nabla T_k(u_n))\,dt \leq C,$$

that is $T_k(u_n)$ is bounded in $L^{p-}\big(0,T;W_0^{1,p(x)}(\Omega)\big)$.

For every $k, h > 0$, using the boundedness of $\nabla T_k(u_n)$ and $\nabla T_{2k}(u_n - T_h(u_n))$ in $(L^{p(\cdot)}(Q))^N$, we draw a subsequence (still denoted by $\{u_n\}$) from $\{u_n\}$ such that

$$\nabla T_k(u_n) \rightharpoonup \nabla T_k(u) \quad \text{weakly in } (L^{p(\cdot)}(Q))^N, \tag{3.5}$$

$$\nabla T_{2k}(u_n - T_h(u_n)) \rightharpoonup \nabla T_{2k}(u - T_h(u)) \quad \text{weakly in } (L^{p(\cdot)}(Q))^N. \tag{3.6}$$

In order to deal with the time derivative of truncations, we will use the regularization method of Landes [24] and use the sequence $(T_k(u))_\mu$ as approximation of $T_k(u)$. For $\mu > 0$, we define the regularization in time of the function $T_k(u)$ given by

$$\big(T_k(u)\big)_\mu(x,t) := \mu \int_{-\infty}^t e^{\mu(s-t)} T_k(u(x,s))\,ds,$$

extending $T_k(u)$ by 0 for $s < 0$. Observe that $(T_k(u))_\mu \in L^{p-}(0,T;W_0^{1,p(\cdot)}(\Omega)) \cap L^\infty(Q)$ with $\nabla(T_k(u))_\mu \in (L^{p(\cdot)}(Q))^N$, it is differentiable for a.e. $t \in (0,T)$ with

$$|(T_k(u))_\mu(x,t)| \leq k(1 - e^{-\mu t}) < k \quad \text{a.e. in } Q,$$

$$\frac{\partial(T_k(u))_\mu}{\partial t} = \mu\big(T_k(u) - (T_k(u))_\mu\big).$$

After computation, we can get

$$\nabla(T_k(u))_\mu \to \nabla T_k(u) \quad \text{strongly in } (L^{p(\cdot)}(Q))^N.$$

Let us take now a sequence $\{\psi_j\}$ of $C_0^\infty(\Omega)$ functions that strongly converge to u_0 in $L^1(\Omega)$, and set

$$\eta_{\mu,j}(u) \equiv (T_k(u))_\mu + e^{-\mu t} T_k(\psi_j).$$

The definition of $\eta_{\mu,j}$, which is a smooth approximation of $T_k(u)$, is needed to deal with a nonzero initial datum (see also [30]). Note that this function has the following properties:

$$\begin{cases} (\eta_{\mu,j}(u))_t = \mu(T_k(u) - \eta_{\mu,j}(u)), \\ \eta_{\mu,j}(u)(0) = T_k(\psi_j), \\ |\eta_{\mu,j}(u)| \leq k, \\ \nabla\eta_{\mu,j}(u) \to \nabla T_k(u) \quad \text{strongly in } (L^{p(\cdot)}(Q))^N, \text{ as } \mu \to +\infty. \end{cases} \tag{3.7}$$

Fix a positive number k. Let $h > k$. We choose

$$w_n = T_{2k}\big(u_n - T_h(u_n) + T_k(u_n) - \eta_{\mu,j}(u)\big)$$

as a test function in (3.2). The use of w_n as a test function to prove the strong convergence of truncations was first introduced in the elliptic case in [25], then adapted to parabolic equations in [30]. If we set $M = 4k+h$, then it is easy to see that $\nabla w_n = 0$ where $|u_n| > M$. Therefore, we may write the weak form of (3.2) as

$$\int_0^T \big\langle \frac{\partial u_n}{\partial t}, w_n \big\rangle \, dt + \int_0^T \int_\Omega |\nabla T_M(u_n)|^{p(x)-2} \nabla T_M(u_n) \cdot \nabla w_n \, dxdt$$
$$= \int_0^T \int_\Omega f_n w_n \, dxdt.$$

In the following, denote $w(n, \mu, j, h)$ all quantities such that

$$\lim_{h\to+\infty} \lim_{j\to+\infty} \lim_{\mu\to+\infty} \lim_{n\to+\infty} w(n, \mu, j, h) = 0.$$

First as far as the first term is concerned, that is

$$\int_0^T \big\langle \frac{\partial u_n}{\partial t}, w_n \big\rangle \, dt.$$

Since $|\eta_{\mu,j}(u)| \leq k$, w_n can be written as

$$w_n = T_{h+k}(u_n - \eta_{\mu,j}(u)) - T_{h-k}(u_n - T_k(u_n)).$$

Applying Lemma 2.1 in [30], we can obtain that

$$\int_0^T \big\langle \frac{\partial u_n}{\partial t}, w_n \big\rangle \, dt \geq w(n, j, h).$$

From the above estimate, we have

$$\int_0^T \int_\Omega |\nabla T_M(u_n)|^{p(x)-2} \nabla T_M(u_n) \cdot \nabla w_n \, dxdt \leq \int_0^T \int_\Omega f_n w_n \, dxdt + w(n, j, h).$$

Splitting the integral in the left-hand side on the sets where $|u_n| \leq k$ and where $|u_n| > k$ and discarding some nonnegative terms, we find

$$\int_0^T \int_\Omega |\nabla T_M(u_n)|^{p(x)-2} \nabla T_M(u_n) \cdot \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - \eta_{\mu,j}(u)) \, dxdt$$

$$\geq \int_0^T \int_\Omega |\nabla T_k(u_n)|^{p(x)-2} \nabla T_k(u_n) \cdot \nabla(T_k(u_n) - \eta_{\mu,j}(u)) \, dxdt$$

$$- \int_{\{|u_n|>k\}} \big| |\nabla T_M(u_n)|^{p(x)-2} \nabla T_M(u_n) \big| |\nabla\eta_{\mu,j}(u)| \, dxdt.$$

It follows from the above inequality that

$$\int_0^T \int_\Omega |\nabla T_k(u_n)|^{p(x)-2} \nabla T_k(u_n) \cdot \nabla(T_k(u_n) - \eta_{\mu,j}(u)) \, dxdt$$
$$\leq \int_{\{|u_n|>k\}} \left||\nabla T_M(u_n)|^{p(x)-2} \nabla T_M(u_n)\right||\nabla \eta_{\mu,j}(u)| \, dxdt$$
$$+ \int_0^T \int_\Omega f_n w_n \, dxdt + w(n,\mu,j,h).$$

Using the fact that $\nabla \eta_{\mu,j}(u) \to \nabla T_k(u)$ strongly in $(L^{p(\cdot)}(Q))^N$ as $\mu \to +\infty$, we conclude that

$$\int_0^T \int_\Omega |\nabla T_k(u_n)|^{p(x)-2} \nabla T_k(u_n) \cdot \nabla(T_k(u_n) - T_k(u)) \, dxdt$$
$$\leq \int_{\{|u_n|>k\}} \left||\nabla T_M(u_n)|^{p(x)-2} \nabla T_M(u_n)\right||\nabla \eta_{\mu,j}(u)| \, dxdt$$
$$+ \int_0^T \int_\Omega f_n w_n \, dxdt + w(n,\mu,j,h).$$

Furthermore, we have

$$\int_0^T \int_\Omega (|\nabla T_k(u_n)|^{p(x)-2} \nabla T_k(u_n) - |\nabla T_k(u)|^{p(x)-2} \nabla T_k(u))$$
$$\cdot \nabla(T_k(u_n) - T_k(u)) \, dxdt$$
$$\leq \int_{\{|u_n|>k\}} \left||\nabla T_M(u_n)|^{p(x)-2} \nabla T_M(u_n)\right||\nabla \eta_{\mu,j}(u)| \, dxdt$$
$$+ \int_0^T \int_\Omega f_n T_{2k}(u_n - T_h(u_n) + T_k(u_n) - \eta_{\mu,j}(u)) \, dxdt$$
$$- \int_0^T \int_\Omega |\nabla T_k(u)|^{p(x)-2} \nabla T_k(u) \cdot \nabla(T_k(u_n) - T_k(u)) \, dxdt + w(n,\mu,j,h)$$
$$= I_1 + I_2 + I_3 + w(n,\mu,j,h). \tag{3.8}$$

Now we show the limits of I_1, I_2 and I_3 are zeros when n, μ and then h tend to infinity respectively.

Limit of I_1. We observe that $|\nabla T_M(u_n)|^{p(x)-2} \nabla T_M(u_n)$ is bounded in $L^{p'(x)}(Q)$, and by the dominated convergence theorem $\chi_{\{|u_n|>k\}}|\nabla \eta_{\mu,j}(u)|$ converges strongly in $L^{p(x)}(Q)$ to $\chi_{\{|u|>k\}}|\nabla T_k(u)|$, which is zero, as n and μ tends to infinity. Thus we obtain

$$\lim_{\mu \to +\infty} \lim_{n \to +\infty} I_1$$
$$= \lim_{\mu \to +\infty} \lim_{n \to +\infty} \int_{\{|u_n|>k\}} \left||\nabla T_M(u_n)|^{p(x)-2} \nabla T_M(u_n)\right||\nabla T_k(u)| \, dxdt$$
$$= 0. \tag{3.9}$$

Limit of I_2. Notice that

$$I_2 \leq \int_0^T \int_\Omega |f_n - f||T_{2k}(u_n - T_h(u_n) + T_k(u_n) - \eta_{\mu,j}(u))| \, dxdt$$

$$+ \int_0^T \int_\Omega |fT_{2k}(u_n - T_h(u_n) + T_k(u_n) - \eta_{\mu,j}(u))| \, dxdt$$

$$\leq 2k \int_0^T \int_\Omega |f_n - f| \, dx$$

$$+ \int_0^T \int_\Omega |fT_{2k}(u_n - T_h(u_n) + T_k(u_n) - \eta_{\mu,j}(u))| \, dxdt.$$

Since f_n is strongly compact in $L^1(Q)$, using (3.3), the definition of $\eta_{\mu,j}$ and the Lebesgue dominated convergence theorem, we have

$$\lim_{h \to +\infty} \lim_{\mu \to +\infty} \lim_{n \to +\infty} |I_2| \leq \lim_{h \to +\infty} \int_0^T \int_\Omega |fT_{2k}(u - T_h(u))| \, dxdt = 0. \qquad (3.10)$$

Limit of I_3. Recalling (3.5), we have

$$\lim_{n \to +\infty} I_3 = 0. \qquad (3.11)$$

Therefore, passing to the limits in (3.8) as n, μ, j, and then h tend to infinity, by means of (3.9), (3.10) and (3.11), we deduce that

$$\lim_{n \to +\infty} E(n) = 0,$$

where

$$E(n) = \int_0^T \int_\Omega \left(|\nabla T_k(u_n)|^{p(x)-2} \nabla T_k(u_n) - |\nabla T_k(u)|^{p(x)-2} \nabla T_k(u) \right)$$

$$\cdot \nabla (T_k(u_n) - T_k(u)) \, dxdt.$$

We recall the following well-known inequalities: for any two real vectors $a, b \in \mathbb{R}^N$,

$$(a|a|^{p-2} - b|b|^{p-2})(a - b) \geq c(p)|a - b|^p, \text{ if } p \geq 2$$

and for every $\varepsilon \in (0, 1]$,

$$|a - b|^p \leq c(p)\varepsilon^{(p-2)/p}(a|a|^{p-2} - b|b|^{p-2})(a - b) + \varepsilon|b|^p, \text{ if } 1 < p < 2,$$

where $c(p) = \frac{2^{1-p}}{p-1}$ when $p \geq 2$ and $c(p) = \frac{3^{2-p}}{p-1}$ when $1 < p < 2$.

Therefore, we have

$$\int_{\{(x,t) \in Q : p(x) \geq 2\}} |\nabla T_k(u_n) - \nabla T_k(u)|^{p(x)} \, dxdt \leq 2^{p_+ - 1}(p_+ - 1)E(n) \qquad (3.12)$$

and

$$\int_{\{(x,t) \in Q : 1 < p(x) < 2\}} |\nabla T_k(u_n) - \nabla T_k(u)|^{p(x)} \, dxdt$$

$$\leq \frac{3^{2-p_-}}{p_- - 1} \cdot \varepsilon^{(p_- - 2)/p_-} E(n) + \varepsilon \int_0^T \int_\Omega |\nabla T_k(u)|^{p(x)} \, dxdt. \qquad (3.13)$$

Since $E(n) \to 0$ as $n \to +\infty$, then using the arbitrariness of ε and $\nabla T_k(u)$ is bounded in $(L^{p(\cdot)}(Q))^N$, we conclude that

$$\lim_{n \to +\infty} \int_0^T \int_\Omega |\nabla T_k(u_n) - \nabla T_k(u)|^{p(x)} \, dxdt = 0,$$

which implies that, for every $k > 0$,

$$\nabla T_k(u_n) \to \nabla T_k(u) \quad \text{strongly in } (L^{p(\cdot)}(Q))^N \qquad (3.14)$$

and
$$|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n) \to |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u) \quad \text{in } (L^{p'(\cdot)}(Q))^N. \tag{3.15}$$
Thanks to Lemma 2.2, we know that
$$T_k(u_n) \to T_k(u) \quad \text{strongly in } L^{p-}\left(0,T;W_0^{1,p(\cdot)}(\Omega)\right).$$

Step 3. Show that u is a renormalized solution.

For given $a, k > 0$, define the function $T_{k,a}(s) = T_a(s - T_k(s))$ as
$$T_{k,a}(s) = \begin{cases} s - k\,\text{sign}(s) & \text{if } k \le |s| < k+a, \\ a\,\text{sign}(s) & \text{if } |s| \ge k+a, \\ 0 & \text{if } |s| \le k. \end{cases}$$

Using $T_{k,a}(u_n)$ as a test function in (3.2), we find
$$\int_{\{|u_n|>k\}}\Theta_a(u_n \mp k)(T)\,dx - \int_{\{|u_{0n}|>k\}}\Theta_a(u_{0n} \mp k)\,dx$$
$$+ \int_{\{k\le|u_n|\le k+a\}}|\nabla u_n|^{p(x)-2}\nabla u_n \cdot \nabla u_n\,dxdt \le \int_\Omega f_n T_{k,a}(u_n)\,dxdt,$$
which yields that
$$\int_{\{k\le|u_n|\le k+a\}}|\nabla u_n|^{p(x)}\,dxdt \le a\left(\int_{\{|u_n|>k\}}|f_n|\,dxdt + \int_{\{|u_{0n}|>k\}}|u_{0n}|\,dx\right).$$
Recalling the convergence of $\{u_n\}$ in $C([0,T];L^1(\Omega))$, we have
$$\lim_{k\to+\infty}\text{meas}\{(x,t)\in Q : |u_n| > k\} = 0 \quad \text{uniformly with respect to } n.$$
Therefore, passing to the limit first in n then in k, we conclude that
$$\lim_{k\to+\infty}\int_{\{(x,t)\in Q:k\le|u(x,t)|\le k+a\}}|\nabla u|^{p(x)}\,dxdt = 0.$$
Choosing $a = 1$, we obtain the renormalized condition, i.e.,
$$\lim_{k\to+\infty}\int_{\{(x,t)\in Q:k\le|u(x,t)|\le k+1\}}|\nabla u|^{p(x)}\,dxdt = 0.$$

Let $S \in W^{2,\infty}(\mathbb{R})$ be such that $\text{supp}S' \subset [-M,M]$ for some $M > 0$. For every $\varphi \in C^\infty(\bar{Q})$ with $\varphi(x,T) = 0$, $S'(u_n)\varphi$ is a test function in (3.2). It yields
$$\int_0^T\int_\Omega \frac{\partial S(u_n)}{\partial t}\varphi\,dxdt + \int_0^T\int_\Omega [S'(u_n)|\nabla u_n|^{p(x)-2}\nabla u_n \cdot \nabla\varphi$$
$$+ S''(u_n)|\nabla u_n|^{p(x)}\varphi]\,dxdt = \int_0^T\int_\Omega f_n S'(u_n)\varphi\,dxdt. \tag{3.16}$$

First we consider the first term on the left-hand side of (3.16). Since S is bounded and continuous, (3.3) implies that $S(u_n)$ converges to $S(u)$ a.e. in Q and weakly-* in $L^\infty(Q)$. Then $\frac{\partial S(u_n)}{\partial t}$ converges to $\frac{\partial S(u)}{\partial t}$ in $D'(Q)$ as $n \to +\infty$, that is
$$\int_0^T\int_\Omega \frac{\partial S(u_n)}{\partial t}\varphi\,dxdt \to \int_0^T\int_\Omega \frac{\partial S(u)}{\partial t}\varphi\,dxdt.$$

For the other terms on the left-hand side of (3.16), because of $\text{supp}S' \subset [-M,M]$ we know
$$S'(u_n)|\nabla u_n|^{p(x)-2}\nabla u_n = S'(u_n)|\nabla T_M(u_n)|^{p(x)-2}\nabla T_M(u_n)$$

and
$$S''(u_n)|\nabla u_n|^{p(x)} = S''(u_n)|\nabla T_M(u_n)|^{p(x)}.$$
Using (3.3), (3.14) and (3.15), we have
$$S'(u_n)|\nabla T_M(u_n)|^{p(x)-2}\nabla T_M(u_n) \to S'(u)|\nabla T_M(u)|^{p(x)-2}\nabla T_M(u) \text{ in } (L^{p'(\cdot)}(Q))^N$$
and
$$S''(u_n)|\nabla T_M(u_n)|^{p(x)} \to S''(u)|\nabla T_M(u)|^{p(x)} \quad \text{in } L^1(Q).$$

Noting that
$$S'(u)|\nabla T_M(u)|^{p(x)-2}\nabla T_M(u) = S'(u)|\nabla u|^{p(x)-2}\nabla u,$$
$$S''(u)|\nabla T_M(u)|^{p(x)} = S''(u)|\nabla u|^{p(x)},$$

we deduce
$$S'(u_n)|\nabla u_n|^{p(x)-2}\nabla u_n \to S'(u)|\nabla u|^{p(x)-2}\nabla u \quad \text{in } (L^{p'(\cdot)}(Q))^N$$
and
$$S''(u_n)|\nabla u_n|^{p(x)} \to S''(u)|\nabla u|^{p(x)} \quad \text{in } L^1(Q).$$

For the right-hand side of (3.16), thanks to the strong convergence of f_n, it is easy to pass to the limits. Therefore, we obtain

$$-\int_\Omega S(u_0)\varphi(x,0)\,dx - \int_0^T\int_\Omega S(u)\frac{\partial\varphi}{\partial t}\,dxdt$$
$$+\int_0^T\int_\Omega [S'(u)|\nabla u|^{p(x)-2}\nabla u\cdot\nabla\varphi + S''(u)|\nabla u|^{p(x)}\varphi]\,dxdt$$
$$=\int_0^T\int_\Omega fS'(u)\varphi\,dxdt.$$

This completes the proof of the existence of renormalized solutions.

(2) Uniqueness of renormalized solutions.

Now we prove the uniqueness of renormalized solutions for problem (1.1) by choosing an appropriate test function motivated by [9] and [6]. Let u and v be two renormalized solutions for problem (1.1). Fix a positive number k. For $\sigma > 0$, let S_σ be the function defined by

$$\begin{cases} S_\sigma(r) = r & \text{if } |r| < \sigma, \\ S_\sigma(r) = (\sigma + \frac{1}{2}) \mp \frac{1}{2}(r \mp (\sigma+1))^2 & \text{if } \sigma \le \pm r \le \sigma+1, \\ S_\sigma(r) = \pm(\sigma + \frac{1}{2}) & \text{if } \pm r > \sigma+1. \end{cases} \qquad (3.17)$$

It is obvious that

$$\begin{cases} S_\sigma'(r) = 1 & \text{if } |r| < \sigma, \\ S_\sigma'(r) = \sigma+1-|r| & \text{if } \sigma \le |r| \le \sigma+1, \\ S_\sigma'(r) = 0 & \text{if } |r| > \sigma+1. \end{cases}$$

It is easy to check $S_\sigma \in W^{2,\infty}(\mathbb{R})$ with $\mathrm{supp} S_\sigma' \subset [-\sigma-1, \sigma+1]$ and $\mathrm{supp} S_\sigma'' \subset [\sigma, \sigma+1] \cup [-\sigma-1, -\sigma]$. Therefore, we may take $S = S_\sigma$ in (1.3) to have

$$\int_0^T\int_\Omega \frac{\partial S_\sigma(u)}{\partial t}\varphi\,dxdt + \int_0^T\int_\Omega [S_\sigma'(u)|\nabla u|^{p(x)-2}\nabla u\cdot\nabla\varphi$$

$$+S''_\sigma(u)|\nabla u|^{p(x)}\varphi]\,dxdt = \int_0^T \int_\Omega fS'_\sigma(u)\varphi\,dxdt$$

and

$$\int_0^T \int_\Omega \frac{\partial S_\sigma(v)}{\partial t}\varphi\,dxdt + \int_0^T \int_\Omega [S'_\sigma(v)|\nabla v|^{p(x)-2}\nabla v \cdot \nabla\varphi$$

$$+S''_\sigma(v)|\nabla v|^{p(x)}\varphi]\,dxdt = \int_0^T \int_\Omega fS'_\sigma(v)\varphi\,dxdt.$$

We plug $\varphi = T_k(S_\sigma(u) - S_\sigma(v))$ as a test function in the above equalities and subtract them to obtain that

$$J_0 + J_1 + J_2 = J_3, \tag{3.18}$$

where

$$J_0 = \int_0^T \Big\langle \frac{\partial(S_\sigma(u) - S_\sigma(v))}{\partial t}, T_k(S_\sigma(u) - S_\sigma(v)) \Big\rangle\,dt,$$

$$J_1 = \int_0^T \int_\Omega \big(S'_\sigma(u)|\nabla u|^{p(x)-2}\nabla u - S'_\sigma(v)|\nabla v|^{p(x)-2}\nabla v\big)$$
$$\cdot\nabla T_k(S_\sigma(u) - S_\sigma(v))\,dxdt,$$

$$J_2 = \int_0^T \int_\Omega [S''_\sigma(u)|\nabla u|^{p(x)} - S''_\sigma(v)|\nabla v|^{p(x)}]T_k(S_\sigma(u) - S_\sigma(v))\,dxdt,$$

$$J_3 = \int_0^T \int_\Omega f(S'_\sigma(u) - S'_\sigma(v))T_k(S_\sigma(u) - S_\sigma(v))\,dxdt.$$

We estimate J_0, J_1, J_2 and J_3 one by one. Recalling the definition of $\Theta_k(r)$, J_0 can be written as

$$J_0 = \int_\Omega \Theta_k(S_\sigma(u) - S_\sigma(v))(T)\,dx - \int_\Omega \Theta_k(S_\sigma(u) - S_\sigma(v))(0)\,dx.$$

Due to the same initial condition for u and v, and the properties of Θ_k, we get

$$J_0 = \int_\Omega \Theta_k(S_\sigma(u) - S_\sigma(v))(T)\,dx \geq 0.$$

Writing

$$J_1 = \int_0^T \int_\Omega \big[|\nabla S_\sigma(u)|^{p(x)-2}\nabla S_\sigma(u) - |\nabla S_\sigma(v)|^{p(x)-2}\nabla S_\sigma(v)\big]$$
$$\cdot\nabla T_k(S_\sigma(u) - S_\sigma(v))\,dxdt$$

$$+ \int_0^T \int_\Omega \big[S'_\sigma(u) - S'_\sigma(u)|S'_\sigma(u)|^{p(x)-2}\big]|\nabla u|^{p(x)-2}\nabla u$$
$$\cdot\nabla T_k(S_\sigma(u) - S_\sigma(v))\,dxdt$$

$$- \int_0^T \int_\Omega \big[S'_\sigma(v) - S'_\sigma(v)|S'_\sigma(v)|^{p(x)-2}\big]|\nabla v|^{p(x)-2}\nabla v$$
$$\cdot\nabla T_k(S_\sigma(u) - S_\sigma(v))\,dxdt$$

$$:= J_1^1 + J_1^2 + J_1^3,$$

and setting $\sigma \geq k$, we have

$$J_1^1 \geq \int_{\{|u-v|\leq k\}\cap\{|u|,|v|\leq k\}} (|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v) \cdot \nabla(u-v) \, dxdt. \qquad (3.19)$$

Recalling $\mathrm{supp} S_\sigma' \subset [-\sigma-1, \sigma+1]$ and $\mathrm{supp} S_\sigma'' \subset [\sigma, \sigma+1] \cup [-\sigma-1, -\sigma]$, we obtain

$$
\begin{aligned}
|J_1^2| \quad \leq \quad & 2\Big(\int_{\{\sigma\leq|u|\leq\sigma+1\}} |\nabla u|^{p(x)} \, dxdt \\
& + \int_{\{\sigma\leq|u|\leq\sigma+1\}\cap\{|v|\leq\sigma+1\}\cap\{|S_\sigma(u)-S_\sigma(v)|\leq k\}} |\nabla u|^{p(x)-1}|\nabla v| \, dxdt \Big) \\
\leq \quad & 2\Big(\int_{\{\sigma\leq|u|\leq\sigma+1\}} |\nabla u|^{p(x)} \, dxdt + \int_{\{\sigma\leq|u|\leq\sigma+1\}\cap\{\sigma-k\leq|v|\leq\sigma+1\}} |\nabla u|^{p(x)-1}|\nabla v| \, dxdt \Big) \\
\leq \quad & C\Big(\int_{\{\sigma\leq|u|\leq\sigma+1\}} |\nabla u|^{p(x)} \, dxdt + \int_{\{\sigma-k\leq|v|\leq\sigma+1\}} |\nabla v|^{p(x)} \, dxdt \Big).
\end{aligned}
$$

And we may get the similar estimate for J_1^3. Furthermore, we have

$$|J_2| \leq C\Big(\int_{\{\sigma\leq|u|\leq\sigma+1\}} |\nabla u|^{p(x)} \, dxdt + \int_{\{\sigma\leq|v|\leq\sigma+1\}} |\nabla v|^{p(x)} \, dxdt \Big).$$

From the above estimates and (i) in Definition 1.1, we obtain

$$\lim_{\sigma\to+\infty} (|J_1^2| + |J_1^3| + |J_2|) = 0.$$

Observing

$$f(S_\sigma'(u) - S_\sigma'(v)) \to 0 \quad \text{strongly in } L^1(Q)$$

as $\sigma \to +\infty$ and using the Lebesgue dominated convergence theorem, we deduce that

$$\lim_{\sigma\to+\infty} |J_3| = 0.$$

Therefore, sending $\sigma \to +\infty$ in (3.18) and recalling (3.19), we have

$$\int_{\{|u|\leq\frac{k}{2}, |v|\leq\frac{k}{2}\}} (|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v) \cdot \nabla(u-v) \, dxdt = 0,$$

which implies $\nabla u = \nabla v$ a.e. on the set $\{|u| \leq \frac{k}{2}, |v| \leq \frac{k}{2}\}$. Since k is arbitrary, we conclude that $\nabla u = \nabla v$ a.e. in Q. Then, set $\xi_n = T_1(T_{n+1}(u) - T_{n+1}(v))$. We have $\xi_n \in L^{p^-}(0, T; W_0^{1,p(x)}(\Omega))$ and

$$
\nabla \xi_n = \begin{cases}
0 & \text{on } \{|u| \leq n+1, |v| \leq n+1\}, \\
\nabla u \chi_{\{|u-T_{n+1}(v)|\leq 1\}} & \text{on } \{|u| \leq n+1, |v| > n+1\}, \\
-\nabla v \chi_{\{|T_{n+1}(u)-v|\leq 1\}} & \text{on } \{|u| > n+1, |v| \leq n+1\},
\end{cases}
$$

such that

$$\int_Q |\nabla \xi_n|^{p(x)} \, dxdt \leq \int_{\{n\leq|u|\leq n+1\}} |\nabla u|^{p(x)} \, dxdt + \int_{\{n\leq|v|\leq n+1\}} |\nabla v|^{p(x)} \, dxdt.$$

Thanks to Lemma 2.2 and (i) in Definition 1.1, we deduce that $\xi_n \to 0$ strongly in $L^{p^-}(0, T; W_0^{1,p(x)}(\Omega))$. Since $\xi_n \to T_1(u-v)$ a.e. in Q, we conclude that $T_1(u-v) = 0$, hence $u = v$ a.e. in Q. Therefore we obtain the uniqueness of renormalized solutions. This completes the proof of Theorem 1.1. $\square$

Next, we prove that the renormalized solution u is also an entropy solution of problem (1.1) and the entropy solution of problem (1.1) is unique.

Proof of Theorem 1.2.

(1) The renormalized solution is an entropy solution.

Now we choose $v_n = T_k(u_n - \phi)$ as a test function in (3.2) for $k > 0$ and $\phi \in C^1(\bar{Q})$ with $\phi|_\Gamma = 0$. We note that, if $L = k + \|\phi\|_{L^\infty(Q)}$ and $n > L$, then

$$\int_0^T \int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla T_k(u_n - \phi) \, dxdt$$
$$= \int_0^T \int_\Omega |\nabla T_L(u_n)|^{p(x)-2} \nabla T_L(u_n) \cdot \nabla T_k(T_L(u_n) - \phi) \, dxdt$$

and

$$\int_0^T \langle (u_n)_t, T_k(u_n - \phi) \rangle \, dt$$
$$+ \int_0^T \int_\Omega |\nabla T_L(u_n)|^{p(x)-2} \nabla T_L(u_n) \cdot \nabla T_k(T_L(u_n) - \phi) \, dxdt$$
$$= \int_0^T \int_\Omega f_n T_k(u_n - \phi) \, dxdt.$$

Since $(u_n)_t = (u_n - \phi)_t + \phi_t$, we have

$$\int_0^T \langle (u_n)_t, T_k(u_n - \phi) \rangle \, dt$$
$$= \int_\Omega \Theta_k(u_n - \phi)(T) \, dx - \int_\Omega \Theta_k(u_n - \phi)(0) \, dx + \int_0^T \langle \phi_t, T_k(u_n - \phi) \rangle \, dt,$$

which yields that

$$\int_\Omega \Theta_k(u_n - \phi)(T) \, dx - \int_\Omega \Theta_k(u_n - \phi)(0) \, dx + \int_0^T \langle \phi_t, T_k(T_L(u_n) - \phi) \rangle \, dt$$
$$+ \int_0^T \int_\Omega |\nabla T_L(u_n)|^{p(x)-2} \nabla T_L(u_n) \cdot \nabla T_k(T_L(u_n) - \phi) \, dxdt$$
$$= \int_0^T \int_\Omega f_n T_k(u_n - \phi) \, dxdt. \tag{3.20}$$

Recalling u_n converges to u in $C([0,T]; L^1(\Omega))$, hence $\forall \, t \le T, u_n(t) \to u(t)$ in $L^1(\Omega)$. Since Θ_k is Lipschitz continuous, we get

$$\int_\Omega \Theta_k(u_n - \phi)(T) \, dx \to \int_\Omega \Theta_k(u - \phi)(T) \, dx$$

and

$$\int_\Omega \Theta_k(u_n - \phi)(0) \, dx \to \int_\Omega \Theta_k(u_0 - \phi(0)) \, dx,$$

as $n \to +\infty$.

Using the strong convergence of f_n, (3.5) and (3.15), we can pass to the limits as n tends to infinity for the other terms to conclude

$$\int_\Omega \Theta_k(u - \phi)(T)\, dx - \int_\Omega \Theta_k(u_0 - \phi(0))\, dx + \int_0^T \langle \phi_t, T_k(u - \phi) \rangle\, dt$$

$$+ \int_0^T \int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla T_k(u - \phi)\, dx = \int_\Omega f T_k(u - \phi)\, dx,$$

for all $k > 0$ and $\phi \in C^1(\bar{Q})$ with $\phi|_\Gamma = 0$. Therefore, we finish the proof of the existence of entropy solutions.

(2) Uniqueness of entropy solutions.

Suppose that u and v are two entropy solutions of problem (1.1). Let $\{u_n\}$ be a sequence constructed in (3.2), which satisfies $\nabla T_k(u_n)$ strongly converges to $\nabla T_k(u)$ in $(L^{p(\cdot)}(Q))^N$, for every $k > 0$. Choosing $S_\sigma(u_n)$ as a test function in (1.4) for entropy solution v, we have

$$\int_\Omega \Theta_k(v - S_\sigma(u_n))(T)\, dx - \int_\Omega \Theta_k(u_0 - S_\sigma(u_{0n}))\, dx$$

$$+ \int_0^T \langle (u_n)_t, S'_\sigma(u_n) T_k(v - S_\sigma(u_n)) \rangle\, dt$$

$$+ \int_0^T \int_\Omega |\nabla v|^{p(x)-2} \nabla v \cdot \nabla T_k(v - S_\sigma(u_n))\, dxdt$$

$$= \int_0^T \int_\Omega f T_k(v - S_\sigma(u_n))\, dxdt. \tag{3.21}$$

In order to deal with the third term on the left-hand side of (3.21), we take $S'_\sigma(u_n)\Psi$ with $\Psi = T_k(v - S_\sigma(u_n))$ as a test function for problem (3.2) to obtain

$$\int_0^T \langle (u_n)_t, S'_\sigma(u_n)\Psi \rangle\, dt + \int_0^T \int_\Omega S''_\sigma(u_n)\Psi |\nabla u_n|^{p(x)}\, dxdt$$

$$+ \int_0^T \int_\Omega S'_\sigma(u_n) |\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla \Psi\, dxdt$$

$$= \int_0^T \int_\Omega f_n S'_\sigma(u_n)\Psi\, dxdt. \tag{3.22}$$

Thus we deduce from (3.21) and (3.22) that

$$\int_\Omega \Theta_k(v - S_\sigma(u_n))(T)\, dx - \int_\Omega \Theta_k(u_0 - S_\sigma(u_{0n}))\, dx$$

$$- \int_0^T \int_\Omega S''_\sigma(u_n) T_k(v - S_\sigma(u_n)) |\nabla u_n|^{p(x)}\, dxdt$$

$$- \int_0^T \int_\Omega S'_\sigma(u_n) |\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla T_k(v - S_\sigma(u_n))\, dxdt$$

$$+ \int_0^T \int_\Omega |\nabla v|^{p(x)-2} \nabla v \cdot \nabla T_k(v - S_\sigma(u_n))\, dxdt$$

$$= \int_0^T \int_\Omega f T_k(v - S_\sigma(u_n))\, dxdt - \int_0^T \int_\Omega f_n S'_\sigma(u_n) T_k(v - S_\sigma(u_n))\, dxdt.$$

We will pass to the limit as $n \to +\infty$ and $\sigma \to +\infty$ successively. Let us denote A_3 for the third term on the left-hand side of the above equality for simplicity. Recalling $\operatorname{supp} S_\sigma'' \subset [\sigma, \sigma+1] \cup [-\sigma-1, -\sigma]$, we have

$$|A_3| \leq k \Big(\int_{\{\sigma \leq |u_n| \leq \sigma+1\}} |\nabla u_n|^{p(x)} \, dxdt \Big).$$

Observe that $S_\sigma'(u_n)|\nabla u_n|^{p(x)-2}\nabla u_n = S_\sigma'(u_n)|\nabla T_{\sigma+1}(u_n)|^{p(x)-2}\nabla T_{\sigma+1}(u_n)$, then we get

$$\int_\Omega \Theta_k(v - S_\sigma(u_n))(T) \, dx - \int_\Omega \Theta_k(u_0 - S_\sigma(u_{0n})) \, dx$$
$$+ \int_0^T \int_\Omega (|\nabla v|^{p(x)-2}\nabla v - S_\sigma'(u_n)|\nabla T_{\sigma+1}(u_n)|^{p(x)-2}\nabla T_{\sigma+1}(u_n))$$
$$\cdot \nabla T_k(v - S_\sigma(u_n)) \, dxdt$$
$$\leq \int_0^T \int_\Omega (f - f_n S_\sigma'(u_n)) T_k(v - S_\sigma(u_n)) \, dxdt$$
$$+ k \Big(\int_{\{\sigma \leq |u_n| \leq \sigma+1\}} |\nabla u_n|^{p(x)} \, dxdt \Big).$$

Thanks to the fact that $\nabla T_k(u_n) \to \nabla T_k(u)$ strongly in $(L^{p(\cdot)}(Q))^N$ and the Lebesgue dominated convergence theorem, letting $n \to +\infty$, we obtain

$$\int_\Omega \Theta_k(v - S_\sigma(u))(T) \, dx - \int_\Omega \Theta_k(u_0 - S_\sigma(u_0)) \, dx$$
$$+ \int_0^T \int_\Omega (|\nabla v|^{p(x)-2}\nabla v - S_\sigma'(u)|\nabla T_{\sigma+1}(u)|^{p(x)-2}\nabla T_{\sigma+1}(u))$$
$$\cdot \nabla T_k(v - S_\sigma(u)) \, dxdt$$
$$\leq \int_0^T \int_\Omega f(1 - S_\sigma'(u)) T_k(v - S_\sigma(u)) \, dxdt$$
$$+ k \Big(\int_{\{\sigma \leq |u| \leq \sigma+1\}} |\nabla u|^{p(x)} \, dxdt \Big). \tag{3.23}$$

Let us denote A_3' for the third term on the left-hand side of (3.23). Then we can write A_3' as

$$
\begin{aligned}
A_3' &= \int_0^T \int_\Omega (|\nabla v|^{p(x)-2}\nabla v - S_\sigma'(u)|\nabla u|^{p(x)-2}\nabla u) \cdot \nabla T_k(v - S_\sigma(u)) \, dxdt \\
&= \int_0^T \int_\Omega (|\nabla v|^{p(x)-2}\nabla v - |\nabla S_\sigma(u)|^{p(x)-2}\nabla S_\sigma(u)) \cdot \nabla T_k(v - S_\sigma(u)) \, dxdt \\
&\quad + \int_0^T \int_\Omega [|S_\sigma'(u)|^{p(x)-2} S_\sigma'(u) - S_\sigma'(u)]|\nabla u|^{p(x)-2}\nabla u \\
&\qquad \cdot \nabla T_k(v - S_\sigma(u)) \, dxdt \\
&= I_1 + I_2.
\end{aligned}
$$

Recalling the definition of S_σ, we have

$$|I_2| \leq 2 \Big(\int_{\{\sigma \leq |u| \leq \sigma+1\}} |\nabla u|^{p(x)} \, dxdt + \int_{\{\sigma \leq |u| \leq \sigma+1\} \cap \{|v - S_\sigma(u)| \leq k\}} |\nabla u|^{p(x)-1}\nabla v \, dxdt \Big)$$

$$\leq \; 2\Big(\int_{\{\sigma\leq|u|\leq\sigma+1\}}|\nabla u|^{p(x)}\,dxdt + \int_{\{\sigma\leq|u|\leq\sigma+1\}\cap\{\sigma-k\leq|v|\leq\sigma+k+1\}}|\nabla u|^{p(x)-1}\nabla v\,dxdt\Big)$$

$$\leq \; C\Big(\int_{\{\sigma\leq|u|\leq\sigma+1\}}|\nabla u|^{p(x)}\,dxdt + \int_{\{\sigma-k\leq|v|\leq\sigma+k+1\}}|\nabla v|^{p(x)}\,dxdt\Big). \qquad (3.24)$$

Now we let $\sigma \to +\infty$. Since

$$|\Theta_k(v-S_\sigma(u))(T)| \leq k(|v(T)|+|u(T)|), \;\; |\Theta_k(u_0-S_\sigma(u_0))| \leq k|u_0|,$$

by the Lebesgue dominated convergence theorem, we have

$$\int_\Omega \Theta_k(u_0-S_\sigma(u_0))\,dx \to 0, \int_\Omega \Theta_k(v-S_\sigma(u))(T)\,dx \to \int_\Omega \Theta_k(v-u)(T)\,dx.$$

According to the fact that

$$\lim_{k\to+\infty}\int_{\{(x,t)\in Q:k\leq|u(x,t)|\leq k+1\}}|\nabla u|^{p(x)}\,dxdt = 0$$

and Fatou's lemma, we deduce from (3.23) and (3.24) that

$$\int_\Omega \Theta_k(v-u)(T)\,dx$$

$$+ \int_{\{|u|\leq\frac{k}{2},|v|\leq\frac{k}{2}\}}(|\nabla v|^{p(x)-2}\nabla v - |\nabla u|^{p(x)-2}\nabla u)\cdot\nabla(v-u)\,dxdt \leq 0.$$

Using the positivity of Θ_k, we have $\nabla u = \nabla v$ a.e. in Q, for all k. Similar to the case of renormalized solutions, we conclude that $u = v$ a.e. in Q. Therefore we obtain the uniqueness of entropy solutions. This completes the proof of Theorem 1.2. $\square$

Remark 3.1. Furthermore, we may improve the integrability of the renormalized solution or entropy solution u for problem (1.1) by assuming that $p_- > 2 - \frac{1}{N+1}$. Then we can prove that

$$\|u\|_{L^q(0,T;W_0^{1,q}(\Omega))} \leq C,$$

with

$$1 \leq q < \frac{p_-(N+1)-N}{N+1}.$$

Recalling (i) in Definition 1.1, $L^{p(\cdot)}(Q) \hookrightarrow L^{p_-}(Q)$ and Lemma 2.2, we get

$$\|\nabla u\|_{p_-,B_m} \;=\; |\nabla u|_{p_-,B_m} \leq C|\nabla u|_{p(\cdot),B_m}$$

$$\leq \; C\max\Big\{\Big(\int_{B_m}|\nabla u|^{p(x)}\,dxdt\Big)^{1/p_-}, \Big(\int_{B_m}|\nabla u|^{p(x)}\,dxdt\Big)^{1/p_+}\Big\}$$

$$\leq \; C,$$

where

$$B_m = \{(x,t)\in Q : m \leq |u(x,t)| < m+1\}.$$

Following the arguments in [12] and $u \in C([0,T];L^1(\Omega))$, we can conclude that

$$\|u\|_{L^q(0,T;W_0^{1,q}(\Omega))} \leq C.$$

Acknowledgment

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