ENTROPY SOLUTIONS FOR A NON-UNIFORMLY PARABOLIC EQUATION

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Abstract. In this paper we prove the existence and uniqueness of entropy solutions for the initial-boundary value problem of a non-uniformly parabolic equation. Moreover, we establish a comparison result. Some well-known parabolic equations are the special cases of this equation.

1. Introduction

Suppose that Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz boundary $\partial\Omega$, and T is a positive number. Denote $\Omega_T = \Omega \times (0, T], \Sigma = \partial\Omega \times (0, T]$. In this paper we study the following non-uniformly parabolic initial-boundary value problem

$$\begin{cases} u_t - \operatorname{div}\big(D_\xi \Phi(\nabla u)\big) = f & \text{in } \Omega_T, \\ u(x,t) = 0 & \text{on } \Sigma, \\ u(x,0) = u_0(x) & \text{on } \Omega, \end{cases} \tag{1.1}$$

where $\Phi : \mathbb{R}^N \mapsto \mathbb{R}_+$ is a C^1 nonnegative, strictly convex function, $D_\xi \Phi : \mathbb{R}^N \to \mathbb{R}$ represents the gradient of $\Phi(\xi)$ with respect to ξ and ∇u represents the gradient with respect to the spatial variables x. Without loss of generality we may assume that $\Phi(0) = 0$.

Our main assumptions are that $\Phi(\xi)$ satisfies the super-linear condition (or 1-coercive condition, see [15], Chapter E)

$$\lim_{|\xi|\to\infty} \frac{\Phi(\xi)}{|\xi|} = \infty, \tag{1.2}$$

and the symmetric condition: there exists a positive number $C > 0$ such that

$$\Phi(-\xi) \leq C\Phi(\xi), \quad \xi \in \mathbb{R}^N. \tag{1.3}$$

In this paper we assume that

$$u_0 \in L^1(\Omega) \quad \text{and} \quad f \in L^1(\Omega_T). \tag{1.4}$$

Under our assumptions, it is reasonable to work with entropy solutions or renormalized solutions, which need less regularity than the usual weak solutions. The notion of entropy solutions was first proposed by Bénilan et al. in [2] for the nonlinear elliptic problems. It was then adapted to the study of some nonlinear elliptic and parabolic problems. We refer to [4, 3, 1, 19] for details.

There are numerous examples of $\Phi(\xi)$ satisfying structure assumptions (1.2) and (1.3). The well-known are listed as follows.

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Example 1

$$\Phi(\xi) = \frac{1}{p}|\xi|^p, \quad p > 1.$$

Example 2

$$\Phi(\xi) = \frac{1}{p_1}|\xi_1|^{p_1} + \frac{1}{p_2}|\xi_2|^{p_2} + \cdots + \frac{1}{p_N}|\xi_N|^{p_N}, \quad p_i > 1,\ i = 1, 2, \ldots, N,$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_N)$. (See [17], Chapter 2.)

Example 3

$$\Phi(\xi) = |\xi| \log(1 + |\xi|)$$

(See [12] and [6], Chapter 4.)

Example 4

$$\Phi(\xi) = |\xi| L_k(|\xi|),$$

where $L_i(s) = \log(1 + L_{i-1}(s))$ $(i = 1, 2, \ldots, k)$ and $L_0(s) = \log(1 + s)$ for $s \geq 0$ (See [14].)

Example 5

$$\Phi(\xi) = e^{\frac{|\xi|^2}{2}} - 1.$$

(See [18], [9] and [16].)

Let T_k denote the truncation function at height $k \geq 0$:

$$T_k(r) = \min\{k, \max\{r, -k\}\} = \begin{cases} k & \text{if } r \geq k, \\ r & \text{if } |r| < k, \\ -k & \text{if } r \leq -k, \end{cases}$$

and its primitive $\Theta_k : \mathbb{R} \to \mathbb{R}^+$ by

$$\Theta_k(r) = \int_0^r T_k(s)\, ds = \begin{cases} \frac{r^2}{2} & \text{if } |r| \leq k, \\ k|r| - \frac{k^2}{2} & \text{if } |r| \geq k. \end{cases}$$

It is obvious that $\Theta_k(r) \geq 0$ and $\Theta_k(r) \leq k|r|$.

Next we define the very weak gradient of a measurable function u with $T_k(u) \in L^1(0, T; W_0^{1,1}(\Omega))$. As a matter of the fact, working as in Lemma 2.1 of [2] we can prove the following result:

Proposition 1.1. *For every measurable function u on Ω_T such that $T_k(u)$ belongs to $L^1(0, T; W_0^{1,1}(\Omega))$ for every $k > 0$, there exists a unique measurable function $v : \Omega_T \to \mathbb{R}^N$, such that*

$$\nabla T_k(u) = v\chi_{\{|u|<k\}}, \qquad \text{almost everywhere in } \Omega_T \text{ and for every } k > 0,$$

where χ_E denotes the characteristic function of a measurable set E. Moreover, if u belongs to $L^1(0, T; W_0^{1,1}(\Omega))$, then v coincides with the weak gradient of u.

From the above Proposition, we denote $v = \nabla u$, which is called the very weak gradient of u. The notion of the very weak gradient allows us to give the following definition of entropy solutions for problem (1.1). Denote $z = (x, t), dz = dxdt$.

Definition 1.2. *A function $u \in C([0, T]; L^1(\Omega))$ with $T_k(u) \in L^1(0, T; W_0^{1,1}(\Omega))$ is an entropy solution to problem* (1.1) *if the following conditions are satisfied:*

(i) $\displaystyle\int_{\Omega_T} D_\xi \Phi(\nabla T_k(u)) \cdot \nabla T_k(u)\, dz < +\infty;$

(ii) *For every $k > 0$ and every function $\phi \in C^1(\bar{\Omega}_T)$ with $\phi|_\Sigma = 0$,*

$$\int_\Omega \Theta_k(u-\phi)(T)\,dx - \int_\Omega \Theta_k(u_0 - \phi(0))\,dx + \int_0^T \langle \phi_t, T_k(u-\phi)\rangle\,dt$$
$$+ \int_{\Omega_T} D_\xi(\Phi(\nabla u)) \cdot \nabla T_k(u-\phi)\,dz \le \int_{\Omega_T} f T_k(u-\phi)\,dz \tag{1.5}$$

holds.

Now we state our main results. The first theorem is about the existence and uniqueness of entropy solutions. The second one is about the comparison principle.

Theorem 1.3. *Under structure assumptions* (1.2), (1.3) *and integrability condition* (1.4), *there exists a unique entropy solution for problem* (1.1).

Theorem 1.4. *Let $u_0, v_0 \in L^1(\Omega)$, $f, g \in L^1(\Omega_T)$ such that $u_0 \le v_0$ and $f \le g$. If u is the entropy solution of problem* (1.1) *and v is the entropy solution of problem* (1.1) *with u_0, f being replaced by v_0, g, then $u \le v$ a.e. in Ω_T.*

The rest of this paper is organized as follows. In Section 2, we state some basic results that will be used later. We will prove the main results in Section 3. In the following sections C will represent a generic constant that may change from line to line even if in the same inequality.

2. Preliminaries

Let $\Phi(\xi)$ be a nonnegative convex function. We define the polar function of $\Phi(\xi)$ as

$$\Psi(\eta) = \sup_{\xi \in \mathbb{R}^N} \{\eta \cdot \xi - \Phi(\xi)\}, \tag{2.1}$$

which is also known as the Legendre transform of $\Phi(\xi)$. It is obvious that $\Psi(\eta)$ is a convex function. In the following we will list several lemmas.

Definition 2.1. (See [15], Definition 4.1.3) *Let $C \subset \mathbb{R}^N$ be convex. The mapping $F : C \to \mathbb{R}^N$ is said to be monotone [rest. strictly monotone] on C when, for all x and x' in C,*

$$\langle F(x) - F(x'), x - x' \rangle \ge 0,$$
$$[\textit{resp. } \langle F(x) - F(x'), x - x' > 0\rangle \quad \textit{whenever} \quad x \ne x'].$$

Lemma 2.2. (See [15], Theorem 4.1.4) *Let f be a function differentiable on an open set $\Omega \subset \mathbb{R}^N$ and let C be a convex subset of Ω. Then, f is convex [resp. strictly convex] on C if and only if its gradient ∇f is monotone [resp. strictly monotone] on C.*

Lemma 2.3. *Suppose that $\Phi(\xi)$ is a convex C^1 function with $\Phi(0) = 0$. Then we have, for all $\xi, \zeta \in \mathbb{R}^N$,*

$$\Phi(\xi) \le \xi \cdot D\Phi(\xi), \tag{2.2}$$
$$(D\Phi(\xi) - D\Phi(\zeta)) \cdot (\xi - \zeta) \ge 0. \tag{2.3}$$

Lemma 2.4. ([5]) *Suppose that $\Phi(\xi)$ is a nonnegative convex C^1 function and $\Psi(\eta)$ is its polar function. Then we have, for $\xi, \eta, \zeta \in \mathbb{R}^N$,*

$$\xi \cdot \eta \le \Phi(\xi) + \Psi(\eta), \tag{2.4}$$
$$\Psi(D\Phi(\zeta)) + \Phi(\zeta) = D\Phi(\zeta) \cdot \zeta. \tag{2.5}$$

Lemma 2.5. (See [13], Chapter 3) *Suppose that $\Phi(\xi)$ is a nonnegative convex function with $\Phi(0) = 0$, which satisfies* (1.2). *Then its polar function $\Psi(\eta)$ in* (2.1) *is a well-defined, nonnegative function in $\mathbb{R}^N$, which also satisfies* (1.2).

Lemma 2.6. (See [20], Chapter 4) *Let $D \subset \mathbb{R}^N$ be measurable with finite Lebesgue measure and $f_k \in L^1(D)$ and $g_k \in L^1(D)\,(k = 1, 2, \dots)$, and*

$$|f_k(x)| \le g_k(x), \quad a.e.\ x \in D,\ k = 1, 2, \dots.$$

If

$$\lim_{k\to\infty} f_k(x) = f(x), \quad \lim_{k\to\infty} g_k(x) = g(x),\ a.e.\ x \in D,$$

and

$$\lim_{k\to\infty} \int_D g_k(x)\,dx = \int_D g(x)\,dx < +\infty,$$

then we have

$$\lim_{k\to\infty} \int_D f_k(x)\,dx = \int_D f(x)\,dx.$$

Lemma 2.7. *Let $D \subset \mathbb{R}^N$ be measurable with finite Lebesgue measure, and let $\{f_n\}$ be a sequence of functions in $L^p(D)(p \ge 1)$ such that*

$$f_n \rightharpoonup f \quad \text{weakly in } L^p(D),$$
$$f_n \to g \quad a.e.\ \text{in } D.$$

Then $f = g$ a.e. in D.

Proof. The result can be found in ([10], Proposition 9.1c). Here we give a slightly different proof.

Since $f_n \to g$ a.e. in D, we have $|f_n|^p \to |g|^p$ a.e. in D. It follows from Fatou's lemma that

$$\int_D |g|^p\,dx \le \liminf_{n\to\infty} \int_D |f_n|^p\,dx < +\infty.$$

Denote $h_n = f_n - g$. Then we know that

$$h_n \to 0 \quad \text{a.e. in } D, \tag{2.6}$$
$$h_n \rightharpoonup h = f - g \quad \text{weakly in } L^p(D). \tag{2.7}$$

We will show that $h = 0$ a.e. in D. For every $\varepsilon > 0$, from (2.6) and Egorov theorem, there exists a subset $E_\varepsilon \subset D$ such that $|D \backslash E_\varepsilon| < \varepsilon$ and $h_n \to 0$ uniformly in E_ε. Then there exists $N > 0$ such that

$$|h_n| < \varepsilon \quad \text{in } E_\varepsilon, \quad \text{for all } n > N.$$

Recalling (2.7), we get

$$\int_{E_\varepsilon} h_n\phi\,dx = \int_D h_n\phi\chi_{E_\varepsilon}\,dx \to \int_{E_\varepsilon} h\phi\,dx,$$

for any $\phi \in L^q(D)$ with $1/p + 1/q = 1$. On the other hand, the Lebesgue dominated convergence theorem implies that

$$\int_{E_\varepsilon} h_n\phi\,dx \to 0.$$

Therefore, we obtain

$$\int_{E_\varepsilon} h\phi\,dx = 0.$$

By choosing $\phi = |h|^{p-1}$, if $p > 1$ and $\phi = \mathrm{sgn}(h)$, if $p = 1$, we get $h = 0$ a.e. in E_ε, for all $\varepsilon > 0$. This finishes the proof. $\square$

Lemma 2.8. (See [8], Chapter 3 and [21]) *Suppose that $\Phi(\xi)$ is a nonnegative convex function satisfying* (1.2). *Let $D \subset \mathbb{R}^N$ be a measurable with finite Lebesgue measure $|D|$ and let $\{f_k\} \subset L^1(D; \mathbb{R}^N)$ be a sequence satisfying that*

$$\int_D \Phi(f_k)\, dx \le C, \tag{2.8}$$

where C is a positive constant. Then there exist a subsequence $\{f_{k_j}\} \subset \{f_k\}$ and a function $f \in L^1(D; \mathbb{R}^N)$ such that

$$f_{k_j} \rightharpoonup f \quad \text{weakly in } \ L^1(D; \mathbb{R}^N) \ \text{ as } \ j \to \infty \tag{2.9}$$

and

$$\int_D \Phi(f)\, dx \le \liminf_{j\to\infty} \int_D \Phi(f_{k_j})\, dx \le C. \tag{2.10}$$

For the convenience of the readers, let us recall the definition of weak solutions for problem (1.1) and the main results in [5].

Definition 2.9. *A function $u : \bar{\Omega} \times [0, T] \to \mathbb{R}$ is a weak solution of problem* (1.1) *if the following conditions are satisfied:*

(i) $u \in C([0, T]; L^2(\Omega)) \cap L^1(0, T; W_0^{1,1}(\Omega))$ *with*

$$\int_{\Omega_T} D_\xi \Phi(\nabla u) \cdot \nabla u\, dz < +\infty;$$

(ii) *For any $\varphi \in C^1(\bar{\Omega}_T)$ with $\varphi(\cdot, T) = 0$ and $\varphi(\cdot, t)|_{\partial\Omega} = 0$, we have*

$$-\int_\Omega u_0(x)\varphi(x, 0)\, dx + \int_{\Omega_T} \big[- u\varphi_t + D_\xi \Phi(\nabla u) \cdot \nabla\varphi \big]\, dz = \int_{\Omega_T} f\varphi\, dz. \tag{2.11}$$

Lemma 2.10. (See [5], Theorem 1.2) *Let the structure assumptions* (1.2) *and* (1.3) *be satisfied. If $u_0 \in L^2(\Omega)$ and $f = 0$, then there exists a unique weak solution for the initial-boundary value problem* (1.1).

Remark 2.11. *If we assume $f \in L^2(\Omega_T)$ the existence and uniqueness of weak solutions of problem* (1.1) *can be obtained working as in the proof of Lemma* 2.10.

Remark 2.12. *Let u be a weak solution in Definition* 2.9. *By using the approximation technique* (see [7], Chapter 3 or [11], Chapter 2) *we have, for every $\varphi \in C^1(\bar{\Omega}_T)$ with $\varphi(\cdot, t)|_{\partial\Omega} = 0$, each $t \in [0, T]$,*

$$\int_\Omega u\varphi\, dx\Big|_0^t + \int_0^t \int_\Omega \big[- u\varphi_t + D_\xi \Phi(\nabla u) \cdot \nabla\varphi \big]\, dx d\tau = \int_0^t \int_\Omega f\varphi\, dx d\tau. \tag{2.12}$$

Remark 2.13. *Let u be a weak solution in Definition* 2.9 *with $f = 0$. We can formally choose u as a test function in* (2.12) *to obtain an energy type estimate. That is, for a.e. $t \in [0, T]$,*

$$\frac{1}{2}\|u(t)\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega D_\xi \Phi(\nabla u) \cdot \nabla u\, dx d\tau = \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}.$$

This can be done by an approximation argument. Indeed, we first extend solution $u(x,t)$ to the initial value $u_0(x)$ when $t<0$. We next mollify u in the spatial directions to have an approximation C^∞ sequence u_ε, then introduce the time average of $u_\varepsilon(x,t)$,

$$\phi_{\varepsilon,h} = \frac{1}{2h}\int_{t-h}^{t+h} u_\varepsilon(x,\tau)\,d\tau.$$

As $u \in C([0,T];L^2(\Omega)) \cap L^1(0,T;W_0^{1,1}(\Omega))$ in Definition 2.9, we know that $\phi_{\varepsilon,h}(x,t) \in C^1(\bar{\Omega}_T)$ with $\phi_{\varepsilon,h}(\cdot,t)|_{\partial\Omega} = 0$, and may choose it as a test function φ in (2.12). Sending first $\varepsilon \to 0$, and then $h \to 0$, by a careful calculation we conclude that, for a.e. $t \in [0,T]$,

$$\int_\Omega u\phi_{\varepsilon,h}\,dx\Big|_0^t - \int_0^t\int_\Omega u[\phi_{\varepsilon,h}]_t\,dxd\tau \to \frac{1}{2}\int_\Omega |u(x,t)|^2\,dx - \frac{1}{2}\int_\Omega |u_0(x)|^2\,dx,$$
$$\int_0^t\int_\Omega D_\xi\Phi(\nabla u)\cdot\nabla\phi_{\varepsilon,h}\,dxd\tau \to \int_0^t\int_\Omega D_\xi\Phi(\nabla u)\cdot\nabla u\,dxd\tau.$$

We refer to the proof of Corollary 1.4 in [5] for more details.

3. The proofs of main results

Now we are ready to prove the main results. First we prove the existence and uniqueness of entropy solutions for problem (1.1). Some of the reasoning is based on the ideas developed in [2] and [19].

Proof of Theorem 1.3.

(1) Existence of entropy solutions.

We first introduce the approximate problems. Let $\{f_n\} \subset C_0^\infty(\Omega_T)$ and $\{u_{0n}\} \subset C_0^\infty(\Omega)$ be two sequences of functions strongly convergent respectively to f in $L^1(\Omega_T)$ and to u_0 in $L^1(\Omega)$ such that

$$\|f_n\|_{L^1(\Omega_T)} \le \|f\|_{L^1(\Omega_T)},\ \|u_{0n}\|_{L^1(\Omega)} \le \|u_0\|_{L^1(\Omega)}. \tag{3.1}$$

Let us consider the approximate problems

$$\begin{cases} (u_n)_t - \operatorname{div}\big(D_\xi\Phi(\nabla u_n)\big) = f_n & \text{in} \quad \Omega_T, \\ u_n = 0 & \text{on} \quad \Sigma, \\ u_n(x,0) = u_{0n} & \text{on} \quad \Omega. \end{cases} \tag{3.2}$$

By virtue of Lemma 2.10 (see also Remark 2.11) we can find $u_n \in C([0,T];L^2(\Omega)) \cap L^1(0,T;W_0^{1,1}(\Omega))$, that is a weak solution of problem (3.2) in the sense of Definition 2.9. Moreover, $(u_n)_t \in L^1(0,T;W^{-1,1}(\Omega)) + L^2(\Omega_T)$ and

$$\int_{\Omega_T} D_\xi\Phi(\nabla u_n)\cdot\nabla u_n\,dz < +\infty. \tag{3.3}$$

Our aim is to prove that a subsequence of these approximate solutions $\{u_n\}$ converges to a measurable function u, which is an entropy solution of problem (1.1). We will divide the proof into several steps. Although some of the arguments are not new, we present a self-contained proof for the sake of clarity and readability.

Using an approximation argument as in Remark 2.12 and Remark 2.13, we can choose $T_k(u_n)\chi_{(0,t)}$ as a test function in (3.2) to have

$$\begin{aligned}&\int_\Omega \Theta_k(u_n)(t)\,dx-\int_\Omega \Theta_k(u_{0n})\,dx\\&+\int_0^t\int_\Omega D_\xi\Phi(\nabla T_k(u_n))\cdot\nabla T_k(u_n)\,dxds=\int_0^t\int_\Omega f_nT_k(u_n)\,dxds.\end{aligned}\tag{3.4}$$

It follows from the definition of $\Theta_k(r)$ and (3.1) that

$$\begin{aligned}&\int_0^t\int_\Omega D_\xi\Phi(\nabla T_k(u_n))\cdot\nabla T_k(u_n)\,dxds+\int_\Omega \Theta_k(u_n)(t)\,dx\\&\le k\big(\|f_n\|_{L^1(\Omega_T)}+\|u_{0n}\|_{L^1(\Omega)}\big)\le k\big(\|f\|_{L^1(\Omega_T)}+\|u_0\|_{L^1(\Omega)}\big).\end{aligned}\tag{3.5}$$

Recalling (2.2), we have

$$\int_{\Omega_T}\Phi(\nabla T_k(u_n))\,dz\le\int_{\Omega_T}D_\xi\Phi(T_k(\nabla u_n))\cdot\nabla T_k(u_n)\,dz\le Ck,\tag{3.6}$$

which implies from (1.2) that

$$\int_{\Omega_T}|\nabla T_k(u_n)|\,dz\le C(k+1),\tag{3.7}$$

that is $T_k(u_n)$ is bounded in $L^1\big(0,T;W_0^{1,1}(\Omega)\big)$.

If we choose $k=1$ in the inequality (3.5), then for a.e. $t\in[0,T]$,

$$\int_\Omega \Theta_1(u_n(t))\,dx\le\|f\|_{L^1(\Omega_T)}+\|u_0\|_{L^1(\Omega)}.$$

Moreover,

$$\int_\Omega |u_n(t)|\,dx\le\text{meas}(\Omega)+\|f\|_{L^1(\Omega_T)}+\|u_0\|_{L^1(\Omega)}.$$

Thus we obtain

$$\|u_n\|_{L^\infty(0,T;L^1(\Omega))}\le C.\tag{3.8}$$

Step 1. We shall prove that $\{u_n\}$ converges in $C([0,T];L^1(\Omega))$ and we shall find a subsequence which is almost everywhere convergent in Ω_T.

Let m and n be two integers, then from (3.2) we can write the weak form as

$$\begin{aligned}&\int_0^T\langle(u_n-u_m)_t,\phi\rangle\,dt+\int_{\Omega_T}[D_\xi\Phi(\nabla u_n)-D_\xi\Phi(\nabla u_m)]\cdot\nabla\phi\,dz\\&=\int_{\Omega_T}(f_n-f_m)\phi\,dz,\end{aligned}\tag{3.9}$$

for all $\phi\in C_0^1(\bar\Omega_T)$. Recalling (2.4), (2.5), (1.3) and (3.3), we observe that

$$\begin{aligned}|D_\xi\Phi(\nabla u_n)\cdot\nabla u_m|&\le\Phi(\nabla u_m)+\Phi(-\nabla u_m)+\Psi(D_\xi\Phi(\nabla u_n))\\&\le(C+1)\Phi(\nabla u_m)+D_\xi\Phi(\nabla u_n)\cdot\nabla u_n\\&\le(C+1)D_\xi\Phi(\nabla u_m)\cdot\nabla u_m+D_\xi\Phi(\nabla u_n)\cdot\nabla u_n\in L^1(\Omega_T).\end{aligned}$$

Denote

$$\alpha_{n,m}=\int_{\Omega_T}|f_n-f_m|\,dz+\int_\Omega|u_{0n}-u_{0m}|\,dx.\tag{3.10}$$

We know that

$$\lim_{n,m\to\infty} \alpha_{n,m} = 0.$$

Using an approximation argument as above, we conclude that $w = T_1(u_n - u_m)\chi_{(0,t)}$ with $t \le T$ can be a test function in (3.9). From (2.3), discarding the positive term we get

$$\begin{aligned}\int_\Omega \Theta_1(u_n - u_m)(t)\,dx &\le \int_\Omega \Theta_1(u_{0n} - u_{0m})\,dx + \|f_n - f_m\|_{L^1(\Omega_T)} \\ &\le \|u_{0n} - u_{0m}\|_{L^1(\Omega)} + \|f_n - f_m\|_{L^1(\Omega_T)} = \alpha_{n,m}.\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}&\int_{\{|u_n-u_m|<1\}} \frac{|u_n - u_m|^2(t)}{2}\,dx + \int_{\{|u_n-u_m|\ge 1\}} \frac{|u_n - u_m|(t)}{2}\,dx \\ &\le \int_\Omega [\Theta_1(u_n - u_m)](t)\,dx \le \alpha_{n,m}.\end{aligned}$$

It follows that

$$\begin{aligned}\int_\Omega |u_n - u_m|(t)\,dx &= \int_{\{|u_n-u_m|<1\}} |u_n - u_m|(t)\,dx + \int_{\{|u_n-u_m|\ge 1\}} |u_n - u_m|(t)\,dx \\ &\le \Big(\int_{\{|u_n-u_m|<1\}} |u_n - u_m|^2(t)\,dx\Big)^{\frac12} \mathrm{meas}(\Omega)^{\frac12} + 2\alpha_{n,m} \\ &\le (2\mathrm{meas}(\Omega))^{\frac12} \alpha_{n,m}^{\frac12} + 2\alpha_{n,m}.\end{aligned}$$

Thus we get

$$\|u_n - u_m\|_{C([0,T];L^1(\Omega))} \to 0 \quad \text{as } n, m \to +\infty,$$

i.e., $\{u_n\}$ is a Cauchy sequence in $C([0,T];L^1(\Omega))$. Then u_n converges to u in $C([0,T];L^1(\Omega))$. We find an a.e. convergent subsequence (still denoted by $\{u_n\}$) in Ω_T such that

$$u_n \to u \quad \text{a.e. in } \Omega_T. \tag{3.11}$$

Recalling (3.6) and Lemma 2.8, we may draw a subsequence (we also denote it by the original sequence for simplicity) such that

$$\nabla T_k(u_n) \rightharpoonup \eta_k, \quad \text{weakly in } L^1(\Omega_T)$$

and

$$\int_{\Omega_T} \Phi(\eta_k)\,dz \le Ck.$$

In view of (3.11), we conclude that $\eta_k = \nabla T_k(u)$ a.e. in Ω_T.

Step 2. We shall prove that the sequence $\{\nabla u_n\}$ converges almost everywhere in Ω_T to ∇u (up to a subsequence).

We first claim that $\{\nabla u_n\}$ is a Cauchy sequence in measure. Let $\delta > 0$, and denote

$$\begin{aligned}E_1 &:= \{(x,t) \in \Omega_T : |\nabla u_n| > h\} \cup \{|\nabla u_m| > h\}, \\ E_2 &:= \{(x,t) \in \Omega_T : |u_n - u_m| > 1\}\end{aligned}$$

and

$$E_3 := \{(x,t) \in \Omega_T : |\nabla u_n| \le h, |\nabla u_m| \le h, |u_n - u_m| \le 1, |\nabla u_n - \nabla u_m| > \delta\},$$

where h will be chosen later. It is obvious that

$$\{(x,t) \in \Omega_T : |\nabla u_n - \nabla u_m| > \delta\} \subset E_1 \cup E_2 \cup E_3.$$

For $k \geq 0$, we can write

$$\begin{aligned}&\{(x,t) \in \Omega_T : |\nabla u_n| \geq h\}\\ &\subset \{(x,t) \in \Omega_T : |u_n| \geq k\} \cup \{(x,t) \in \Omega_T : |\nabla T_k(u_n)| \geq h\}.\end{aligned}$$

Thus, applying (3.8), (1.2) and (3.7), there exist constants $C > 0$ such that

$$\text{meas}\{(x,t) \in \Omega_T : |\nabla u_n| \geq h\} \leq \frac{C}{k} + \frac{Ck}{h},$$

when h is large appropriately. By choosing $k = Ch^{\frac{1}{2}}$, we deduce that

$$\text{meas}\{(x,t) \in \Omega_T : |\nabla u_n| \geq h\} \leq Ch^{-\frac{1}{2}}.$$

Let $\varepsilon > 0$. We may let $h = h(\varepsilon)$ large enough such that

$$\text{meas}(E_1) \leq \varepsilon/3, \text{ for all } n, m \geq 0. \tag{3.12}$$

On the other hand, by Step 1, we know that $\{u_n\}$ is a Cauchy sequence in $L^1(\Omega_T)$. Then there exists $N_1(\varepsilon) \in \mathbb{N}$ such that

$$\text{meas}(E_2) \leq \varepsilon/3, \quad \text{for all } n, m \geq N_1(\varepsilon). \tag{3.13}$$

Moreover, since Φ is C^1 and strictly convex, then from Lemma 2.2 and Definition 2.1, there exists a real valued function $m(h, \delta) > 0$ such that

$$(D\Phi(\xi) - D\Phi(\zeta)) \cdot (\xi - \zeta) \geq m(h, \delta) > 0, \tag{3.14}$$

for all $\xi, \zeta \in \mathbb{R}^N$ with $|\xi|, |\zeta| \leq h, |\xi - \zeta| \geq \delta$. By taking $T_1(u_n - u_m)$ as a test function in (3.9), we obtain

$$\begin{aligned}m(h,\delta)\text{meas}(E_3) &\leq \int_{E_3} [D_\xi\Phi(\nabla u_n) - D_\xi\Phi(\nabla u_m)] \cdot (\nabla u_n - \nabla u_m)\, dz\\ &\leq \int_{\Omega_T} [D_\xi\Phi(\nabla u_n) - D_\xi\Phi(\nabla u_m)] \cdot \nabla T_1(u_n - u_m)\, dz\\ &\leq \int_{\Omega_T} |f_n - f_m|\, dz + \int_\Omega |u_{0n} - u_{0m}|\, dx = \alpha_{n,m},\end{aligned}$$

which implies that

$$\text{meas}(E_3) \leq \frac{\alpha_{n,m}}{m(h,\delta)} \leq \varepsilon/3,$$

for all $n, m \geq N_2(\varepsilon, \delta)$. It follows from (3.12) and (3.13) that

$$\text{meas}\{(x,t) \in \Omega_T : |\nabla u_n - \nabla u_m| > \delta\} \leq \varepsilon, \quad \text{for all } n, m \geq \max\{N_1, N_2\},$$

that is $\{\nabla u_n\}$ is a Cauchy sequence in measure. Then we may choose a subsequence (denote it by the original sequence) such that

$$\nabla u_n \to v \quad \text{a.e. in } \Omega_T.$$

Thus, from Proposition 1.1 and $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ weakly in $L^1(\Omega_T)$, we deduce from Lemma 2.7 that v coincides with the very weak gradient of u. Therefore, we have

$$\nabla u_n \to \nabla u \quad \text{a.e. in } \Omega_T. \tag{3.15}$$

Step 3. We shall prove that u is an entropy solution.

Now we choose $v_n = T_k(u_n - \phi)$ as a test function in (3.2) for $k > 0$ and $\phi \in C^1(\bar{\Omega}_T)$ with $\phi|_\Sigma = 0$. We note that, if $L = k + \|\phi\|_{L^\infty(\Omega_T)}$, then

$$\begin{aligned}&\int_{\Omega_T} D_\xi \Phi(\nabla u_n) \cdot \nabla T_k(u_n - \phi)\, dz \\ &= \int_{\Omega_T} D_\xi \Phi(\nabla T_L(u_n)) \cdot \nabla T_k(T_L(u_n) - \phi)\, dz\end{aligned}$$

and

$$\begin{aligned}&\int_0^T \langle (u_n)_t, T_k(u_n - \phi)\rangle\, dt + \int_{\Omega_T} D_\xi \Phi(\nabla T_L(u_n)) \cdot \nabla T_k(T_L(u_n) - \phi)\, dz \\ &= \int_{\Omega_T} f_n T_k(u_n - \phi)\, dz.\end{aligned}$$

Since $(u_n)_t = (u_n - \phi)_t + \phi_t$, we have

$$\begin{aligned}\int_0^T \langle (u_n)_t, T_k(u_n - \phi)\rangle\, dt = &\int_\Omega \Theta_k(u_n - \phi)(T)\, dx - \int_\Omega \Theta_k(u_n - \phi)(0)\, dx \\ &+ \int_0^T \langle \phi_t, T_k(u_n - \phi)\rangle\, dt,\end{aligned}$$

which yields that

$$\begin{aligned}&\int_\Omega \Theta_k(u_n - \phi)(T)\, dx - \int_\Omega \Theta_k(u_n - \phi)(0)\, dx + \int_0^T \langle \phi_t, T_k(u_n - \phi)\rangle\, dt \\ &+ \int_{\Omega_T} D_\xi \Phi(\nabla T_L(u_n)) \cdot \nabla T_k(T_L(u_n) - \phi)\, dz = \int_{\Omega_T} f_n T_k(u_n - \phi)\, dz. \qquad (3.16)\end{aligned}$$

Recalling u_n converges to u in $C([0,T]; L^1(\Omega))$, we have $u_n(t) \to u(t)$ in $L^1(\Omega)$, for all $t \leq T$. Since Θ_k is Lipschitz continuous, we get

$$\int_\Omega \Theta_k(u_n - \phi)(T)\, dx \to \int_\Omega \Theta_k(u - \phi)(T)\, dx$$

and

$$\int_\Omega \Theta_k(u_n - \phi)(0)\, dx \to \int_\Omega \Theta_k(u_0 - \phi(0))\, dx,$$

as $n \to +\infty$.

The fourth term on the left hand side of (3.16) can be written as

$$\begin{aligned}&\int_{\Omega_T} D_\xi \Phi(\nabla T_L(u_n)) \cdot \nabla T_k(T_L(u_n) - \phi)\, dz \\ &= \int_{\{|T_L(u_n) - \phi| \leq k\}} D_\xi \Phi(\nabla T_L(u_n)) \cdot \nabla T_L(u_n)\, dz \\ &\quad - \int_{\{|T_L(u_n) - \phi| \leq k\}} D_\xi \Phi(\nabla T_L(u_n)) \cdot \nabla \phi\, dz.\end{aligned}$$

From (2.2), we have

$$D_\xi \Phi(\nabla T_L(u_n)) \cdot \nabla T_L(u_n) \geq 0,$$

it follows from Fatou's lemma and (3.15) that

$$\int_{\{|T_L(u) - \phi| \leq k\}} D_\xi \Phi(\nabla T_L(u)) \cdot \nabla T_L(u)\, dz$$

$$\leq \liminf_{n\to\infty} \int_{\{|T_L(u_n)-\phi|\leq k\}} D_\xi \Phi(\nabla T_L(u_n)) \cdot \nabla T_L(u_n)\, dz.$$

In view of (3.5) and (2.5), we know that

$$\int_{\Omega_T} \Psi(D_\xi \Phi(\nabla T_L(u_n)))\, dz \leq C. \tag{3.17}$$

Applying Lemma 2.5, Lemma 2.8 and (3.15), we conclude that (up to a subsequence)

$$D_\xi \Phi(\nabla T_L(u_n)) \rightharpoonup D_\xi \Phi(\nabla T_L(u)) \quad \text{weakly in } L^1(\Omega_T). \tag{3.18}$$

Denote

$$\xi_n = D_\xi \Phi(\nabla T_L(u_n)), E_n = \{(x,t) \in \Omega_T : |T_L(u_n) - \phi| \leq k\}$$

and

$$E = \{(x,t) \in \Omega_T : |T_L(u) - \phi| \leq k\}$$

for simplicity. We can write

$$\int_{E_n} \xi_n \cdot \nabla \phi\, dz = \int_E \xi_n \cdot \nabla \phi\, dz + \int_{E_n \setminus E} \xi_n \cdot \nabla \phi\, dz := I_1 + I_2.$$

From (3.18), we have

$$\lim_{n\to\infty} I_1 = \int_{\{|T_L(u)-\phi|\leq k\}} D_\xi \Phi(\nabla T_L(u)) \cdot \nabla \phi\, dz.$$

Recalling Lemma 2.5, we know that Ψ also satisfies the super-linear condition (1.2). Then for every $\varepsilon > 0$, there exists a constant $M > 0$ such that

$$|s| \leq \varepsilon \Psi(s), \quad \text{for all } |s| > M.$$

It follows that from (3.17) that

$$\begin{aligned}
|I_2| &\leq C(\|\nabla \phi\|_{L^\infty(\Omega_T)}) \int_{\Omega_T} |\xi_n| \chi_{E_n \setminus E}\, dz \\
&= C\Big(\int_{\{|\xi_n|\leq M\}} |\xi_n| \chi_{E_n\setminus E}\, dz + \int_{\{|\xi_n|\geq M\}} |\xi_n| \chi_{E_n \setminus E}\, dz \Big) \\
&\leq C\Big(M \text{meas}(E_n \setminus E) + \varepsilon \int_{\Omega_T} \Psi(\xi_n)\, dz \Big) \\
&\leq CM \text{meas}(E_n \setminus E) + C\varepsilon.
\end{aligned}$$

Moreover, by the arbitrariness of ε, we get

$$\lim_{n\to\infty} |I_2| = 0.$$

Thus we obtain

$$\int_{\{|T_L(u_n)-\phi|\leq k\}} D_\xi \Phi(\nabla T_L(u_n)) \cdot \nabla \phi\, dz \to \int_{\{|T_L(u)-\phi|\leq k\}} D_\xi \Phi(\nabla T_L(u)) \cdot \nabla \phi\, dz.$$

Using the strong convergence of f_n, (3.11) and the Lebesgue dominated convergence theorem, we can pass to the limits as $n \to \infty$ in the other terms of (3.16) to conclude

$$\begin{aligned}
&\int_\Omega \Theta_k(u-\phi)(T)\, dx - \int_\Omega \Theta_k(u_0 - \phi(0))\, dx + \int_0^T \langle \phi_t, T_k(u-\phi)\rangle\, dt \\
&+ \int_{\Omega_T} D_\xi \Phi(\nabla u) \cdot \nabla T_k(u-\phi)\, dz \leq \int_{\Omega_T} f T_k(u-\phi)\, dz,
\end{aligned}$$

for all $k > 0$ and $\phi \in C^1(\bar{\Omega}_T)$ with $\phi|_\Sigma = 0$. Therefore, we finish the proof of the existence of entropy solutions.

(2) Uniqueness of entropy solutions.

Now we prove the uniqueness of entropy solutions for problem (1.1) by choosing appropriate test functions. Suppose that v is another entropy solution of problem (1.1), we will show that $u = v$ a.e. in Ω_T. For $\sigma > 0, 0 < \varepsilon \le 1$, define the function $S_{\sigma,\varepsilon}$ in $W^{2,\infty}(\mathbb{R})$ by

$$\begin{cases} S_{\sigma,\varepsilon}(r) = r & \text{if } |r| \le \sigma, \\ S_{\sigma,\varepsilon}(r) = (\sigma + \dfrac{\varepsilon}{2}) - \dfrac{r}{|r|}\dfrac{1}{2\varepsilon}\big(r - \dfrac{r}{|r|}(\sigma+\varepsilon)\big)^2 & \text{if } \sigma < |r| < \sigma + \varepsilon, \\ S_{\sigma,\varepsilon}(r) = \dfrac{r}{|r|}(\sigma + \dfrac{\varepsilon}{2}) & \text{if } |r| \ge \sigma + \varepsilon. \end{cases}$$

It is obvious that

$$\begin{cases} S'_{\sigma,\varepsilon}(r) = 1 & \text{if } |r| \le \sigma, \\ S'_{\sigma,\varepsilon}(r) = \dfrac{1}{\varepsilon}(\sigma + \varepsilon - |r|) & \text{if } \sigma < |r| < \sigma + \varepsilon, \\ S'_{\sigma,\varepsilon}(r) = 0 & \text{if } |r| \ge \sigma + \varepsilon. \end{cases}$$

Choosing $\phi = S_{\sigma,\varepsilon}(u_n)\chi_{(0,t)}$ as a test function in (1.5) for entropy solution v, we have

$$\begin{aligned} &\int_\Omega \Theta_k(v - S_{\sigma,\varepsilon}(u_n))(t)\,dx - \int_\Omega \Theta_k(u_0 - S_{\sigma,\varepsilon}(u_{0n}))\,dx \\ &\quad + \int_0^t \langle (u_n)_t, S'_{\sigma,\varepsilon}(u_n)T_k(v - S_{\sigma,\varepsilon}(u_n))\rangle\,ds \\ &\quad + \int_0^t\int_\Omega D_\xi\Phi(\nabla v)\cdot\nabla T_k(v - S_{\sigma,\varepsilon}(u_n))\,dxds \\ &\le \int_0^t\int_\Omega fT_k(v - S_{\sigma,\varepsilon}(u_n))\,dxds. \end{aligned} \tag{3.19}$$

In order to deal with the third term of (3.19), we take $S'_{\sigma,\varepsilon}(u_n)T_k(v - S_{\sigma,\varepsilon}(u_n))\chi_{(0,t)}$ as a test function for problem (3.2) to have

$$\begin{aligned} &\int_0^t \langle (u_n)_t, S'_{\sigma,\varepsilon}(u_n)T_k(v - S_{\sigma,\varepsilon}(u_n))\rangle\,ds \\ &\quad + \int_0^t\int_\Omega S''_{\sigma,\varepsilon}(u_n)T_k(v - S_{\sigma,\varepsilon}(u_n))D_\xi\Phi(\nabla u_n)\cdot\nabla u_n\,dxds \\ &\quad + \int_0^t\int_\Omega S'_{\sigma,\varepsilon}D_\xi\Phi(\nabla u_n)\cdot\nabla T_k(v - S_{\sigma,\varepsilon}(u_n))\,dxds \\ &= \int_0^t\int_\Omega f_nS'_{\sigma,\varepsilon}(u_n)T_k(v - S_{\sigma,\varepsilon}(u_n))\,dxds. \end{aligned} \tag{3.20}$$

Combining (3.19) and (3.20), we get

$$\begin{aligned} &\int_\Omega \Theta_k(v - S_{\sigma,\varepsilon}(u_n))(t)\,dx - \int_\Omega \Theta_k(u_0 - S_{\sigma,\varepsilon}(u_{0n}))\,dx \\ &\quad - \int_0^t\int_\Omega S''_{\sigma,\varepsilon}(u_n)T_k(v - S_{\sigma,\varepsilon}(u_n))D_\xi\Phi(\nabla u_n)\cdot\nabla u_n\,dxds \end{aligned}$$

$$
\begin{aligned}
&-\int_0^t\int_\Omega S'_{\sigma,\varepsilon}(u_n)D_\xi\Phi(\nabla u_n)\cdot\nabla T_k(v-S_{\sigma,\varepsilon}(u_n))\,dxds\\
&+\int_0^t\int_\Omega D_\xi\Phi(\nabla v)\cdot\nabla T_k(v-S_{\sigma,\varepsilon}(u_n))\,dxds\\
\le&\int_0^t\int_\Omega fT_k(v-S_{\sigma,\varepsilon}(u_n))\,dxds\\
&-\int_0^t\int_\Omega f_nS'_{\sigma,\varepsilon}(u_n)T_k(v-S_{\sigma,\varepsilon}(u_n))\,dxds.
\end{aligned}
$$

We will pass to the limits as $\varepsilon\to 0$, $n\to\infty$ and $\sigma\to\infty$ successively. We begin with $\varepsilon\to 0$. Let us denote A_1 to A_7 these seven terms, then we get

$$
A_1+A_2+A_4+A_5\le A_6+A_7+|A_3|. \tag{3.21}
$$

Since $|\Theta_k(v-S_{\sigma,\varepsilon}(u_n))(t)|\le k(|v|+|T_{\sigma+1}(u_n)|)(t)$, $S'_{\sigma,\varepsilon}(r)\le T'_{\sigma+1}(r)$ and $|\nabla T_k(v-S_{\sigma,\varepsilon}(u_n))|\le[|\nabla T_{\sigma+k+1}(v)|+|\nabla T_{\sigma+1}(u_n)|]$, the four terms of the left hand side and the two terms of the right hand side in (3.21) pass to the limit for $\varepsilon\to 0$ by the Lebesgue dominated convergence theorem.

Now we estimate $|A_3|$. Let $R_{\sigma,\varepsilon}$ be an even function such that $R_{\sigma,\varepsilon}(r)=r-S_{\sigma,\varepsilon}(r)$ for $r\ge 0$. Then we choose $R'_{\sigma,\varepsilon}(u_n)\chi_{(0,t)}$ as a test function in (3.2) to have

$$
\begin{aligned}
&\int_\Omega R_{\sigma,\varepsilon}(u_n)(t)\,dx-\int_\Omega R_{\sigma,\varepsilon}(u_{0n})\,dx\\
&+\int_0^t\int_\Omega R''_{\sigma,\varepsilon}(u_n)D_\xi\Phi(\nabla u_n)\cdot\nabla u_n\,dxds=\int_0^t\int_\Omega f_nR'_{\sigma,\varepsilon}(u_n)\,dxds.
\end{aligned}
$$

Since $R_{\sigma,\varepsilon}(r)\ge 0$, $R_{\sigma,\varepsilon}(r)\le|r|$ on the set $\{|r|>\sigma\}$ and $|S''_{\sigma,\varepsilon}(r)|=R''_{\sigma,\varepsilon}(r)$, we obtain that

$$
\begin{aligned}
\int_0^t\int_\Omega |S''_{\sigma,\varepsilon}(u_n)|D_\xi\Phi(\nabla u_n)\cdot\nabla u_n\,dxds&=\int_0^t\int_\Omega R''_{\sigma,\varepsilon}(u_n)D_\xi\Phi(\nabla u_n)\cdot\nabla u_n\,dxds\\
&\le\int_{\{|u_n|>\sigma\}}|f_n|dz+\int_{\{|u_{0n}|>\sigma\}}|u_{0n}|dx.
\end{aligned}
$$

Thus

$$
|A_3|\le k\Big(\int_{\{|u_n|>\sigma\}}|f_n|dz+\le\int_{\{|u_{0n}|>\sigma\}}|u_{0n}|dx\Big).
$$

Recalling that Φ is a C^1 nonnegative convex function and 0 is the minimum point, we conclude that $D\Phi(0)=0$ and $T'_\sigma(u_n)D_\xi\Phi(\nabla u_n)=D_\xi\Phi(\nabla T_\sigma(u_n))$. Then by letting $\varepsilon\to 0$ in (3.21), we obtain

$$
\begin{aligned}
&\int_\Omega\Theta_k(v-T_\sigma(u_n))(t)\,dx-\int_\Omega\Theta_k(u_0-T_\sigma(u_{0n}))\,dx\\
&\quad+\int_0^t\int_\Omega(D_\xi\Phi(\nabla v)-D_\xi\Phi(\nabla T_\sigma(u_n)))\cdot\nabla T_k(v-T_\sigma(u_n))\,dxds\\
&\le\int_0^t\int_\Omega(f-f_nT'_\sigma(u_n))T_k(v-T_\sigma(u_n))\,dxds\\
&\quad+k\Big(\int_{\{|u_n|>\sigma\}}|f_n|\,dxdt+\int_{\{|u_{0n}|>\sigma\}}|u_{0n}|\,dx\Big).
\end{aligned}
$$

Thanks to the fact that $\nabla T_\sigma(u_n) \to \nabla T_\sigma(u)$ a.e. in Ω_T as $n \to \infty$, Fatou's Lemma and the Lebesgue dominated convergence theorem, sending $n \to \infty$ in the above inequality, we have

$$\begin{aligned}
&\int_\Omega \Theta_k(v - T_\sigma(u))(t)\,dx - \int_\Omega \Theta_k(u_0 - T_\sigma(u_0))\,dx \\
&\quad + \int_0^t\int_\Omega (D_\xi\Phi(\nabla v) - D_\xi\Phi(\nabla T_\sigma(u)) \cdot \nabla T_k(v - T_\sigma(u))\,dxds \\
&\le \int_0^t\int_\Omega f(1 - T'_\sigma(u))T_k(v - T_\sigma(u))\,dxds \\
&\quad + k\Big(\int_{\{|u|>\sigma\}} |f|\,dz + \int_{\{|u_0|>\sigma\}} |u_0|\,dx\Big).
\end{aligned}$$

Now we let $\sigma \to \infty$. Since

$$|\Theta_k(v - T_\sigma(u))(t)| \le k(|v(t)| + |u(t)|),\ |\Theta_k(u_0 - T_\sigma(u_0))| \le k|u_0|,$$

by the Lebesgue dominated convergence theorem, we obtain

$$\int_\Omega \Theta_k(u_0 - T_\sigma(u_0))\,dx \to 0,\ \int_\Omega \Theta_k(v - T_\sigma(u))(t)\,dx \to \int_\Omega \Theta_k(v - u)(t)\,dx$$

and

$$\int_{\{|u|>\sigma\}} |f|\,dz + \int_{\{|u_0|>\sigma\}} |u_0|\,dx \to 0.$$

Therefore, we deduce that

$$\int_\Omega \Theta_k(v - u)(t)\,dx + \int_0^t\int_\Omega (D_\xi\Phi(\nabla v) - D_\xi\Phi(\nabla u)) \cdot \nabla T_k(v - u)\,dxds \le 0,$$

which implies that

$$\int_\Omega \Theta_k(v - u)(t)\,dx + \int_{\{|u|\le\frac{k}{2},|v|\le\frac{k}{2}\}} (D_\xi\Phi(\nabla v) - D_\xi\Phi(\nabla u)) \cdot \nabla(v - u)\,dxds \le 0.$$

Using the nonnegativity of the two terms in the above inequality, we conclude that $u = v$ a.e. in Ω_T. Therefore we obtain the uniqueness of entropy solutions. This completes the proof of Theorem 1.3. □

Next, we begin to prove the comparison result.

Proof of Theorem 1.4. First, we suppose that $u_0, v_0 \in L^2(\Omega)$ and $f, g \in L^2(\Omega_T)$. Then we use an approximation argument. By Remark 2.11, we can obtain two weak solutions u and v under Definition 2.9 for problems (1.1) and

$$\begin{cases} v_t - \operatorname{div}\big(D_\xi\Phi(\nabla v)\big) = g & \text{in } \Omega_T, \\ v = 0 & \text{on } \Sigma, \\ v(x,0) = v_0(x) & \text{on } \Omega. \end{cases} \tag{3.22}$$

Making use of the approximation argument, we choose $(u - v)^+\chi_{(0,t)}$ as a test function and subtract the resulting equalities to get

$$\begin{aligned}
&\int_0^t\int_\Omega (u - v)_t(u - v)^+\,dxds + \int_0^t\int_\Omega (D_\xi\Phi(\nabla u) - D_\xi\Phi(\nabla v)) \cdot \nabla(u - v)^+\,dxds \\
&= \int_0^t\int_\Omega (f - g)(u - v)^+\,dxds \le 0.
\end{aligned}$$

Moreover, from (2.3) we have

$$\frac{1}{2}\int_0^t\int_\Omega \frac{d}{dt}[(u-v)^+]^2\,dxds = \frac{1}{2}\int_\Omega [(u-v)^+]^2(t)\,dx - \frac{1}{2}\int_\Omega [(u_0-v_0)^+]^2\,dx \le 0.$$

Recalling $u_0 \le v_0$, we conclude that

$$(u-v)^+ = 0 \quad \text{a.e. in } \Omega_T.$$

Thus we obtain $u \le v$ a.e. in Ω_T.

Now we consider u and v as the entropy solution of problems (1.1) and (3.22) with L^1 data. Find four sequences of functions $\{f_n\}, \{g_n\} \subset C_0^\infty(\Omega_T)$ and $\{u_{0n}\}, \{v_{0n}\} \subset C_0^\infty(\Omega)$ strongly converging respectively to f, g in $L^1(\Omega_T)$ and to u_0, v_0 in $L^1(\Omega)$ such that

$$\begin{aligned} f_n &\le g_n, & u_{0n} &\le v_{0n}, \\ \|f_n\|_{L^1(\Omega_T)} &\le \|f\|_{L^1(\Omega_T)}, & \|g_n\|_{L^1(\Omega_T)} &\le \|g\|_{L^1(\Omega_T)}, \\ \|u_{0n}\|_{L^1(\Omega)} &\le \|u_0\|_{L^1(\Omega)}, & \|v_{0n}\|_{L^1(\Omega)} &\le \|v_0\|_{L^1(\Omega)}. \end{aligned}$$

Thus we use Theorem 1.3 to construct two approximation sequences $\{u_n\}$ and $\{v_n\}$ of entropy solutions u and v, and apply the comparison result above to obtain $u_n \le v_n$ a.e. in Ω_T. Moreover, by the uniqueness of entropy solutions, we know $u_n \to u$ and $v_n \to v$ a.e. in Ω_T. Therefore, we conclude that $u \le v$ a.e. in Ω_T. This completes the proof of Theorem 1.4. □

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