On some Finsler metrics of constant (or scalar) flag curvature *

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Abstract

This paper gives a lot of new Finsler metrics of scalar curvature. In particular, we show at least there is an $\frac{n(n-1)}{2}$-dimensional family of new Finsler metrics of constant flag curvature.

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1 Introduction

It is one of important problems in Finsler geometry to study and classification Finsler metrics of constant (or scalar) flag curvature because the flag curvature is the most important Riemannian quantity in Finsler geometry and it is an analogue of sectional curvature in Riemannian geometry [1, 2]. Furthermore, Finsler metrics of constant curvature (or scalar curvature and dimension $n \geq 3$) are the natural extension of Riemannian metrics of constant sectional curvature.

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Recently, a significant progress has been made in classifying Randers metrics of constant flag curvature [2]. Bao-Robles-Shen' work includes the generalizing Funk metric and many other classical examples of constant flag curvature Randers metrics [5, 12, 13]. In the spirit of Bao-Robles-Shen, many Finslerian geometers have made efforts in the study of $(\alpha,\, \beta)$-metrics (in particular, Randers metrics), of constant (or scalar) flag curvature. For instance, in [5, 6, 8], authors locally and globally classify a large class of Randers metrics of scalar curvature and in [10, 11] authors manufacture $(\alpha,\, \beta)$-metrics of constant (and scalar) curvature, which include famous Finsler metric of zero flag curvature due to Berwald [4].

$(\alpha,\, \beta)$-metrics $F := \alpha\phi(\frac{\beta}{\alpha})$ (for definition, see Section 2) form a special class of Finsler metrics where α is a Riemannian metric, β is a 1-form and $\phi = \phi(s)$ is a positive smooth function. They are "computable" although the computation sometimes runs into very complicated situation. When $\phi = 1 + s$, we get Randers metrics.

The main purpose of this paper is to obtain a lot of new Finsler metrics of scalar curvature by the shortest time problem (see Theorem 5.1) generalizing result previously only known in the case of two-dimensional Funk metric [12]. In particular, we show at least there is an $\frac{n(n-1)}{2}$-dimensional family of new Finsler metrics of constant flag curvature. Precisely we obtain the following:

Theorem 1.1 *Let*

$$\Phi := \frac{[(1 + \langle a,\, x\rangle)(\sqrt{(1 - |x|^2)|y|^2 + \langle x,\, y\rangle^2} + \langle x,\, y\rangle) + (1 - |x|^2)\langle a,\, y\rangle]^2}{(1 - |x|^2)^2\sqrt{(1 - |x|^2)|y|^2 + \langle x,\, y\rangle^2}}$$

be an $(\alpha,\, \beta)$ metric on the open subset $\mathcal{U}$ at origin in $\mathbb{R}^N$. Assume that V is a vector field on $\mathcal{U}$ defined by (4.2) where Q is skew-symmetric and satisfies that (4.3) and $\Phi(x,\, V_x) < 1$. Then Finsler metric F given by

$$\Phi\left(x,\, \frac{y}{F(x,\, y)} + V_x\right) = 1, \quad \forall x \in \mathcal{U}, \quad y \in T_x\mathcal{U}$$

is of zero flag curvature.

2 Preliminaries

Recall that a *(local) flow* on a manifold M is a map $\phi : (-\epsilon, \epsilon) \times M \to M$, also denoted by $\phi_t := \phi(t, \cdot)$, satisfying

- $\phi_0 = id : M \to M$
- $\phi_s \circ \phi_t = \phi_{s+t}$ for any $s, t \in (-\epsilon, \epsilon)$ with $s + t \in (-\epsilon, \epsilon)$.

Hence, the lift of a flow ϕ_t on M is again a flow $\breve{\phi}_t$ on the tangent bundle TM,

$$\breve{\phi}_t(x, y) := (\phi_t(x), \phi_{t*}(y)) \tag{2.1}$$

By the relationship of vector fields and flows, (2.1) induces a natural way to lift a vector field u on M to a vector field X_u on TM. In natural coordinate, we have

$$X_u = u^i \frac{\partial}{\partial x^i} + y^j \frac{\partial u^i}{\partial x^j} \frac{\partial}{\partial y^i} \in \Gamma(T(TM_0)) \tag{2.2}$$

where $TM_0 := TM \backslash \{0\}$.

A vector field V on a Finsler manifold (M, F) is said to be a *Killing field* if the corresponding flow ϕ_t is isometric, i.e.,

$$\breve{\phi}_t^* F = F \tag{2.3}$$

equivalently, $X_V(F) = 0$ (cf.[9]).

Lemma 2.1 *Let V be a vector field on a manifold M. For functions f and g on TM and constant σ, the following hold*

$$\text{(i) } \mathrm{X_V(f\,g) = X_V(f)g + f\,X_V(g)}, \tag{2.4}$$

$$\text{(ii) } \mathrm{X_V(f^{\sigma}) = \sigma f^{\sigma-1} X_V(f)}. \tag{2.5}$$

Consider the following function

$$F := \alpha\phi(s), \qquad s = \frac{\beta}{\alpha} \tag{2.6}$$

where $\phi = \phi(s)$ is a positive C^∞ function on $(-b_o, b_o)$ satisfying

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \qquad |s| \leq b \leq b_o.$$

Then by Lemma 1.1.2 in [7], F is a Finsler metric if $\|\beta_x\|_\alpha < b_o$ for any $x \in M$. A Finsler metric in the form (2.6) is called an *(α, β)-metric.*

Lemma 2.2 *Let V be a vector field on a manifold M and $\Phi := \alpha\phi(\beta/\alpha)$ an (α, β)-metric on M. Then*

$$X_V(\Phi) = \left(\phi - \frac{\beta}{\alpha}\right) X_V(\alpha) + \phi' X_V(\beta) \tag{2.7}$$

where

$$\phi = \phi(\beta/\alpha), \quad \phi' = \phi'(\beta/\alpha).$$

Proof: By a straightforward computation one obtains

$$\begin{aligned} \frac{\partial\Phi}{\partial x^i} &= \frac{\partial}{\partial x^i}\left[\alpha\phi\left(\frac{\beta}{\alpha}\right)\right] \\ &= \phi \cdot \frac{\partial\alpha}{\partial x^i} + \alpha\phi' \frac{\frac{\partial\beta}{\partial x^i}\alpha - \frac{\partial\alpha}{\partial x^i}\beta}{\alpha^2} = \left(\phi - \frac{\beta}{\alpha}\right)\frac{\partial\alpha}{\partial x^i} + \phi' \frac{\partial\beta}{\partial x^i}. \end{aligned} \tag{2.8}$$

Similarly, we have

$$\frac{\partial\Phi}{\partial y^i} = \left(\phi - \frac{\beta}{\alpha}\right)\frac{\partial\alpha}{\partial y^i} + \phi' \frac{\partial\beta}{\partial y^i}. \tag{2.9}$$

By using (2.2), (2.8) and (2.9) we have

$$\begin{aligned} X_V(\Phi) &= V^i\left[\left(\phi - \frac{\beta}{\alpha}\right)\frac{\partial\alpha}{\partial x^i} + \phi'\frac{\partial\beta}{\partial x^i}\right] + y^j\frac{\partial V^i}{\partial x^j}\left[\left(\phi - \frac{\beta}{\alpha}\right)\frac{\partial\alpha}{\partial y^i} + \phi'\frac{\partial\beta}{\partial y^i}\right] \\ &= \left(\phi - \frac{\beta}{\alpha}\right)\left(V^i\frac{\partial\alpha}{\partial x^i} + y^j\frac{\partial V^i}{\partial x^j}\frac{\partial\alpha}{\partial y^i}\right) + \phi'\left(V^i\frac{\partial\beta}{\partial x^i} + y^j\frac{\partial V^i}{\partial x^j}\frac{\partial\beta}{\partial y^i}\right) \\ &= \left(\phi - \frac{\beta}{\alpha}\right) X_V(\alpha) + \phi' X_V(\beta). \end{aligned}$$

3 Some lemmas

In this section, we establish the Lemmas required in next section manufacturing Killing fields for a large class of (α, β)-metrics.

Let $\Phi = \alpha\phi(\beta/\alpha)$ be an (α, β)-metric on an open subset $\mathcal{U} \subset \mathbb{R}^N$. Define

$$\alpha := \rho(h)\bar{\alpha}, \qquad \beta := B\rho(h)^{r+1}dh \tag{3.1}$$

where

$$\bar{\alpha} = \frac{\sqrt{|y|^2 + \mu(|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 + \mu|x|^2} \tag{3.2}$$

where

$$\rho(t)=\left[-\frac{2rB^2}{p}(C+\eta t-\frac{1}{2}\mu t^2)\right]^{-\frac{1}{2r}} \tag{3.3}$$

$$h:=\frac{1}{\sqrt{1+\mu|x|^2}}\left\{A+\langle a,\,x\rangle+\frac{\eta|x|^2}{1+\sqrt{1+\mu|x|^2}}\right\} \tag{3.4}$$

where A, B, C and η are constants ($B>0$) and $a\in\mathbb{R}^N$ is a constant vector.

Remark Recently $(\alpha,\,\beta)$-metrics satisfying (3.1) have been used to construct projectively flat $(\alpha,\,\beta)$-metrics [10, 11, 14].

Lemma 3.1 *Let α be a Riemannian metric and β a 1-form satisfying (3.1). Then*

$$\alpha=C_1\lambda^{-\frac{1}{2r}}\bar{\alpha},\quad \beta=C_2\lambda^{-\frac{r+1}{2r}}dh \tag{3.5}$$

where

$$\lambda=C+\frac{\eta}{\omega}\zeta+\frac{\eta^2|x|^2}{\omega(1+\omega)}-\frac{\mu\zeta^2}{2\omega^2}-\frac{\mu\eta|x|^2\zeta}{\omega^2(1+\omega)}-\frac{\mu\eta^2|x|^4}{2\omega^2(1+\omega)^2}, \tag{3.6}$$

$$dh=\frac{\langle a,\,y\rangle}{\omega}+\frac{2\eta\langle x,\,y\rangle}{\omega(1+\omega)}-\frac{\mu\zeta\langle x,\,y\rangle}{\omega^3}-\frac{\mu\eta|x|^2\langle x,\,y\rangle}{\omega^3(1+\omega)}-\frac{\mu\eta|x|^2\langle x,\,y\rangle}{\omega^2(1+\omega)^2}, \tag{3.7}$$

$$C_1=\left(-\frac{2rB^2}{p}\right)^{-\frac{1}{2r}},\quad C_2=B\cdot\left(-\frac{2rB^2}{p}\right)^{-\frac{r+1}{2r}} \tag{3.8}$$

where

$$\omega:=\sqrt{1+\mu|x|^2},\quad \zeta=A+\langle a,\,x\rangle. \tag{3.9}$$

Proof: Plugging (3.9) into (3.4) yields

$$h=\frac{1}{\omega}\left(\zeta+\frac{\eta|x|^2}{1+\omega}\right). \tag{3.10}$$

Substituting this into (3.3) we obtain

$$\begin{aligned}\rho(h) &= \left(-\tfrac{2rB^2}{p}\right)^{-\frac{1}{2r}}\left[C+\tfrac{\eta}{\omega}\left(\zeta+\tfrac{\eta|x|^2}{1+\omega}\right)-\tfrac{\mu}{2\omega^2}\left(\zeta+\tfrac{\eta|x|^2}{1+\omega}\right)^2\right]^{-\frac{1}{2r}}\\ &= C_1\left[C+\tfrac{\eta}{\omega}\zeta+\tfrac{\eta^2|x|^2}{\omega(1+\omega)}-\tfrac{\mu\zeta^2}{2\omega^2}-\tfrac{\mu\eta|x|^2\zeta}{\omega^2(1+\omega)}-\tfrac{\mu\eta^2|x|^4}{2\omega^2(1+\omega)^2}\right]^{-\frac{1}{2r}}=C_1\lambda^{-\frac{1}{2r}}.\end{aligned} \tag{3.11}$$

Plugging (3.11) into (3.1) we have the first equation of (3.5). Also, from (3.11), ones obtain

$$\rho(h)^{r+1} = C_1^{r+1}\lambda^{-\frac{r+1}{2r}}.$$

It follows from (3.1) that

$$\beta = BC_1^{r+1}\lambda^{-\frac{r+1}{2r}}dh = C_2\lambda^{-\frac{r+1}{2r}}dh.$$

Hence the second equation of (3.5) holds. Now we are going to prove (3.7). It is easy to see that

$$d\omega = \frac{\mu\langle x,\, y\rangle}{\omega}, \tag{3.12}$$

$$d\zeta = \langle a,\, y\rangle, \tag{3.13}$$

$$d|x|^2 = 2\langle x,\, y\rangle \tag{3.14}$$

from (3.9). Combining with (3.10) we get

$$\begin{aligned} dh &= d\left[\tfrac{1}{\omega}\left(\zeta + \tfrac{\eta|x|^2}{1+\omega}\right)\right] \\ &= -\tfrac{d\omega}{\omega^2}\left(\zeta + \tfrac{\eta|x|^2}{1+\omega}\right) + \tfrac{1}{\omega}\left[d\zeta + \eta\tfrac{(1+\omega)d|x|^2 - |x|^2 d(1+\omega)}{(1+\omega)^2}\right] \\ &= -\tfrac{\mu\langle x,\, y\rangle}{\omega^3}\left(\zeta + \tfrac{\eta|x|^2}{1+\omega}\right) + \tfrac{1}{\omega}\left[\langle a,\, y\rangle + \tfrac{2\eta\langle x,\, y\rangle}{1+\omega} - \tfrac{\mu\eta|x|^2\langle x,\, y\rangle}{\omega(1+\omega)^2}\right]. \end{aligned} \tag{3.15}$$

Lemma 3.2 *Let $\bar\alpha$ be a Riemannian metric satisfying (3.2). Then*

$$\frac{\partial\bar\alpha}{\partial x^i} = \mu\frac{(2\langle x,\, y\rangle^2 - \omega^2|y|^2)x^i - \omega^2\langle x,\, y\rangle y^i}{\omega^4\sqrt{\omega^2|y|^2 - \mu\langle x,\, y\rangle^2}}. \tag{3.16}$$

$$\frac{\partial\bar\alpha}{\partial y^i} = \frac{\omega^2 y^i - \mu\langle x,\, y\rangle x^i}{\omega^2\sqrt{\omega^2|y|^2 - \mu\langle x,\, y\rangle^2}} \tag{3.17}$$

where ω is defined in (3.9).

Proof: Plugging (3.9) into (3.2) yields

$$\bar\alpha = \frac{\sqrt{\omega^2|y|^2 - \mu\langle x,\, y\rangle^2}}{\omega^2}. \tag{3.18}$$

By simple calculations, we have

$$\frac{\partial\omega^2}{\partial x^i} = 2\mu x^i, \qquad \frac{\partial\omega^2}{\partial y^i} = 0, \tag{3.19}$$

$$\frac{\partial|y|^2}{\partial x^i} = 0, \qquad \frac{\partial|y|^2}{\partial y^i} = 2y^i, \tag{3.20}$$

$$\frac{\partial}{\partial x^i}\langle x,\, y\rangle = y^i, \qquad \frac{\partial}{\partial y^i}\langle x,\, y\rangle = x^i. \tag{3.21}$$

By using these formula we obtain

$$\begin{aligned}
\frac{\partial\bar{\alpha}}{\partial x^i} &= \frac{\partial}{\partial x^i}\frac{\sqrt{\omega^2|y|^2-\mu\langle x,y\rangle^2}}{\omega^2} \\
&= \frac{1}{\omega^4}\left[\frac{\mu x^i|y|^2-\mu\langle x,y\rangle y^i}{\sqrt{\omega^2|y|^2-\mu\langle x,y\rangle^2}}\omega^2 - 2\mu\sqrt{\omega^2|y|^2-\mu\langle x,\, y\rangle^2}x^i\right] \\
&= \mu\frac{(2\langle x,y\rangle^2-\omega^2|y|^2)x^i-\omega^2\langle x,y\rangle y^i}{\omega^4\sqrt{\omega^2|y|^2-\mu\langle x,y\rangle^2}}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial\bar{\alpha}}{\partial y^i} &= \frac{\partial}{\partial y^i}\frac{\sqrt{\omega^2|y|^2-\mu\langle x,y\rangle^2}}{\omega^2} \\
&= \frac{1}{\omega^2}\frac{\partial}{\partial y^i}\sqrt{\omega^2|y|^2-\mu\langle x,\, y\rangle^2} = \frac{\omega^2 y^i-\mu\langle x,y\rangle x^i}{\omega^2\sqrt{\omega^2|y|^2-\mu\langle x,y\rangle^2}}.
\end{aligned}$$

Lemma 3.3 *Suppose that ω and ζ are defined in (3.9). Then*

$$\frac{\partial}{\partial x^i}\frac{\langle a,\, y\rangle}{\omega} = -\frac{\mu\langle a,\, y\rangle x^i}{\omega^3}, \tag{3.22}$$

$$\frac{\partial}{\partial y^i}\frac{\langle a,\, y\rangle}{\omega} = \frac{a^i}{\omega}, \tag{3.23}$$

$$\frac{\partial}{\partial x^i}\frac{\langle x,\, y\rangle}{\omega(1+\omega)} = \frac{y^i}{\omega(1+\omega)} - \mu\frac{\langle x,\, y\rangle(1+2\omega)x^i}{\omega^3(1+\omega)^2}, \tag{3.24}$$

$$\frac{\partial}{\partial y^i}\frac{\langle x,\, y\rangle}{\omega(1+\omega)} = \frac{x^i}{\omega(1+\omega)}, \tag{3.25}$$

$$\frac{\partial}{\partial x^i}\frac{\zeta\langle x,\, y\rangle}{\omega^3} = \frac{\langle x,\, y\rangle a^i + \zeta y^i}{\omega^3} - \mu\frac{3\zeta\langle x,\, y\rangle x^i}{\omega^5}, \tag{3.26}$$

$$\frac{\partial}{\partial y^i}\frac{\zeta\langle x,\, y\rangle}{\omega^3} = \frac{\zeta x^i}{\omega^3}, \tag{3.27}$$

$$\frac{\partial}{\partial x^i}\frac{|x|^2\langle x,\, y\rangle}{\omega^3(1+\omega)} = \frac{2\langle x,\, y\rangle x^i + |x|^2 y^i}{\omega^3(1+\omega)} - \mu\frac{(3+4\omega)|x|^2\langle x,\, y\rangle x^i}{\omega^5(1+\omega)^2}, \tag{3.28}$$

$$\frac{\partial}{\partial y^i}\frac{|x|^2\langle x,\, y\rangle}{\omega^3(1+\omega)} = \frac{|x|^2 x^i}{\omega^3(1+\omega)}, \tag{3.29}$$

$$\frac{\partial}{\partial x^i}\frac{|x|^2\langle x,\, y\rangle}{\omega^2(1+\omega)^2} = \frac{2\langle x,\, y\rangle x^i + |x|^2 y^i}{\omega^2(1+\omega)^2} - \frac{2\mu(1+2\omega)|x|^2\langle x,\, y\rangle x^i}{\omega^4(1+\omega)^3}, \tag{3.30}$$

$$\frac{\partial}{\partial y^i}\frac{|x|^2\langle x,\,y\rangle}{\omega^2(1+\omega)^2}=\frac{|x|^2x^i}{\omega^2(1+\omega)^2}. \tag{3.31}$$

Proof: We only prove (3.22), (3.24), (3.26), (3.28) and (3.30). The others are easy to obtain. By using (3.19) we have

$$\frac{\partial}{\partial x^i}\frac{\langle a,\,y\rangle}{\omega}=\langle a,\,y\rangle\frac{\partial\omega^{-1}}{\partial x^i}=-\frac{\mu\langle a,\,y\rangle x^i}{\omega^3}.$$

From (3.19) and (3.21) we obtain

$$\begin{aligned}\frac{\partial}{\partial x^i}\frac{\langle x,\,y\rangle}{\omega(1+\omega)} &= \frac{\omega(1+\omega)y^i-\langle x,\,y\rangle\left[\frac{\mu x^i}{\omega}(1+\omega)+\mu x^i\right]}{\omega^2(1+\omega)^2}\\ &= \frac{y^i}{\omega(1+\omega)}-\mu\frac{\langle x,\,y\rangle(1+2\omega)x^i}{\omega^3(1+\omega)^2}.\end{aligned}$$

It follows that (3.22) and (3.24) holds. Using (3.9) we get

$$\frac{\partial\zeta}{\partial x^i}=a^i,\quad \frac{\partial\zeta}{\partial y^i}=0. \tag{3.32}$$

A direct calculation using (3.32), (3.19) and (3.21) yields

$$\frac{\partial}{\partial x^i}\frac{\zeta\langle x,\,y\rangle}{\omega^3}=\frac{1}{\omega^6}\left[(\langle x,\,y\rangle a^i+\zeta y^i)\omega^3-3\mu\zeta\langle x,\,y\rangle x^i\right]=\frac{\langle x,\,y\rangle a^i+\zeta y^i}{\omega^3}-\mu\frac{3\zeta\langle x,\,y\rangle x^i}{\omega^5}.$$

This implies (3.26). By a straightforward computation one obtains

$$\frac{\partial|x|^2}{\partial x^i}=2x^i,\qquad \frac{\partial|x|^2}{\partial x^i}=0. \tag{3.33}$$

Together with (3.19) and (3.21) we have

$$\begin{aligned}\frac{\partial}{\partial x^i}\frac{|x|^2\langle x,y\rangle}{\omega^3(1+\omega)} &= \frac{2\langle x,y\rangle x^i+|x|^2y^i}{\omega^3(1+\omega)}-\frac{|x|^2\langle x,y\rangle}{\omega^6(1+\omega)^2}\left[3\mu\omega x^i(1+\omega)+\mu\omega^2x^i\right]\\ &= \frac{2\langle x,y\rangle x^i+|x|^2y^i}{\omega^3(1+\omega)}-\mu\frac{(3+4\omega)|x|^2\langle x,y\rangle x^i}{\omega^5(1+\omega)^2}.\end{aligned}$$

Finally, using (3.19), (3.21) and (3.33), we obtain

$$\begin{aligned}\frac{\partial}{\partial x^i}\frac{|x|^2\langle x,y\rangle}{\omega^2(1+\omega)^2} &= \frac{2\langle x,y\rangle x^i+|x|^2y^i}{\omega^2(1+\omega)^2}-\frac{|x|^2\langle x,y\rangle}{\omega^4(1+\omega)^4}\left[2\mu x^i(1+\omega)^2+2\mu\omega(1+\omega)x^i\right]\\ &= \frac{2\langle x,y\rangle x^i+|x|^2y^i}{\omega^2(1+\omega)^2}-\frac{2\mu(1+2\omega)|x|^2\langle x,y\rangle x^i}{\omega^4(1+\omega)^3}.\end{aligned}$$

Lemma 3.4 *Suppose that λ, ω and ζ are defined in (3.6) and (3.9). Then*

$$\frac{\partial}{\partial x^i}\frac{\zeta}{\omega}=\frac{a^i}{\omega}-\frac{\mu\zeta x^i}{\omega}, \tag{3.34}$$

$$\frac{\partial}{\partial x^i}\frac{|x|^2}{\omega(1+\omega)} = \frac{2x^i}{\omega(1+\omega)} - \frac{\mu(1+2\omega)|x|^2x^i}{\omega^3(1+\omega)^2}, \tag{3.35}$$

$$\frac{\partial}{\partial x^i}\frac{\zeta^2}{\omega^2} = \frac{2\zeta a^i}{\omega^2} - \frac{\mu\zeta^2x^i}{\omega^2}, \tag{3.36}$$

$$\frac{\partial}{\partial x^i}\frac{\zeta|x|^2}{\omega^2(1+\omega)} = \frac{2\zeta x^i + |x|^2a^i}{\omega^2(1+\omega)} - \frac{\mu\zeta(2+3\omega)|x|^2x^i}{\omega^4(1+\omega)^2}, \tag{3.37}$$

$$\frac{\partial}{\partial x^i}\frac{|x|^4}{\omega^2(1+\omega)^2} = \frac{4|x|^2x^i}{\omega^2(1+\omega)^2} - \frac{2\mu(1+2\omega)|x|^4x^i}{\omega^4(1+\omega)^3}, \tag{3.38}$$

$$\frac{\partial\lambda}{\partial y^i} = 0. \tag{3.39}$$

Proof: By using (3.19) and (3.32) we have

$$\frac{\partial}{\partial x^i}\frac{\zeta}{\omega} = \frac{1}{\omega^2}\left(\omega a^i - \mu\zeta\frac{x^i}{\omega}\right) = \frac{a^i}{\omega} - \frac{\mu\zeta x^i}{\omega}$$

and hence

$$\frac{\partial}{\partial x^i}\frac{\zeta^2}{\omega^2} = 2\frac{\zeta}{\omega}\frac{\partial}{\partial x^i}\left(\frac{\zeta}{\omega}\right) = \frac{2\zeta a^i}{\omega} - \frac{\mu\zeta^2x^i}{\omega}.$$

These give (3.34) and (3.36). From (3.19) and (3.33) ones obtain

$$\begin{aligned}\frac{\partial}{\partial x^i}\frac{|x|^2}{\omega(1+\omega)} &= \frac{1}{\omega^2(1+\omega)^2}\left\{2x^i\omega(1+\omega) - |x|^2\left[\frac{\mu x^i}{\omega}(1+\omega) + \mu x^i\right]\right\} \\ &= \frac{2x^i}{\omega(1+\omega)} - \frac{\mu(1+2\omega)|x|^2x^i}{\omega^3(1+\omega)^2}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial x^i}\frac{|x|^4}{\omega^2(1+\omega)^2} &= \frac{4|x|^2x^i}{\omega^2(1+\omega)^2} - \frac{|x|^4}{\omega^2(1+\omega)^2}\left[2\mu x^i(1+\omega)^2 + 2\mu\omega(1+\omega)x^i\right] \\ &= \frac{4|x|^2x^i}{\omega^2(1+\omega)^2} - \frac{2\mu(1+2\omega)|x|^4x^i}{\omega^4(1+\omega)^3}.\end{aligned}$$

These give (3.35) and (3.38). Now we are going to show (3.37). In fact, using (3.19), (3.32) and (3.33), we get

$$\begin{aligned}\frac{\partial}{\partial x^i}\frac{\zeta|x|^2}{\omega^2(1+\omega)} &= \frac{1}{\omega^4(1+\omega)^2}\left\{(2x^i\zeta + |x|^2a^i)\omega^2(1+\omega) - |x|^2\zeta\left[2\mu x^i(1+\omega) + \mu\omega x^i\right]\right\} \\ &= \frac{2\zeta x^i + |x|^2a^i}{\omega^2(1+\omega)} - \frac{\mu\zeta(2+3\omega)|x|^2x^i}{\omega^4(1+\omega)^2}.\end{aligned}$$

Finally, note that λ is independent of y. Hence we obtain (3.39).

4 Killing fields of $(\alpha,\,\beta)$-metrics

In [9], authors give flag curvature a changing formula via the shortest time problem with respect to a homothetic field. In particular, they get scalar flag curvature a preserving property for navigation problem of a Killing field. In order to producing new Finsler metrics with scalar (or constant) flag curvature, this section constructs Killing fields for a large class of $(\alpha,\,\beta)$-metrics.

Lemma 4.1 *Let $\Phi = \alpha\phi(\beta/\alpha)$ be an $(\alpha,\,\beta)$-metric on an open subset $\mathcal{U} \subset \mathbb{R}^n$. Assume that α and β satisfying (3.1). Then*

$$\begin{aligned} X_V(\Phi) &= \left[-\frac{C_1}{2r}\lambda^{-\frac{2r+1}{2r}}\bar{\alpha}\left(\phi - \frac{\beta}{\alpha}\right) + C_2\frac{r+1}{2r}\phi'\lambda^{-\frac{3r+1}{2r}}dh\right]X_V(\lambda) \\ &\quad + C_1\lambda^{-\frac{1}{2r}}\left(\phi - \frac{\beta}{\alpha}\right)X_V(\bar{\alpha}) + C_2\phi'\lambda^{-\frac{r+1}{2r}}X_V(dh) \end{aligned} \tag{4.1}$$

where V is a vector field on $\mathcal{U}$.

Proof: By using (2.4), (2.5), (2.7) and (3.5) we have

$$\begin{aligned} X_V(\Phi) &= \left(\phi - \frac{\beta}{\alpha}\right)X_V(\alpha) + \phi' X_V(\beta) \\ &= C_1\left(\phi - \frac{\beta}{\alpha}\right)X_V(\lambda^{-\frac{1}{2r}}\bar{\alpha}) + C_2\phi' X_V(\lambda^{-\frac{r+1}{2r}}dh) \\ &= C_1\left(\phi - \frac{\beta}{\alpha}\right)\left[-\frac{1}{2r}\lambda^{-\frac{1}{2r}-1}X_V(\lambda)\bar{\alpha} + \lambda^{-\frac{1}{2r}}X_V(\bar{\alpha})\right] \\ &\quad + C_2\phi'\left[-\frac{1+r}{2r}\lambda^{-\frac{1+r}{2r}-1}X_V(\lambda)dh + \lambda^{-\frac{1+r}{2r}}X_V(dh)\right] \\ &= \left[C_1\left(\phi - \frac{\beta}{\alpha}\right)\left(-\frac{1}{2r}\lambda^{-\frac{1+2r}{2r}}\bar{\alpha}\right) - C_2\phi'\left(-\frac{1+r}{2r}\right)\lambda^{-\frac{1+3r}{2r}}dh\right]X_V(\lambda) \\ &\quad + C_1\left(\phi - \frac{\beta}{\alpha}\right)\lambda^{-\frac{1}{2r}}X_V(\bar{\alpha}) + C_2\phi'\lambda^{-\frac{1+r}{2r}}X_V(dh) \end{aligned}$$

This gives (4.1).

Proposition 4.2 *Let $\Phi = \alpha\phi(\beta/\alpha)$ be an $(\alpha,\,\beta)$-metric on an open subset $\mathcal{U} \subset \mathbb{R}^N$. Assume that α and β satisfying (3.1). Let V denote a vector field on $\mathcal{U}$ defined by*

$$V_x = xQ \quad \text{at} \quad x \in \mathcal{U} \tag{4.2}$$

where Q is skew-symmetric and satisfies that

$$Qa^T = 0 \tag{4.3}$$

where $a \in \mathbb{R}^N$ is a constant vector given in (3.4). Then V is of Killing type with respect to Φ.

Proof: Let $Q = (q_{ij})$ and $a = (a^1, \cdots, a^n)$. Since Q is anti-symmetric, we have

$$\Sigma_{i,j} x^i x^j q_{ij} = \Sigma_{j,i} x^j x^i q_{ji} = -\Sigma_{j,i} x^j x^i q_{ij} = -\Sigma_{i,j} x^i x^j q_{ij}.$$

It follows that

$$x^i x^j q_{ij} = 0. \tag{4.4}$$

Similarly, we get

$$y^i y^j q_{ij} = 0. \tag{4.5}$$

The condition (4.3) implies that

$$\Sigma a^i q_{ji} = 0. \tag{4.6}$$

Lemma 4.1 tells us it is sufficient to show

$$X_V(\lambda) = X_V(\bar{\alpha}) = X_V(dh) = 0. \tag{4.7}$$

By (2.2) and (4.2), we have

$$X_V = x^j q_{ji} \frac{\partial}{\partial x^i} + y^j q_{ji} \frac{\partial}{\partial y^i}. \tag{4.8}$$

Together with (3.6), (4.4), (4.6) and Lemma 3.4 we get

$$\begin{aligned} X_V(\lambda) &= x^j q_{ji} \tfrac{\partial \lambda}{\partial x^i} + y^j q_{ji} \tfrac{\partial \lambda}{\partial y^i} \\ &= x^j q_{ji} \left[\eta \tfrac{\partial}{\partial x^i} \tfrac{\zeta}{\omega} + \eta^2 \tfrac{\partial}{\partial x^i} \tfrac{|x|^2}{\omega(1+\omega)} - \tfrac{\mu}{2} \tfrac{\partial}{\partial x^i} \tfrac{\zeta^2}{\omega^2} - \mu\eta \tfrac{\partial}{\partial x^i} \tfrac{\zeta |x|^2}{\omega^2(1+\omega)} - \tfrac{\mu\eta^2}{2} \tfrac{\partial}{\partial x^i} \tfrac{|x|^4}{\omega^2(1+\omega)^2} \right] \\ &= \eta x^j q_{ji} \tfrac{a^i - \mu\zeta x^i}{\omega} - \tfrac{\mu}{2} x^j q_{ji} \tfrac{2\zeta a^i - \mu\zeta^2 x^i}{\omega^2} + \eta^2 x^j q_{ji} \left[\tfrac{2x^i}{\omega(1+\omega)} - \tfrac{\mu(1+2\omega)|x|^2 x^i}{\omega^3(1+\omega)^2} \right] \\ &\quad -\mu\eta x^j q_{ji} \left[\tfrac{2\zeta x^i + |x|^2 a^i}{\omega^2(1+\omega)} - \tfrac{\mu\zeta(2+3\omega)|x|^2 x^i}{\omega^4(1+\omega)^2} \right] \\ &\quad -\tfrac{\mu\eta^2}{2} x^j q_{ji} \left[\tfrac{4|x|^2 x^i}{\omega^2(1+\omega)^2} - \tfrac{2\mu(1+2\omega)|x|^4 x^i}{\omega^4(1+\omega)^3} \right] = 0. \end{aligned}$$

Similarly, one obtains $X_V(\bar{\alpha}) = 0$ from (4.4), (4.5), (4.8) and Lemma 3.3. Finally, from (3.7), (4.4), (4.5), (4.6) and Lemma 3.4 we get $X_V(dh) = 0$. Thus $X_V(\Phi) = 0$, therefore V is a Killing field of Φ.

5 New Finsler metrics with scalar flag curvature from old

In this section we are going to produce new Finsler metrics with scalar flag curvature from a given Finsler metric.

In [11], authors given an explicit construction of polynomial of arbitrary degree $(\alpha,\, \beta)$-metrics with scalar flag curvature and determine their scalar flag curvature. Precisely, they have proved

Theorem 5.1 *Let $\phi(s)$ be a polynomial function defined by*

$$\phi(s) = 1 + 2^n s + 2n\Sigma_{k=0}^{n-1}\frac{(-1)^k C_k^{n-1} s^{2k+2}}{(2k+1)(2k+2)} \tag{5.1}$$

where

$$C_k^m := \frac{m(m-1)\cdots(m-k+1)}{k!}.$$

Then the following polynomial $(\alpha,\, \beta)$ metric on an the open subset at origin in $\mathbb{R}^N$

$$F := \frac{(1+\langle a,\, x\rangle)^{2n}}{(1-|x|^2)^{n+1}}\sqrt{(1-|x|^2)|y|^2+\langle x,\, y\rangle^2}\,\phi\left(\frac{(1-|x|^2)\langle a,\, y\rangle + (1+\langle a,\, x\rangle)\langle x,\, y\rangle}{(1+\langle a,\, x\rangle)\sqrt{(1-|x|^2)|y|^2+\langle x,\, y\rangle^2}}\right)$$

is of scalar curvature with flag curvature

$$\begin{aligned} K \;=\; & -\frac{(n+1)|y|^2}{F^2\omega^2} + \frac{(n^2-1)\langle x,\, y\rangle^2}{F^2\omega^4} - \frac{\zeta^{2n-4}\psi^2\phi''}{2\theta F^3\omega^{2n+2}} \\ & + \frac{\zeta^{2n-2}}{4F^4\omega^{4n+4}}(2n\langle a,\, y\rangle\theta\phi\zeta + \phi'\psi)[4(n+1)F\langle x,\, y\rangle\omega^{2n} + 3\zeta^{2n-2}] \\ & - (n\langle a,\, y\rangle\theta\phi\zeta + \phi'\psi)\frac{(2n-1)\langle a,\, y\rangle\zeta^{2n-3}}{F^3\omega^{2n+2}} \end{aligned}$$

where $n \in \{0,\, 1,\, 2,\, \cdots\}$, $a \in \mathbb{R}^N$ is a constant vector with $|a| < 1$ and ω and ζ are defined in (3.9) and

$$\begin{aligned} \theta : \;&=\; \sqrt{(1-|x|^2)|y|^2+\langle x,\, y\rangle^2} \\ \psi : \;&=\; \zeta^2|y|^2 - 2\langle a,\, y\rangle\langle x,\, y\rangle\zeta - \omega^2\langle a,\, y\rangle^2 \\ \phi^{(i)} : \;&=\; \phi^{(i)}\left(\frac{(1-|x|^2)\langle a,\, y\rangle + (1+\langle a,\, x\rangle)\langle x,\, y\rangle}{(1+\langle a,\, x\rangle)\sqrt{(1-|x|^2)|y|^2+\langle x,\, y\rangle^2}}\right) \end{aligned}$$

where $\phi^{(i)}$ denotes i-order derivative for $\phi(s)$.

Now we take a look at the special case of (3.1)$\sim$ (3.4): when $A = C = 1$, $B = \eta = 0$, $\mu = -1$, $r = -\frac{1}{2n}$,

$$\alpha = \frac{(1+\langle a,\, x\rangle)^{2n}}{(1-|x|^2)^{n+1}}\sqrt{(1-|x|^2)|y|^2+\langle x,\, y\rangle^2} \tag{5.2}$$

$$\beta = \frac{(1+\langle a,\, x\rangle)^{2n-1}}{(1-|x|^2)^{n+1}}\left[(1-|x|^2)\langle a,\, y\rangle + (1+\langle a,\, x\rangle)\langle x,\, y\rangle\right] \tag{5.3}$$

Let $\Phi = \alpha\phi(\beta/\alpha)$ be an $(\alpha,\, \beta)$-metric on an open subset $\mathcal{U} \subset \mathbb{R}^N$. Assume that α and β satisfying (5.2) and (5.3). Proposition 4.2 tells us $V := xQ$ is a Killing field of Φ where Q satisfies $Q^T = -Q$ and $Qa^T = 0$. Suppose that ϕ is given in (5.1). Then Theorem 5.1 implies that Φ is of scalar curvature. Let V be a Killing field of Φ on $\mathcal{U}$ with $\Phi(x,\, V_x) < 1$. Define a new Finsler metric F by [2, 9, 12]

$$\Phi\left(x,\, \frac{y}{F(x,\, y)} + V_x\right) = 1, \quad \forall x \in \mathcal{U}, \quad y \in T_x\mathcal{U}. \tag{5.4}$$

See [13] for a general account. By using Mo-Huang' Theorem 1.1 and Corollary 7.1, we obtain F is also of scalar curvature. Moreover, its scalar flag curvature K_F is given by

$$K_F(x,\, y) = K_\Phi\left(x,\, y - \Phi(x,y)V_x\right). \tag{5.5}$$

Combining this with Theorem 5.1 we have the following:

Theorem 5.2 *Let*

$$\Phi := \frac{(1+\langle a,\, x\rangle)^{2n}}{(1-|x|^2)^{n+1}}\sqrt{(1-|x|^2)|y|^2+\langle x,\, y\rangle^2}\,\phi\left(\frac{(1-|x|^2)\langle a,\, y\rangle + (1+\langle a,\, x\rangle)\langle x,\, y\rangle}{(1+\langle a,\, x\rangle)\sqrt{(1-|x|^2)|y|^2+\langle x,\, y\rangle^2}}\right)$$

be an $(\alpha,\, \beta)$ metric on an the open subset $\mathcal{U}$ at origin in $\mathbb{R}^N$ where

$$\phi(s) = 1 + 2^n s + 2n\Sigma_{k=0}^{n-1}\frac{(-1)^k C_k^{n-1} s^{2k+2}}{(2k+1)(2k+2)}.$$

Assume that V is a vector field on $\mathcal{U}$ defined by (4.2) where Q is skew-symmetric and satisfies that (4.3) and $\Phi(x,\, V_x) < 1$. Then Finsler metric F given by (5.4) is *of scalar curvature with flag curvature*

$$
\begin{aligned}
K_F \;&=\; -\frac{(n+1)|y-\Phi(x,y)V_x|^2}{\Phi^2\omega^2} + \frac{(n^2-1)\langle x,\, y-\Phi(x,y)V_x\rangle^2}{\Phi^2\omega^4} - \frac{\zeta^{2n-4}\psi^2\phi''}{2\theta\Phi^3\omega^{2n+2}} \\
&+\frac{\zeta^{2n-2}}{4\Phi^4\omega^{4n+4}}\left(2n\langle a,\, y-\Phi(x,y)V_x\rangle\theta\phi\zeta+\phi'\psi\right) \\
&\times\left[4(n+1)\Phi\langle x,\, y-\Phi(x,y)V_x\rangle\omega^{2n}+3\zeta^{2n-2}\right] \\
&-\left(n\langle a,\, y-\Phi(x,y)V_x\rangle\theta\phi\zeta+\phi'\psi\right)\frac{(2n-1)\langle a,\, y-\Phi(x,y)V_x\rangle\zeta^{2n-3}}{\Phi^3\omega^{2n+2}}
\end{aligned}
$$

where

$$
\begin{aligned}
\theta :\; &= \sqrt{(1-|x|^2)|y-\Phi(x,y)V_x|^2+\langle x,\, y-\Phi(x,y)V_x\rangle^2} \\
\psi :\; &= \zeta^2|y-\Phi(x,y)V_x|^2-2\langle a,\, y-\Phi(x,y)V_x\rangle\langle x,\, y-\Phi(x,y)V_x\rangle\zeta \\
&\quad -\omega^2\langle a,\, y-\Phi(x,y)V_x\rangle^2 \\
\phi^{(i)} :\; &= \phi^{(i)}\left(\frac{\omega^2\langle a,\, y-\Phi(x,y)V_x\rangle+\zeta\langle x,\, y-\Phi(x,y)V_x\rangle}{\zeta\theta}\right)
\end{aligned}
$$

ω and ζ are defined in (3.9) where $\phi^{(i)}$ denotes i-order derivative for $\phi(s)$.

Remark Note that when $n=0,\ a=0,\ N=2,\ Q=\begin{pmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{pmatrix}$ we recover Shen's construction [12, Theorem 1.3].

Proof of Theorem 1.1: We take $n=1$ in Theorem 5.2. Then Φ is the Mo-Shen-Yang metric with zero flag curvature [10]. Together with (5.5) we obtain F is of zero flag curvature.

Remark We also obtain some other Finsler metrics with scalar flag curvature by using Theorem 5.4, Theorem 5.5 and Theorem 5.6 in [11] and Proposition 4.2.

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