Optimal Decentralized Control of Linear Systems via Groebner Bases and Variable Elimination

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Abstract

We consider the problem of optimal decentralized controller synthesis. There are several classes of such problems for which effective algorithms are known, including the quadratically invariant cases. In this paper, we use Groebner bases and elimination methods to characterize all the possible closed-loop maps which are obtainable by forming a feedback loop with decentralized controllers. We show that this approach allows solution of a strictly wider class of optimal decentralized control problems than the quadratically invariant ones.

1   Introduction

The problem of computing the optimal decentralized controller in the standard framework of a linear system with quadratic cost and Gaussian noise is well-known to be extremely hard, with many previous results in this area. There is no known algorithm which can efficiently compute an optimal decentralized controller for a general linear system. However, given a particular set of information accessible to the controller, called the information structure, and a corresponding linear plant, the synthesis problem may become tractable, depending on characteristics of the dynamics and the information structure. Much effort has been focused on characterizing those information structures and systems for which finding the optimal decentralized controller is easy.

In this paper, we introduce a large class of tractable problems by presenting an algorithm which is able to decide whether a decentralized control problem is easily solvable and find an optimal solution if it is so. Even though the algorithm does not characterize all the classes that are tractable, we believe it is one of the most powerful systematic methods to tackle the decentralized control problems.

Our approach is to analyze the set of closed-loop systems achievable using a decentralized controller, and we present a new algorithm for determining this set when the plant and controller are rational linear time-invariant systems. Using this algorithm, we are able to evaluate whether the set of achievable closed-loop maps is a linear space, and if it is, this allows standard synthesis tools to be applied.

We focus on rational linear systems, and our approach is based on algebraic geometry. Roughly speaking, we would like to solve

$$\begin{aligned} \text{minimize} \quad & \|P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}\| \\ \text{subject to} \quad & K \in S_p \\ & K \text{ is stabilizing} \end{aligned}$$

Here P_{11}, P_{12}, P_{21}, and P_{22} are matrix-valued rational functions, and S_p is the information structure, a set of matrix rational functions with a particular sparsity, thus S_p is the set of controllers which meet the desired decentralization constraint. The set of achievable closed-loop maps is then just

$$\left\{ P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \mid K \in S_p \right\}$$

and we would like to determine whether this set is an affine set. Our approach is to eliminate K, and describe this set by implicit equations so that a matrix rational function lies in this set if and only if its entries satisfy certain polynomial equations. The general theory of elimination is well-known in the field of algebraic geometry, and we bring this to bear in this specific matrix problem.

The main contribution of this paper is as follows. We introduce the idea of using rational elimination theory for finding the set of closed-loop maps. We show that in certain cases, this reduces the set of closed-loop maps to the solution set of a collection of linear equations. We show how to use these linear equations to find the optimal decentralized controller.

Prior Work.   Decentralized control has been studied for quite a long time. Radner [5] showed that for systems without feedback, an optimal controller minimizing quadratic cost for a linear system may be chosen to be linear. For the feedback case, it was shown by Witsenhausen [8] that for a linear system subject to a decentralized information constraint, a nonlinear controller can achieve better performance than any linear controller.

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Even finding an optimal linear controller is hard, and in fact, it was shown that some cases are intractable [1]. Since this work, research has been focused on characterizing the class of easily solvable problems. Ho and Chu [4] generalized Radner's result and defined a class of information structures, called *partially nested*, for which an optimal controller for the LQG problem is linear. Recently, Rotkowitz and Lall [6, 7] showed that the class of *quadratically invariant* problems may be easily solvable via convex programming. However, the class of quadratically invariant problems does not include all the tractable problems in the decentralized control. We will see an example in this paper.

2 Preliminaries

We define some terminology for matrix-valued transfer functions and closed-loop maps. We consider transfer functions for continuous-time systems. Therefore, we regard transfer functions that are defined on the set of the purely imaginary complex numbers, $i\mathbb{R}$. We denote the set of all real-rational functions $G : i\mathbb{R} \to \mathbb{C}^{m \times n}$ by $\mathbb{R}(s)^{m \times n}$. A matrix-valued rational function G is called **proper** if $\lim_{\omega \to \infty} G(i\omega)$ exists and is finite. Furthermore, if $\lim_{\omega \to \infty} G(i\omega) = 0$ then it is called **strictly proper**. Then, we denote the set of real-rational proper transfer function matrices as

$$\mathbb{R}(s)_p^{m \times n} = \left\{ G \in \mathbb{R}(s)^{m \times n} \mid G \text{ is proper} \right\}.$$

Similarly, we let

$$\mathbb{R}(s)_{sp}^{m \times n} = \left\{ G \in \mathbb{R}(s)^{m \times n} \mid G \text{ is strictly proper} \right\}.$$

Suppose $G \in \mathbb{R}(s)^{n \times n}$. If $\det(G) \neq 0$, *i.e.*, $\det(G)$ is a nonzero rational function, then we say that G is **invertible** and $G^{-1} \in \mathbb{R}(s)^{n \times n}$ is a well-defined matrix of rational functions. It follows that if $P_{22} \in \mathbb{R}(s)_{sp}^{n_y \times n_u}$ and $K \in \mathbb{R}(s)_p^{n_u \times n_y}$ then $I - P_{22}K$ is always invertible.

Now, consider a feedback loop formed by P and K, where the real-rational transfer function matrices

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \in \mathbb{R}(s)^{(n_z+n_y) \times (n_w+n_u)}$$

and $K \in \mathbb{R}(s)^{n_u \times n_y}$ represent the transfer function matrices of a given plant and a controller, respectively. We say that the interconnection is **well-defined** if $I - P_{22}K$ is invertible. Note that if $P \in \mathbb{R}(s)_p, P_{22} \in \mathbb{R}(s)_{sp}$, and $K \in \mathbb{R}(s)_p$, then this notion of well-defined is consistent with the conventional notion of *well-posed*. If the interconnection is well-defined, we define a map $f_L(P, K)$, which is called the (lower) **linear fractional transformation** (LFT) of P and K:

$$f_L(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}.$$

We also refer to this as the **closed-loop map**. For convenience, we also define

$$f(P, K) = P_{12}K(I - P_{22}K)^{-1}P_{21},$$

hence $f_L(P, K) = P_{11} + f(P, K)$. We define the set M of controllers K such that $f_L(P, K)$ is well-defined by

$$M = \left\{ K \in \mathbb{R}(s)^{n_u \times n_y} \mid I - P_{22}K \text{ is invertible} \right\}.$$

Note that if $P_{22} \in \mathbb{R}(s)_{sp}$ then $\mathbb{R}(s)_p \subset M$, *i.e.*, for a strictly proper plant any proper controller leads to a well-defined feedback loop. For any set $T \subset M$, we will denote by $f_L(P, T)$ the set of all the closed-loop maps which are achievable by the linear fractional transformation of P and $K \in T$. Finally, we define the upper linear fractional transformation of P and K by

$$f_U(P, K) = P_{22} + P_{21}K(I - P_{11}K)^{-1}P_{12}.$$

Problem Formulation. In this section we formulate a version of the decentralized control problem. Suppose that $S_p \subset \mathbb{R}(s)_p^{n_u \times n_y}$ is a subspace consisting of controllers satisfying a desired decentralization constraint. The subspace S_p is called the **information constraint**. Then, given a plant transfer function matrix $P \in \mathbb{R}(s)_p^{(n_z+n_y) \times (n_w+n_u)}$, where $P_{22} \in \mathbb{R}(s)_{sp}^{n_y \times n_u}$, we would like to solve the following optimization problem:

$$\begin{aligned} \text{minimize} \quad & \|f_L(P, K)\| \\ \text{subject to} \quad & K \in S_p \end{aligned} \tag{1}$$

In the problem (1), $\|\cdot\|$ is any norm on $\mathbb{R}(s)_p^{n_z \times n_w}$, which captures the performance measure of the closed-loop system. Many decentralized control problems can be reformulated as this form. If a feasible solution $K^* \in S_p$ minimizes $\|f_L(P, K)\|$, then we say that K^* is an optimal controller. Also, note that $S_p \subset M$ because $P_{22} \in \mathbb{R}(s)_{sp}$ and $S_p \subset \mathbb{R}(s)_p$.

If we do not have the information constraint subspace S_p in (1), it is well known that the problem may be solved easily by exploiting a simple change of variables, which is known as the *Youla Parametrization*. However, in the presence of the information constraint S_p, the problem is difficult to solve in general cases. One reason why this problem is difficult to solve is because $\|f_L(P, K)\|$, the objective function of the problem (1), is not a convex function of K.

In this paper, we do not consider the constraint that the controller K must stabilize the closed-loop system. For many practical problems of interest this constraint is a weaker one than the constraint that the closed-loop norm be finite, and therfore in such cases the optimal controllers found by our approach will automatically be stabilizing.

3 Elimination Theory

In this section we review some basic concepts from algebraic geometry and the elimination theory. The material in this section is from [3]; see that reference for further details and proofs.

Definition 1. *Given* $I = \langle f_1, \ldots, f_s \rangle \subset \mathbb{F}[x_1, \ldots, x_n]$, *we define the l-th **elimination ideal** I_l to be the ideal of $\mathbb{F}[x_{l+1}, \ldots, x_n]$ defined by*

$$I_l = I \cap \mathbb{F}[x_{l+1}, \ldots, x_n].$$

With a proper monomial order, Groebner bases solve the elimination problem. This is presented in the following theorem called **The Elimination Theorem**.

Theorem 2. *Let $I \subset \mathbb{F}[x_1, \ldots, x_n]$ be an ideal and let G be a Groebner basis of I with respect to lex order where $x_1 > x_2 > \cdots > x_n$. Then, for every $0 \le l \le n$, the set*

$$G_l = G \cap \mathbb{F}[x_{l+1}, \ldots, x_n]$$

is a Groebner basis of the l-th elimination ideal I_l.

Elimination corresponds to projecting a variety onto a lower dimensional subspace. Suppose that we are given $V = \mathbf{V}(f_1, \ldots, f_s) \subset \mathbb{F}^n$. To eliminate the first l variables $x_1, \ldots, x_l$, we will consider the *projection map* $\pi_l : \mathbb{F}^n \to \mathbb{F}^{n-l}$, which sends $(x_1, \ldots, x_n)$ to $(x_{l+1}, \ldots, x_n)$. If we apply π_l to $V \subset \mathbb{F}^n$, then we get $\pi_l(V) \subset \mathbb{F}^{n-l}$. The next theorem, which is known as **The Closure Theorem**, deals with the relation between $\pi_l(V)$ and $\mathbf{V}(I_l)$.

Theorem 3. *Let $\mathbb{F}$ be an algebraically closed field. Let $V = \mathbf{V}(f_1, \ldots, f_s) \subset \mathbb{F}^n$ and I_l be the l-th elimination ideal of $\langle f_1, \ldots, f_s \rangle$. Then, $\mathbf{V}(I_l)$ is the smallest variety containing $\pi_l(V) \subset \mathbb{F}^{n-l}$.*

Finally, we state an important theorem, which will be directly applied to the problem of the decentralized control in the next section. Suppose that we are given a system of rational function equations as follows:

$$x_i = \frac{q_i(t_1, \ldots, t_m)}{d_i(t_1, \ldots, t_m)} \qquad \text{for } i = 1, \ldots, n \qquad (2)$$

where $q_i, d_i \in \mathbb{F}[t_1, \ldots, t_m]$ for all $i = 1, \ldots, n$. This system of equation is called *rational parametrization*. If we let $W = \mathbf{V}(d_1 d_2 \cdots d_n) \subset \mathbb{F}^m$, then it is clear that

$$f(t_1, \ldots, t_m) = \left(\frac{q_1(t_1, \ldots, t_m)}{d_1(t_1, \ldots, t_m)}, \ldots, \frac{q_n(t_1, \ldots, t_m)}{d_n(t_1, \ldots, t_m)} \right)$$

defines a map $f : \mathbb{F}^m - W \to \mathbb{F}^n$.

Theorem 4. *If $\mathbb{F}$ is an infinite field, let $f : \mathbb{F}^m - W \to \mathbb{F}^n$ be the function determined by the rational parametrization (2). Let J be the ideal $J = \langle x_1 d_1 - q_1, \ldots, x_n d_n - q_n, 1 - yd \rangle \subset \mathbb{F}[y, t_1, \ldots, t_m, x_1, \ldots, x_n]$, where $d = d_1 d_2 \cdots d_n$, and let $J_{m+1} = J \cap \mathbb{F}[x_1, \ldots, x_n]$ be the $(m+1)$-th elimination ideal. Then $\mathbf{V}(J_{m+1})$ is the smallest variety in $\mathbb{F}^n$ containing $f(\mathbb{F}^m - W)$.*

4 Characterizing Closed-Loop Maps

For the remainder of this paper we focus on the case when S_p is defined by a sparsity constraint as follows. We define S_p by

$$S_p = \left\{ K \in \mathbb{R}(s)_p^{n_u \times n_y} \;\middle|\; K = \sum_{i=1}^{m} t_i E^i, \; t_i \in \mathbb{R}(s)_p \right.$$
$$\left. \text{for all } i = 1, \ldots, m \right\}, \quad (3)$$

where we let $Z = \{E^1, \ldots, E^m\}$ be a linearly independent set of matrices $E^i \in \{0,1\}^{n_u \times n_y}$ and let each E^i be a binary matrix with exactly one non-zero element. Therefore, we represent the information constraint S_p as the set of all finite linear combinations of basis matrices E^i with coefficients in $\mathbb{R}(s)_p$.

We also define the associated set of transfer function matrices which are not necessarily proper as

$$S = \left\{ K \in \mathbb{R}(s)^{n_u \times n_y} \;\middle|\; K = \sum_{i=1}^{m} t_i E^i, \; t_i \in \mathbb{R}(s) \right.$$
$$\left. \text{for all } i = 1, \ldots, m \right\}. \quad (4)$$

S can be interpreted as the set of real-rational transfer function matrices with the same sparsity pattern as S_p with the properness constraint relaxed. Therefore, $S_p = S \cap \mathbb{R}(s)_p$.

An important point to note is that we consider the set of real-rational functions in s, *i.e.*, $\mathbb{R}(s)$ as a *field* $\mathbb{F}$. Note that Theorem 4 holds for any infinite field $\mathbb{F}$. Certainly, $\mathbb{R}(s)$ is an infinite field, hence we can apply Theorem 4.

4.1 The Controller Elimination Algorithm

Consider the linear fractional transformation

$$f_L(P, K) = P_{11} + P_{12} K (I - P_{22} K)^{-1} P_{21}$$
$$= P_{11} + f(P, K).$$

From linear algebra, we know that if $I - P_{22}K$ is invertible,

$$f(P, K) = \frac{P_{12} K \operatorname{adj}(I - P_{22} K) P_{21}}{\det(I - P_{22} K)}.$$

Note that, from (4),

$$\det(I - P_{22} K) = \det\left(I - \sum_{i=1}^{m} t_i P_{22} E^i \right).$$

Therefore, $\det(I - P_{22}K) \in \mathbb{R}(s)[t_1, \ldots, t_m]$ for any $K \in S$. In other words, $\det(I - P_{22}K)$ is a polynomial in variables $t_1, \ldots, t_m$ with coefficients in $\mathbb{R}(s)$. Similarly, $\operatorname{adj}(I - P_{22}K)$ is a matrix whose elements are also polynomials in variables $t_1, \ldots, t_m$ with coefficients in $\mathbb{R}(s)$. These facts follow from linear algebra.

Now, we define $d(t_1, \ldots, t_m) = \det(I - P_{22}K)$ and $Q(t_1, \ldots, t_m) = P_{12}K \operatorname{adj}(I - P_{22}K)P_{21}$. Then, we have

$$f(P, K) = \frac{Q(t_1, \ldots, t_m)}{d(t_1, \ldots, t_m)}.$$

Let $q_{11}, \ldots, q_{n_z n_w} \in \mathbb{R}(s)[t_1, \ldots, t_m]$ be the entries of $Q = P_{12}K \operatorname{adj}(I - P_{22}K)P_{21}$. Then $X \in f(P, S \cap M)$ if and only if there exists $t_1, \ldots, t_m \in \mathbb{R}(s)$ such that the following equations hold:

$$x_{11} = \frac{q_{11}(t_1, \ldots, t_m)}{d(t_1, \ldots, t_m)}, \ldots, x_{n_z n_w} = \frac{q_{n_z n_w}(t_1, \ldots, t_m)}{d(t_1, \ldots, t_m)},$$

where x_{ij} are the entries of X.

We now proceed to eliminate $t_1, \ldots, t_m$ using Theorem 4, to give the main result of this paper. Theorem 5 below is an easy application of Theorem 4 and the proof is omitted due to space constraints.

Theorem 5. *Suppose $P \in \mathbb{R}(s)^{(n_z+n_y)\times(n_w+n_u)}$ and $S \subset \mathbb{R}(s)^{n_u \times n_y}$ is as defined in (4). Define $d, q_{11}, \ldots, q_{n_z n_w}$ as above, and let J be the ideal*

$$J = \langle x_{11}d - q_{11}, \ldots, x_{n_z n_w}d - q_{n_z n_w}, 1 - yd \rangle$$

so that $J \subset \mathbb{R}(s)[y, t_1, \ldots, t_m, x_{11}, \ldots, x_{n_z n_w}]$. Let $J_{m+1} = J \cap \mathbb{R}(s)[x_{11}, \ldots, x_{n_z n_w}]$ be the $(m+1)$-th elimination ideal of J. Then $\mathbf{V}(J_{m+1})$ is the smallest variety in $\mathbb{R}(s)^{n_z n_w}$ containing $f(P, S \cap M)$.

Controller elimination algorithm. From Theorem 5, we have the following algorithm to find the smallest variety containing $f(P, S \cap M)$. Starting with $P \in \mathbb{R}(s)$ and sparsity basis $Z = \{E^1, \ldots, E^m\}$,

(i) Find the polynomials d and $q_{11}, \ldots, q_{n_z n_w}$ in variables $t_1, \ldots, t_m$ with coefficients in $\mathbb{R}(s)$.

(ii) Let the ideal J be generated by the polynomials $x_{11}d - q_{11}, \ldots, x_{n_z n_w}d - q_{n_z n_w}, 1 - yd$, each in $\mathbb{R}(s)[y, t_1, \ldots, t_m, x_{11}, \ldots, x_{n_z n_w}]$. Using these polynomials to represent the ideal J, find a Groebner basis G of J with respect to the lexicographic ordering where $y > t_1 > \ldots > t_m > x_{11} > \ldots > x_{n_z n_w}$.

(iii) Select those polynomials from the Groebner basis G which generate the $(m+1)$-th elimination ideal $J_{m+1} = J \cap \mathbb{R}(s)[x_{11}, \ldots, x_{n_z n_w}]$.

(iv) $\mathbf{V}(J_{m+1}) = \mathbf{V}(G_{m+1})$ is the smallest variety containing $f(P, S \cap M)$.

We have succeeded in parametrizing the set $f(P, S \cap M)$ with variables $x_{11}, \ldots, x_{n_z n_w}$. In other words, the Groebner basis G_{m+1} of the $(m+1)$-th elimination ideal J_{m+1} contains the polynomial equations that any closed-loop map should satisfy.

If $\mathbf{V}(J_{m+1})$ is a convex set, then we may be able to find an optimal closed-loop map $f_L(P, K^*)$ of the problem (1). Computing an optimal closed-loop map and optimal controller will be discussed in the following sections.

4.2 Finding an Optimal Closed-Loop Map

Even though we succeeded in parametrizing the set of closed-loop maps with variables $x_{11}, \ldots, x_{n_z n_w}$, the smallest variety found by the controller elimination algorithm may fail to be a convex set. In this case, our introduced method fails to change the problem (1) to a tractable convex programming problem, even though the algorithm could characterize the set of closed-loop maps.

Now, suppose that the smallest variety $\mathbf{V}(J_{m+1})$ turns out to be convex, in particular, a linear subspace (or affine set). In other words, the Groebner basis G_{m+1} of the elimination ideal J_{m+1} happens to be a set of affine polynomials in variables $x_{11}, \ldots, x_{n_z n_w}$. In this case, let l denote the number of polynomials of G_{m+1}, i.e., $l = |G_{m+1}|$. Then the Groebner basis of the elimination ideal can be concisely written as $A^i \cdot X + b^i$ for $i = 1, \ldots, l$, where $A \cdot B = \operatorname{trace}(A^T B)$. Therefore, we know that any $X \in \mathbf{V}(J_{m+1})$ should satisfy

$$A^i \cdot X + b^i = 0 \quad \text{for all } i = 1, \ldots, l. \tag{5}$$

Note that $0 \in \mathbf{V}(J_{m+1})$ because $f(P, 0) = 0$. Therefore, $X = 0$ is a solution of (5), hence $b^i = 0$ for all $i = 1, \ldots, l$. We summarize in the following corollary.

Corollary 6. *Suppose $P \in \mathbb{R}(s)^{(n_z+n_y)\times(n_w+n_u)}$ and $S \subset \mathbb{R}(s)^{n_u \times n_y}$ is as defined in (4). Suppose that the variety $\mathbf{V}(J_{m+1})$ is affine, hence we can define $A^1, \ldots, A^l$ as above. If $X \in \mathbb{R}(s)^{n_z \times n_w}$ there exists $K \in S \cap M$ such that $X = P_{12}K(I - P_{22}K)^{-1}P_{21}$ only if $A^i \cdot X = 0$ for all $i = 1, \ldots, l$.*

Now, we can formulate the problem of finding an optimal closed-loop map as follows. For any $P \in \mathbb{R}(s)^{(n_z+n_y)\times(n_w+n_u)}$ we would like to solve

$$
\begin{aligned}
\text{minimize} \quad & \|X + P_{11}\| \\
\text{subject to} \quad & A^i \cdot X = 0 \quad \text{for all } i = 1, \ldots, l \\
& X \in \mathbb{R}(s)^{n_u \times n_y}
\end{aligned}
\tag{6}
$$

4.3 Proper Controllers

In the previous section, we reformulated the original problem (1) as a convex problem (6). However, one caveat should be mentioned. The elimination theorem assumes that the variables $t_1, \ldots, t_m$ can have all the values in the field $\mathbb{R}(s)$, i.e., $t_1, \ldots, t_m$ can be non-proper real-rational functions. Therefore, the smallest variety $\mathbf{V}(J_{m+1})$ from the elimination algorithm includes the set $f(P, S \cap M)$, which is a much bigger set than $f(P, S_p)$. In other words,

$$f(P, S_p) \subset f(P, S \cap M) \subset \mathbf{V}(J_{m+1}),$$

and the inclusions are strict in general. Recall that we want to characterize the set $f(P, S_p)$.

However, note that if $P \in \mathbb{R}(s)_p, P_{22} \in \mathbb{R}(s)_{sp}$, and $S_p \subset \mathbb{R}(s)_p$, then $f(P, S_p) \subset \mathbb{R}(s)_p$. This can be easily proved (See [9], for example). Recall that we have $P \in \mathbb{R}(s)_p, P_{22} \in \mathbb{R}(s)_{sp}$, and $S_p \subset \mathbb{R}(s)_p$ in the original problem (1). Therefore, when we try to find an optimal closed-loop map $f_L(P, K^*)$ with $K^* \in S_p$, we can restrict $\mathbf{V}(J_{m+1})$ to $\mathbf{V}(J_{m+1}) \cap \mathbb{R}(s)_p$ without losing any closed-loop map $f_L(P, K)$ with $K \in S_p$. In other words, we have

$$f(P, S_p) \subset \mathbf{V}(J_{m+1}) \cap \mathbb{R}(s)_p.$$

Therefore, Corollary 6 can be restated for the case of proper controllers as the following:

Corollary 7. Suppose $P \in \mathbb{R}(s)_p^{(n_z+n_y) \times (n_w+n_u)}$, $P_{22} \in \mathbb{R}(s)_{sp}^{n_y \times n_u}$, and $S_p \subset \mathbb{R}(s)_p^{n_u \times n_y}$ is as defined in (3). Suppose that the variety $\mathbf{V}(J_{m+1})$ is affine and define $A^1, \ldots, A^l$ as above. If $X \in \mathbb{R}(s)_p^{n_z \times n_w}$ there exists $K \in S_p$ such that $X = P_{12}K(I - P_{22}K)^{-1}P_{21}$ only if $A^i \cdot X = 0$ for all $i = 1, \ldots, l$.

Recall we would like to solve the optimization problem

$$\text{minimize} \quad \|P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}\|$$
$$\text{subject to} \quad K \in S_p$$

When $\mathbf{V}(J_{m+1})$ is affine, we solve instead the following optimization problem:

$$\text{minimize} \quad \|X + P_{11}\|$$
$$\text{subject to} \quad A^i \cdot X = 0 \quad \text{for all } i = 1, \ldots, l \quad (7)$$
$$X \in \mathbb{R}(s)_p^{n_u \times n_y}$$

Note that the above problem (7) is a convex optimization problem. Therefore, we may be able to solve this problem efficiently [2]. If X^* is an optimal solution of (7), then $X^* + P_{11}$ is a strong candidate for an optimal closed-loop map $f_L(P, K^*)$.

4.4 Finding an Optimal Controller

In the previous section, we reformulated the decentralized control problem (1) as a convex optimization problem (7) if $\mathbf{V}(J_{m+1})$ turned out to be a linear subspace. However, there can be still some gap between the two problems, (1) and (7). In other words, even though X^* is an optimal solution of (7), it may be the case that $X^* \notin f(P, S_p)$. This is because we only have $f(P, S_p) \subset \mathbf{V}(J_{m+1}) \cap \mathbb{R}(s)_p$ and the inclusion is not guaranteed to be equality. If the inclusion becomes equality, then the two problems are equivalent and we can solve (1).

Unfortunately, we can not guarantee that we always have the equality $f(P, S_p) = \mathbf{V}(J_{m+1}) \cap \mathbb{R}(s)_p$ even though the smallest variety $\mathbf{V}(J_{m+1})$ becomes a linear subspace. However, we believe that $f(P, S_p) = \mathbf{V}(J_{m+1}) \cap \mathbb{R}(s)_p$ in most cases of (1). In particular,

we discuss the case of quadratically invariant problems in the next section and show that quadratic invariance with an additional technical condition gives this equality.

Note that, even though $f(P, S_p) \subsetneq \mathbf{V}(J_{m+1}) \cap \mathbb{R}(s)_p$, given an optimal solution X^* of (7), if we find a controller $K^* \in S_p$ such that $X^* = f(P, K^*)$, then (1) is completely solved and K^* is an optimal controller. In general, we do not know how to find a K given $f(P, K)$: however, we have a partial result, which follows from Lemma 10.4 of [9]. The proof is omitted.

Theorem 8. Let $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$ and K be rational transfer function matrices. If P and $f_L(P, K)$ are proper, $\det P(\infty) \neq 0$, $\det \left(P - \begin{bmatrix} f_L(P, K) & 0 \\ 0 & 0 \end{bmatrix} \right)(\infty) \neq 0$, and P_{12} and P_{21} are square and invertible for almost all s, then K is proper and $K = f_U(P^{-1}, f_L(P, K))$.

4.5 Quadratic Invariance

In this section, we consider the quadratically invariant problems. We prove that under an additional technical assumption, quadratically invariant problems can be solved by the controller elimination algorithm. First, we introduce the following definitions and results from [6].

Definition 9. Suppose $S_p \subset \mathbb{R}(s)_p^{n_u \times n_y}$ and $P_{22} \in \mathbb{R}(s)_{sp}^{n_y \times n_u}$. The set S_p is called **quadratically invariant** under P_{22} if

$$KP_{22}K \in S_p \quad \text{for all } K \in S_p.$$

Definition 10. Given $P_{22} \in \mathbb{R}(s)^{n_y \times n_u}$, we define a map $h : M \to \mathbb{R}(s)^{n_u \times n_y}$ by

$$h(K) = -K(I - P_{22}K)^{-1} \quad \text{for all } K \in M.$$

Theorem 11. Suppose $P_{22} \in \mathbb{R}(s)_{sp}^{n_y \times n_u}$ and $S_p \subset \mathbb{R}(s)_p^{n_u \times n_y}$ is a sparsity constraint. Then

S_p is quadratically invariant under $P_{22} \iff h(S_p) = S_p$.

Now, we introduce a technical condition for our results.

Definition 12. Suppose that $S_p \subset \mathbb{R}(s)_p^{n_u \times n_y}$ is a sparsity constraint, $P \in \mathbb{R}(s)_p^{(n_z+n_y) \times (n_w+n_u)}$, and $P_{22} \in \mathbb{R}(s)_{sp}^{n_y \times n_u}$. We say that S_p is **proper preserved** under P if

$$\left\{ \sum_{i=1}^m t_i(P_{12}E^iP_{21}) \;\middle|\; t_i \in \mathbb{R}(s) \right\} \cap \mathbb{R}(s)_p =$$

$$\left\{ \sum_{i=1}^m t_i(P_{12}E^iP_{21}) \;\middle|\; t_i \in \mathbb{R}(s)_p \right\}.$$

Note that the condition of being proper preserved may be easily checked by inspecting the elements of $P_{12}E^iP_{21}$.

Finally, we state a theorem which shows that the controller elimination algorithm can solve problems that are both quadratically invariant and proper preserved. The proof is omitted due to space constraints.

Theorem 13. *Suppose that $P \in \mathbb{R}(s)_p^{(n_z+n_y)\times(n_w+n_u)}$, $P_{22} \in \mathbb{R}(s)_{sp}^{n_y \times n_u}$ and the information constraint $S_p \subset \mathbb{R}(s)_p^{n_u \times n_y}$ is a sparsity constraint. Suppose also that S_p is quadratically invariant under P_{22} and is proper preserved under P. Then*

$$f(P, S_p) = \mathbf{V}(J_{m+1}) \cap \mathbb{R}(s)_p.$$

5 Example

We consider an example, which is *not* quadratically invariant, hence we have no systematic method to find an optimal controller except the controller elimination algorithm.

Suppose that we are given a plant transfer function matrix P as follows:

$$P_{11} = \begin{bmatrix} \frac{1}{s+4} & 0 \\ 0 & \frac{1}{s+5} \end{bmatrix}, \qquad P_{12} = \begin{bmatrix} 1 & \frac{1}{s+6} \\ 0 & 1 \end{bmatrix},$$

$$P_{21} = \begin{bmatrix} 1 & \frac{1}{s+7} \\ \frac{1}{s+8} & 0 \\ 0 & 1 \end{bmatrix}, \qquad P_{22} = \begin{bmatrix} \frac{1}{s+1} & 0 \\ \frac{1}{s+2} & 0 \\ 0 & \frac{1}{s+3} \end{bmatrix}.$$

Suppose also that the information constraint S_p is given as $S_p = \left\{ K \in \mathbb{R}(s)_p^{2\times3} \mid K_{12} = K_{13} = K_{21} = 0 \right\}$. Therefore, any $K \in S_p$ can be represented as a linear combination of basis matrices:

$$K = t_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + t_3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $t_1, t_2, t_3 \in \mathbb{R}(s)_p$.

Executing the controller elimination algorithm gives us the elimination ideal

$$J_4 = \left\langle \left(-\frac{1}{s+7}\right)x_{11} + \left(\frac{1}{(s+6)(s+7)}\right)x_{21} + \right.$$
$$\left. x_{12} + \left(-\frac{s^2+11s+24}{(s+3)(s+6)(s+8)}\right)x_{22} \right\rangle.$$

Therefore, we have

$$f_L(P, S_p) \subset \left\{ \begin{bmatrix} \frac{1}{s+4} & 0 \\ 0 & \frac{1}{s+5} \end{bmatrix} + \begin{bmatrix} \frac{1}{s+6} & 0 \\ 1 & 0 \end{bmatrix} a + \right.$$
$$\begin{bmatrix} 1 & 0 \\ 0 & -\frac{(s+3)(s+6)(s+8)}{(s+7)(s^2+11s+24)} \end{bmatrix} b +$$
$$\left. \begin{bmatrix} 0 & \frac{s^2+11s+24}{(s+3)(s+6)(s+8)} \\ 0 & 1 \end{bmatrix} c \ \middle| \ a,b,c \in \mathbb{R}(s)_p \right\}.$$

We can find an optimal closed-loop map $f_L(P, K^*)$ in the right-hand side set, which minimizes $\mathcal{H}_2$ norm. It turns out that the optimal value is 0.0779 and we can find the optimal controller K^* from $f(P, K^*)$. Details are omitted due to space constraints.

6 Summary and Conclusion

We presented the controller elimination algorithm, based on algebraic geometry, which can solve a broad class of decentralized control problems. The main idea of the algorithm is that we can eliminate the controller variables to find the smallest variety containing all possible closed-loop maps. We also showed that the algorithm can solve quadratically invariant problems, subject to a certain technical condition. We presented an example, whose system is not quadratically invariant, illustrating the proposed algorithm.

The approach of this paper has been to characterize which decentralized control problems have simple sets of achievable closed-loop maps, and in particular when this set is affine. As an abstract question, this is inherently a problem of elimination for rational functions, and hence Groebner bases are a natural tool, and these provide very strong conditions in the field of algebraic geometry. This paper transfers that approach to the specific linear fractional rational functions which are of importance in control. While for general rational functions Groebner bases are a cornerstone of any elimination approach, it may be possible to develop stronger elimination methods for specific control problems, and that is a question for future research.

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