An Explicit State-Space Solution for a Decentralized Two-Player Optimal Linear-Quadratic Regulator

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Abstract

We develop controller synthesis algorithms for decentralized control problems, where individual subsystems are connected over a network. We focus on the simplest information structure, consisting of two interconnected linear systems, and construct the optimal controller subject to a decentralization constraint via a spectral factorization approach. We provide explicit state-space formulae for the optimal controller, characterize its order, and show that its states are those of a particular optimal estimator.

1 Introduction

We are interested in controller synthesis algorithms for distributed control problems, where individual subsystems are connected over a network. Examples of such problems include formation flight, networked control, teams of vehicles, and many other applications involving multiple agents interacting to achieve a global objective.

Many recent papers address such controller synthesis problems. While some problem formulations are currently intractable [2], or have nonlinear optimal controllers [16], others have optimal linear controllers [8] which may be computed via convex optimization [10].

In this paper, we focus on a very specific information structure, consisting of two interconnected systems with dynamics such that player 1's state affects player 2's state. Our objective is to find a pair of controllers such that player 1 has access only to the first state, whereas player 2 can measure both states. The controller should be chosen to minimize an expected quadratic cost.

This problem is known to have a linear optimal controller which may be found via convex optimization [15, 3, 10, 8]. Unfortunately, most existing convex formulations of this problem are infinite-dimensional. Specifically, the approach in [8] requires a change of variables via the Youla parameterization, and optimization over this parameter. The parameter itself is a linear stable

system, and a standard parameterization would be via a basis for the impulse response function. This is in contrast to the centralized case, for which there are explicit state-space formulae.

In this paper, we provide explicit formulae for the optimal controllers for this system. We show that both controllers separate naturally into a composition of controller and estimator, and each has the same number of states as player 2. Such formulae offer the practical advantages of computational reliability and simplicity, as well as providing understanding and interpretation of the controller structure. Also, it establishes the order of the optimal controller for this system, an open problem for general decentralized systems. Our approach makes use of spectral factorization, and for simplicity we focus on the finite-horizon time-varying case. We anticipate that the spectral factorization methods used here extend naturally to more general networks, and the results in this paper are a first step towards general state-space solutions.

Previous Work. Since the general decentralized problem is currently intractable, most work has been centered around classifying systems that can be reformulated as convex problems [3, 5, 6, 1]. These results were unified and generalized under the concept of quadratic invariance [9]. For systems represented by graph structures and sparsity constraints, necessary and sufficient conditions for quadratic invariance of such systems was provided in [13]. Similar results were achieved in [12] from a poset-based framework.

Many different approaches have been taken to try and find numerical solutions to these problems. Some methods were suggested, though not implemented, in [15]. For the problem considered here, [11] provides a solution based on semidefinite programming. Other SDP based approaches have been provided in [7, 17]. For the quadratic case, vectorization [8] provides a finite-dimensional approach, but this loses the intrinsic structure and results in high-order controllers.

However, in none of these approaches have explicit state-space formulae been derived. In this paper, we take a spectral factorization approach, similar to [14], to derive explicit state-space formulae for the two-player problem. As a result, we can efficiently and analytically compute the optimal controllers for this distributed problem.

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Moreover, we gain insight into the form of the solution which previous approaches do not provide.

2   Problem Formulation

We consider two interconnected systems with overall dynamics

$$
\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} A_{11}(t) & 0 \\ A_{21}(t) & A_{22}(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
$$
$$
+ \begin{bmatrix} B_{11}(t) & 0 \\ B_{21}(t) & B_{22}(t) \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}
$$
$$
+ \begin{bmatrix} H_1(t+1) & 0 \\ 0 & H_2(t+1) \end{bmatrix} \begin{bmatrix} w_1(t+1) \\ w_2(t+1) \end{bmatrix} \quad (1)
$$

for $t = 0, \ldots, N-1$ with initial condition $x(0) = H(0)w(0)$ and output

$$
z(t) = \begin{bmatrix} C_1(t) & C_2(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} D_1(t) & D_2(t) \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \quad (2)
$$

Here $x_i(t)$ is the state of system i at time t, $u_i(t)$ is the control action taken by system i, and $w_i(t)$ is external noise, which is assumed IID Gaussian with unit covariance. We let $A(t), B(t), C(t), D(t), H(t)$ be the matrices defined in (1) and (2), and assume for all t

$$
C^T(t)D(t) = 0 \qquad D^T(t)D(t) > 0
$$

Our objective is to find controllers, where $u_1(t)$ is a function of $x_1(0), \ldots, x_1(t)$, and $u_2(t)$ is a function of $x(0), \ldots, x(t)$. That is, player 1 has access only to the first state, whereas player 2 can measure both states. This information structure is common knowledge for both players. The cost function of interest is

$$
\mathrm{E} \sum_{t=0}^{N} \|z(t)\|^2
$$

which is equal to the usual LQR cost, when $C^T C$ is denoted by Q and $D^T D$ is denoted by R. Note that this framework allows us to couple the states x_1 and x_2 in the cost, since $C^T C$ need not be block diagonal.

We use the following notation throughout this paper. For a sequence of vectors $x(0), x(1), \ldots, x(N)$ and matrices $A(0), \ldots, A(N)$ we denote

$$
x = \begin{bmatrix} x(0) \\ \vdots \\ x(N) \end{bmatrix} \qquad A = \begin{bmatrix} A(0) & & \\ & \ddots & \\ & & A(N) \end{bmatrix} \quad (3)
$$

Define the shift matrix

$$
Z = \begin{bmatrix} 0 & & & \\ I & 0 & & \\ & \ddots & \ddots & \\ & & I & 0 \end{bmatrix} \quad (4)
$$

The dimensions of Z will be defined by the context. If X is a block diagonal matrix, we use X^+ to denote the block diagonal matrix $X^+ = Z^T X Z$.

The system dynamics (1) are then equivalent to

$$
\begin{aligned} x &= ZAx + ZBu + Hw \\ z &= Cx + Du \end{aligned} \quad (5)
$$

Using this notation, we define $\widetilde{A}$ to be

$$
\widetilde{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11}(0) & & A_{12}(0) & \\ & \ddots & & \ddots \\ A_{21}(0) & & A_{22}(0) & \\ & \ddots & & \ddots \end{bmatrix}
$$

Note that A and $\widetilde{A}$ are related by permutation. We define this permutation matrix to be $\mathbb{P}$, where its dimensions are implied by the context, so that $A = \mathbb{P}\widetilde{A}\mathbb{P}^T$. In many cases, we will define a matrix by its tilde notation.

We'll define for convenience $\mathcal{T} = (I - ZA)^{-1}$. We use the term **_lower_** to mean lower triangular, and we define lower$(\cdot)$ to be the projection of any matrix M to its lower triangular component, so that $(\mathrm{lower}(M))_{ij} = M_{ij}$ if $i \geq j$, and zero otherwise. From (5), we have

$$
\begin{aligned} z &= \mathcal{P}_{11}w + \mathcal{P}_{12}u \\ x &= \mathcal{P}_{21}w + \mathcal{P}_{22}u \end{aligned}
$$

where

$$
\begin{bmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{21} & \mathcal{P}_{22} \end{bmatrix} = \begin{bmatrix} C(I - ZA)^{-1}H & C(I - ZA)^{-1}ZB + D \\ (I - ZA)^{-1}H & (I - ZA)^{-1}ZB \end{bmatrix}
$$

The sparsity structure of $\mathcal{P}_{22}$ is that it is block lower triangular, and each block is itself a 2×2 block lower triangular matrix. Let S be the set of such matrices. Note that a matrix $T \in S$ if and only if

$$
\widetilde{T} = \begin{bmatrix} \tilde{T}_{11} & 0 \\ \tilde{T}_{21} & \tilde{T}_{22} \end{bmatrix}
$$

and each $\tilde{T}_{ij}$ is lower triangular.

In this problem, we are interested in an information structure whereby player 1 determines the input u_1 based on causal measurements of x_1, and player 2 determines u_2 based causally on both x_1 and x_2. The set of linear maps with this information structure is also S.

Suppose the controller is $\mathcal{K} \in S$ so that $u = \mathcal{K}x$, and the closed-loop map from w to z is given by

$$
z = \big(\mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}\mathcal{P}_{21} \big)w
$$

Then, our optimization problem is

$$
\begin{aligned} & \text{minimize} && \|\mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}\mathcal{P}_{21}\|_F^2 \\ & \text{subject to} && \mathcal{K} \in S \end{aligned} \quad (6)
$$

3 Main Results.

Having established our notation and problem formulation, we now present the optimal solution for (6). We will develop the proof for this result in the following two sections.

Theorem 1. *Let X be the unique block diagonal solution to the Riccati equation*

$$X = C_2^T C_2 + A_{22}^T X^+ A_{22} \\ - A_{22}^T X^+ B_{22}(D_2^T D_2 + B_{22}^T X^+ B_{22})^{-1} B_{22}^T X^+ A_{22} \quad (7)$$

and J be the associated matrix

$$J = (D_2^T D_2 + B_{22}^T X^+ B_{22})^{-1} B_{22}^T X^+ A_{22} \quad (8)$$

Also, let Y be the unique block diagonal solution to the Riccati equation

$$Y = C^T C + A^T Y^+ A \\ - A^T Y^+ B(D^T D + B^T Y^+ B)^{-1} B^T Y^+ A \quad (9)$$

with K the associated matrix

$$K = (D^T D + B^T Y^+ B)^{-1} B^T Y^+ A \quad (10)$$

Let

$$A^K(t) = A_{22}(t) - B_{21}(t)K_{12}(t) - B_{22}(t)K_{22}(t)$$
$$B^K(t) = A_{21}(t) - B_{21}(t)K_{11}(t) - B_{22}(t)K_{21}(t)$$

The optimal controllers are:

- *Controller 1 has realization*

$$q_1(t+1) = A^K(t)q_1(t) + B^K(t)x_1(t)$$
$$u_1(t) = -K_{12}(t)q_1(t) - K_{11}(t)x_1(t)$$

- *Controller 2 has realization*

$$q_2(t+1) = A^K(t)q_2(t) + B^K(t)x_1(t)$$
$$u_2(t) = \big(J(t) - K_{22}(t)\big)q_2(t) \\ - K_{21}x_1(t) - J(t)x_2(t)$$

Note that the Riccati equations (7) and (9) are simply Riccati recursions written in block diagonal notation; hence, unique solutions are guaranteed. Having established the form of the optimal controller, a number of remarks are in order.

With the inclusion of q_1 and q_2, the optimal controller is not a static gain, despite the fact that we have state feedback in each subsystem and player 2 has complete state information. Contrast this result with the classical LQR controller in which the optimal centralized controller would be the static gain K. In fact, both controllers have dynamics, and each has the same number of states as system 2.

It will be shown in Section 6 that q_1 and q_2 in the optimal controllers are in fact the minimum-mean square error estimate of x_2 given the history of x_1. Letting $\eta(t) = \mathrm{E}\big(x_2(t) \mid x_1(t), \ldots, x_1(0)\big)$ the optimal control policy can be written as

$$u_1(t) = -K_{11}(t)x_1(t) - K_{12}(t)\eta(t) \quad (11)$$
$$u_2(t) = -K_{21}(t)x_1(t) - K_{22}(t)\eta(t) + J(t)\big(\eta(t) - x_2(t)\big)$$

Thus, the optimal policy is, in fact, attempting to perform the optimal centralized policy, though using η instead of x_2. However, there is an additional term in u_2 which represents the error between x_2 and its estimate η. We also see that in the case where x_2 is deterministic, so that $\eta = x_2$, then the optimal distributed controller reduces to the optimal centralized solution, as it should.

In contrast to the optimal solution presented above, consider the following heuristic solutions.

$\mathcal{K}_{\mathrm{proj}}$ The optimal controller is not a static gain. Thus, one might take the optimal centralized gain, K, and project it onto the set S; basically, zeroing K_{12} and setting $q_2(t) = x_2(t)$.

$\mathcal{K}_{\mathrm{est}}$ Since player 2 has complete state information, we drop the estimation occurring in player 2 and use the centralized solution; setting $q_2(t) = x_2(t)$.

While the first heuristic $\mathcal{K}_{\mathrm{proj}}$ is a fairly naive approach, it is a common misconception that $\mathcal{K}_{\mathrm{est}}$ is an optimal controller; that is, player 2 does not require estimation since it has complete knowledge of the state. In general, such a controller is strictly suboptimal.

Although we have not placed any explicit constraints on the communication channels, some immediate results of this controller follow. In particular, in many cases the dimension of u_2 is much smaller than that of x_1. As a result, passing $-K_{21}(t)x_1(t) - (K_{22}(t) - J(t))\eta(t)$ to player 2, instead of $x_1(t)$, can save bandwidth on the communication channel and is beneficial in cases where memory or computation is expensive in player 2.

4 Quadratic Invariance and Change of Variables

It is straightforward to show that the set S is *quadratically invariant* with respect to $\mathcal{P}_{22}$, (in this case because S is an algebra) so the optimal controller with this information pattern is linear [8] and may be found via convex programming [10].

Note $\mathcal{P}_{21}, \mathcal{P}_{21}^{-1} \in S$ and $I - \mathcal{P}_{22}\mathcal{K}$ is invertible for all $\mathcal{K} \in S$, so we can use the change of variables $\mathcal{Q} = \mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}\mathcal{P}_{21}$ to solve the equivalent, convex optimization problem

$$\begin{aligned} \text{minimize} \quad & \|\mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{Q}\|_F^2 \\ \text{subject to} \quad & \mathcal{Q} \in S \end{aligned} \quad (12)$$

This change of variables is bijective as required, so after finding the optimal $\mathcal{Q}$ the optimal $\mathcal{K}$ is given by

$$\mathcal{K} = (I + \mathcal{Q}\mathcal{P}_{21}^{-1}\mathcal{P}_{22})^{-1}\mathcal{Q}\mathcal{P}_{21}^{-1}$$

Define $S^\perp$ according to the usual inner product, so that $T \in S^\perp$ if and only if $\operatorname{trace} T^T B = 0$ for all $B \in S$. Note that $T \in S^\perp$ if and only if

$$\widetilde{T} = \begin{bmatrix} \tilde{T}_{11} & \tilde{T}_{12} \\ \tilde{T}_{21} & \tilde{T}_{22} \end{bmatrix}$$

and $\tilde{T}_{11}$, $\tilde{T}_{21}$, $\tilde{T}_{22}$ are all strictly upper triangular.

Lemma 2. *Suppose S is a subspace of $\mathbb{R}^{m \times n}$, $A \in \mathbb{R}^{p \times m}$ and $B \in \mathbb{R}^{p \times n}$. Then $Q \in S$ is optimal for*

$$\begin{aligned} minimize \quad & \|B + AQ\|_F^2 \\ subject\ to \quad & Q \in S \end{aligned} \tag{13}$$

if and only if $A^T B + A^T A Q \in S^\perp$.

Proof. This result follows from the KKT conditions, and the details are omitted due to space constraints. ∎

Lemma 2 implies that $\mathcal{Q} \in S$ is optimal for problem (12) if and only if

$$\mathcal{P}_{12}^T \mathcal{P}_{11} + \mathcal{P}_{12}^T \mathcal{P}_{12} \mathcal{Q} \in S^\perp \tag{14}$$

5 Solution of the Optimality Conditions

If we permute the optimality condition (14), it has the following structure

where $\Lambda \in S^\perp$. For the centralized version of this control problem, the set S is replaced by the set of lower triangular matrices, and condition (14) may then be solved via spectral factorization of $\mathcal{P}_{12}^T \mathcal{P}_{12}$. For the decentralized version, we will need to decompose the problem into two separate problems, as in the following lemma.

Lemma 3. *Suppose F is permuted and partitioned so that*

$$\widetilde{F} = \begin{bmatrix} \tilde{F}_{11} & \tilde{F}_{12} \\ \tilde{F}_{21} & \tilde{F}_{22} \end{bmatrix}$$

and similarly for G and Q. Suppose $Q \in S$. Then, $G + FQ \in S^\perp$ if and only if the following conditions both hold:

(i) $\tilde{G}_{22} + \tilde{F}_{22}\tilde{Q}_{22}$ is strictly upper

(ii) $\mathbb{P}\left(\begin{bmatrix} \tilde{G}_{11} \\ \tilde{G}_{21} \end{bmatrix} + \widetilde{F}\begin{bmatrix} \tilde{Q}_{11} \\ \tilde{Q}_{21} \end{bmatrix}\right)$ is strictly upper

Proof. This follows from the definition of S. ∎

Both conditions in Lemma 3 have the form $B + AQ$ is strictly upper, which has the following structure

where Ω is a strictly upper triangular matrix. This equation is solved via a triangular factorization of A, as in the following lemma.

Lemma 4. *Suppose A, B are square matrices and $A > 0$ with factorization $A = L^T F L$ where L is lower and invertible, and F is diagonal. Then there exists a unique lower Q such that $B + AQ$ is strictly upper, given by*

$$Q = -L^{-1}F^{-1}\operatorname{lower}(L^{-T}B)$$

Proof. We have $L^T FLQ + B$ is strictly upper if and only if $FLQ + L^{-T}B$ is strictly upper, which holds if and only if $\operatorname{lower}(FLQ + L^{-T}B) = 0$, from which the result follows. ∎

The required factorization is a standard spectral factorization result, as below.

Lemma 5. *Suppose A, B, C and D are block diagonal matrices of appropriate dimensions with $C^T D = 0$ and $D^T D > 0$. Let $\mathcal{G} = C(I - ZA)^{-1}ZB + D$, and let P be the unique block diagonal matrix satisfying*

$$\begin{aligned} P = C^T C &+ A^T P^+ A \\ &- A^T P^+ B(D^T D + B^T P^+ B)^{-1}B^T P^+ A \end{aligned} \tag{16}$$

and define the block diagonal matrices

$$F = D^T D + B^T P^+ B \qquad K = F^{-1}B^T P^+ A$$

and the lower triangular matrix

$$\mathcal{L} = I + K(I - ZA)^{-1}ZB$$

Then, $\mathcal{G}^T \mathcal{G} = \mathcal{L}^T F \mathcal{L}$ and

$$\mathcal{L}^{-T}\mathcal{G}^T C\mathcal{T} = FK\mathcal{T} + B^T Z^T (I + Z(A - BK))^{-T}P \tag{17}$$

Proof. This result is standard. A simple proof follows the approach in [4]. Note that the Riccati equation (16) may be expressed as a recursion, and so it always has a unique block diagonal solution. ∎

Now we can solve conditions of the form (15), explicitly in state-space, as follows.

Lemma 6. *Suppose A, B, C, D and H are block diagonal matrices of appropriate dimensions with $C^T D = 0$ and $D^T D > 0$. Let*

$$\mathcal{G} = C(I - ZA)^{-1}ZB + D \qquad \mathcal{E} = C(I - ZA)^{-1}H$$

Then there exists a unique lower triangular $\mathcal{Q}$ such that $\mathcal{G}^T\mathcal{G}\mathcal{Q} + \mathcal{G}^T\mathcal{E}$ is strictly upper, given by

$$\mathcal{Q} = -K(I - Z(A - BK))^{-1}H$$

where K is as in Lemma 5.

Proof. From Lemma 5 we have $\mathcal{G}^T\mathcal{G} = \mathcal{L}^T F \mathcal{L}$, where $\mathcal{L}$ and F are defined in the Lemma. Then from Lemma 4 we have existence and uniqueness of $\mathcal{Q}$, and

$$\mathcal{Q} = -\mathcal{L}^{-1}F^{-1}\,\mathrm{lower}(\mathcal{L}^{-T}\mathcal{G}^T\mathcal{E})$$

Now using (17) we have $\mathcal{Q} = -\mathcal{L}^{-1}K\mathcal{T}H$, and the result follows. ∎

We partition the identity matrix as $I = \begin{bmatrix} E_1 & E_2 \end{bmatrix}$, where the dimensions conform to the context in which they are used. To solve the optimization problem (12), we can now apply Lemma 3 to find $\mathcal{Q}$ satisfying the optimality condition (14), as follows.

Lemma 7. *Suppose $\mathcal{Q} \in S$. Let X, Y, J, K be defined by the Riccati equations (7–10). Then the unique optimal $\mathcal{Q}$ for (12) is given by*

$$\widetilde{\mathcal{Q}} = \begin{bmatrix} \widetilde{\mathcal{Q}}_{11} & 0 \\ \widetilde{\mathcal{Q}}_{21} & \widetilde{\mathcal{Q}}_{22} \end{bmatrix}$$

where

$$\widetilde{\mathcal{Q}}_{22} = -J\big(I - Z(A_{22} - B_{22}J)\big)^{-1}H_2 \qquad (18)$$

$$\begin{bmatrix} \widetilde{\mathcal{Q}}_{11} \\ \widetilde{\mathcal{Q}}_{21} \end{bmatrix} = -\widetilde{K}\big(I - \widetilde{Z}(\widetilde{A} - \widetilde{B}\widetilde{K})\big)^{-1} \begin{bmatrix} H_1 \\ 0 \end{bmatrix} \qquad (19)$$

Proof. This result follows by applying Lemma 3 to the optimality condition (14). In the notation of the Lemma, we have $F = \mathcal{P}_{12}^T\mathcal{P}_{12}$ and $G = \mathcal{P}_{12}^T\mathcal{P}_{11}$. Condition (i) of Lemma 3 is that $\tilde{F}_{22}\widetilde{\mathcal{Q}}_{22} + \tilde{G}_{22}$ is strictly upper. We have $\tilde{F}_{22} = \mathcal{G}^T\mathcal{G}$ and $\tilde{G}_{22} = \mathcal{G}^T\mathcal{E}$ where

$$\mathcal{G} = \mathcal{P}_{12}\mathbb{P}E_2 = C_2(I - ZA_{22})^{-1}B_{22} + D_2$$
$$\mathcal{E} = \mathcal{P}_{11}\mathbb{P}E_2 = C_2(I - ZA_{22})^{-1}H_2$$

Hence, condition (i) is equivalent to the requirement that $\mathcal{G}^T\mathcal{E} + \mathcal{G}^T\mathcal{G}\widetilde{\mathcal{Q}}_{22}$ is strictly upper. Since, $C_2^T D_2 = 0$ and $D_2^T D_2 > 0$, Lemma 6 gives (18).

Similarly, to obtain (19), let $\mathcal{G} = \mathcal{P}_{12}$ and $\mathcal{E} = \mathcal{P}_{11}\mathbb{P}E_1$. Then, condition (ii) of Lemma 3 implies

$$\mathcal{G}^T\mathcal{G}\mathbb{P}\begin{bmatrix} \widetilde{\mathcal{Q}}_{11} \\ \widetilde{\mathcal{Q}}_{21} \end{bmatrix} + \mathcal{G}^T\mathcal{E} \quad \text{is strictly upper}$$

Note that $H\mathbb{P}E_1$ is block diagonal, and since $\mathcal{Q} \in S$, it follows that

$$\mathbb{P}\begin{bmatrix} \widetilde{\mathcal{Q}}_{11} \\ \widetilde{\mathcal{Q}}_{21} \end{bmatrix} \quad \text{is lower}$$

and so Lemma 6 implies

$$\mathbb{P}\begin{bmatrix} \widetilde{\mathcal{Q}}_{11} \\ \widetilde{\mathcal{Q}}_{21} \end{bmatrix} = -K\big(I - Z(A - BK)\big)^{-1}H\mathbb{P}E_1$$

from which (19) follows. ∎

Using these results, we can find the optimal controller for this distributed system.

Theorem 8. *Suppose J and K are the block diagonal matrices defined in Lemma 7. Let K be permuted and partitioned so that*

$$\widetilde{K} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

Then, the unique optimal $\mathcal{K} \in S$ for (6) is given by

$$\widetilde{\mathcal{K}} = -\begin{bmatrix} K_{11} + K_{12}\Phi & 0 \\ K_{21} + \big(K_{22} - J\big)\Phi & J \end{bmatrix} \qquad (20)$$

where

$$\Phi = (I - ZE_2^T(\widetilde{A} - \widetilde{B}\widetilde{K})E_2)^{-1}ZE_2^T(\widetilde{A} - \widetilde{B}\widetilde{K})E_1 \quad (21)$$

Proof. From Lemma 7, we have the unique optimal $\mathcal{Q} \in S$ for (12), given by (18) and (19). The unique optimal $\mathcal{K} \in S$ for (6) can therefore be found from

$$\mathcal{K} = \mathcal{Q}\mathcal{P}_{21}^{-1}(I + \mathcal{P}_{22}\mathcal{Q}\mathcal{P}_{21}^{-1})^{-1}$$

The result follows from algebraic manipulations, which are omitted here due to space constraints. ∎

Proof of Theorem 1. This result follows directly from Theorem 8, where $A^K = E_2^T(\widetilde{A} - \widetilde{B}\widetilde{K})E_2$ and $B^K = E_2^T(\widetilde{A} - \widetilde{B}\widetilde{K})E_1$. ∎

6 Estimation Structure

Having determined the optimal controller for our problem, we turn now to analyzing this result. To lighten notation, let $M(t) = A_{22}(t) - B_{22}(t)J(t)$ and $N(t) = A(t) - B(t)K(t)$. From (21), define $\eta = \Phi x_1$. Hence, we obtain the following state-space system

$$\eta(t+1) = N_{22}(t)\eta(t) + N_{21}(t)x_1(t)$$

with initial condition $\eta(0) = 0$. As a result, the optimal policy is given by (11). Combining this with the dynamics in (1), the closed-loop dynamics of the system become

$$\begin{bmatrix} x_1(t+1) \\ \eta(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} N_{11}(t) & N_{12}(t) & 0 \\ N_{21}(t) & N_{22}(t) & 0 \\ N_{21}(t) & N_{22}(t) - M(t) & M(t) \end{bmatrix}\begin{bmatrix} x_1(t) \\ \eta(t) \\ x_2(t) \end{bmatrix}$$
$$+ \begin{bmatrix} H_1(t+1) & 0 \\ 0 & 0 \\ 0 & H_2(t+1) \end{bmatrix}\begin{bmatrix} w_1(t+1) \\ w_2(t+1) \end{bmatrix} \quad (22)$$

With the closed-loop system in mind, we now attempt to construct the minimum-mean square error estimator of $x_2(t)$ based on measurements of $x_1(0), \ldots, x_1(t)$ and $\eta(0), \ldots, \eta(t)$, given by the conditional mean

$$\mathrm{E}\big(x_2(t) \mid x_1(0), \ldots, x_1(t), \eta(0), \ldots, \eta(t)\big)$$

To this end, consider the following lemma.

Lemma 9. *Suppose x_1, x_2 represent the state of the following autonomous system driven by noise*

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} A_{11}(t) & 0 \\ A_{21}(t) & A_{22}(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} w_1(t+1) \\ w_2(t+1) \end{bmatrix}$$

where $x(0) = w(0)$, and $w_1(t), w_2(t)$ are independent Gaussians for all $t \geq 0$. Define $\mu(t)$ such that

$$\mu(t) = \mathrm{E}\big(x_2(t) \mid x_1(0) = z(0), \ldots, x_1(t) = z(t)\big)$$

Then, $\mu(0) = 0$, and for each $t \geq 0$,

$$\mu(t+1) = A_{22}(t)\mu(t) + A_{21}(t)z(t)$$

Proof. The proof, which we omit due to space constraints, follows from Bayes' law and the fact that $A_{12}(t) = 0$ for all t. ∎

With this lemma, we obtain a very simple representation for the optimal controller.

Theorem 10. *Suppose x_1, x_2, η are the states of the autonomous system in (22). Then,*

$$\eta(t) = \mathrm{E}\big(x_2(t) \mid x_1(0), \ldots, x_1(t)\big)$$

Proof. From (22), we see that the state transition matrix is lower triangular. Thus, we can use the results of Lemma 9 to get

$$\mu(t+1) = M(t)\mu(t) + \begin{bmatrix} N_{21}(t) & N_{22}(t) - M(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ \eta(t) \end{bmatrix}$$

$$= \eta(t+1) + M(t)\big(\mu(t) - \eta(t)\big)$$

where we have used the definition of $\eta(t+1)$ in the last expression. Now, since $\mu(0) = \eta(0) = 0$, we inductively see that $\mu(t) = \eta(t)$ for all t. Lastly, since $\eta(t)$ can be deterministically computed given $x_1(t), \ldots, x_1(0)$, we have

$$\eta(t) = \mathrm{E}\big(x_2(t) \mid x_1(0), \ldots, x_1(t), \eta(0), \ldots, \eta(t)\big)$$

$$= \mathrm{E}\big(x_2(t) \mid x_1(0), \ldots, x_1(t)\big)$$

as desired. ∎

7   Conclusion

In this paper, we have analytically derived the optimal state-space controller for a two player distributed system. This was accomplished via a spectral factorization technique. Analysis of the solution showed that the optimal distributed controller involved an estimator, and the order of the optimal controllers was established.

This work is a first step toward explicit state-space solutions for more general distributed control problems. The advantage of our technique used herein is that it extends naturally to more general distributed control structures. Our future work will involve these non-trivial extensions to more general structures, as well as the natural extension to infinite horizon and output feedback problems.

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