Hat Guessing Games and the Use of Coding for Decentralized Control

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Abstract

We study a class of decentralized team decision problems over discrete state spaces with non classical information structures. We present a simple class of problems, where an optimal solution can be obtained via coding. For the example presented, we explicitly construct a coding scheme, called the binary sum coding scheme, and show that it is optimal. This class of problems is motivated by a famous mathematical puzzle called *the hats problem*.

1 Introduction and Prior Work

The study of team decision problems as well as the role of information patterns in decision making was initiated by Marschak in his work on the theory of organizations [12]. Radner [16] provided a mathematical framework to study such problems and provided a solution for a special class of team decision problems.

The role of information structures in decision making was emphasized in the work of Witsenhausen [19]. Specifically, it was shown that with non-classical information structures, a non-linear controller may outperform the best linear controller even in the case of linear quadratic Gaussian (LQG) systems. This was one of the first examples that highlighted the role of information in decision making. In that problem, there are two decision makers. The objective of the decision makers is to minimize a quadratic cost function. Witsenhausen showed that, from the point of view of the first decision maker, there are two opposing choices. One is to use the action to encode information, and signal to the other player. We refer to this option as *coding* or *signaling*. Although signaling helps the second player to make a better decision, choosing a large action for signaling can increase the cost function, and so the other option is to control the action to directly minimize the cost. It is this tension between control and signaling that is exploited by Witsenhausen to construct a non-linear controller. In fact, it was shown in [15] that a simple non linear controller based on quantization and maximum likelihood decoding can perform arbitrarily better than the best linear controller. In other words, a controller in which the first decision maker *signals* its information to the second decision maker can outperform the best linear controller.

Ho and Chu [8] identified a condition on information structures (called *partially nested*) for which a linear controller is optimal. It was also shown in [3] that when the cost function does not contain product term between decision variables, a linear controller is indeed optimal for LQG team decision problems. The work by Ho and Chu provided a class of decentralized control problems for which coding is unnecessary.

The objective of this paper is to find and analyze a decentralized control problem where the coding objective is aligned with the cost minimization objective. This allows one to use a coding scheme to design an optimal decentralized control policy.

This connection between signaling and decision making has been recognized earlier. One of the earliest works in this field was in the context of non-cooperative game theory [4]. The author considered a two person non-cooperative game where the control of the first player is also the information to the second player. It was shown that the players behave as if they have the same information. This *transparent* information constraint is attained by means of coding. The idea of using action to signal intent to other players has also been used to explain the role of *reputation* in economic theory [14]. In the context of control theory, this connection was first alluded to in [9].

In this paper, we study a team decision problem where the use of coding allows us to explicitly construct an optimal controller. The problem is inspired by a famous mathematical puzzle called the *hats problem*: There are N prisoners standing in a line, each given a hat colored black or white. Each prisoner cannot see his own hat or the hats of prisoners behind him, but he can see the hats of all the prisoners standing in front of him in the line. The prisoners are then to guess the color of the hat above their own head. The guess of each prisoner is heard by all the other prisoners. If it is true, the pris-

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oner is freed, otherwise he is executed. The goal is to find a guessing strategy which minimizes the number of prisoners executed.

While the model considered in our paper allows for communication between users (through their guesses), there is the other symmetric variant of the hats problem in which no communication is allowed between the prisoners but every prisoner can see the hats of all the other prisoners. In particular, [10] considers the symmetric variant of the hats problem and explores its connections with Hamming codes. Under a similar setting, [5] and [6] present worst-case analysis for different versions of the symmetric hats problem. In this work we consider a stochastic framework in which the hats are drawn according to a given distribution and the expected number of correct guesses is to be maximized. Our problem has some similarities with the Witsenhausen's example in terms of the tension between signaling and control. Each prisoner can choose his guess to maximize his own probability of correct guess or he can give some information about other hats colors through his guess signal. In terms of information pattern, the problem is similar to that of [4]. However, compared to that work, we consider an N player team decision problem. Furthermore, we provide an explicit coding scheme and show that it is optimal. In contrast to [8], the team decision problem considered here does not satisfy the notion of partially nested information structure. Although the problem looks similar to Witsenhausen's counterexample, it differs in two major aspects: first the problem is defined on a discrete space, second the cost of action is implicit in our problem and third there is no noise in the signaling channel. Surprisingly, this allows us to construct an optimal controller while the optimal solution for Witsenhausen's problem is not yet known.

This problem can be viewed as a bridge between control theory and network coding theory. On the control side, it provides valuable insights into the design of decision rules in the presence of non-classical information structures. In particular, we hope to use this problem as a starting point in understanding a larger class of team decision problems that can be effectively solved using coding mechanisms. On the network coding side, this work introduces a new paradigm for effective use of network coding, namely decentralized team decision. Current research in this area includes, but is not limited to [7], [11], [13], [17], [18] and [20].

The rest of the paper is organized as follows: Section 2 presents the problem formulation. In Section 3, we will propose the optimal coding scheme called the binary sum coding. The proof of the optimality is given in Section 4. Section 5 provides an extension of the problem when the observations at each player are noisy. Section 6 concludes the paper.[1]

Notation. In the remainder of the paper we use bold letters (e.g. $\mathbf{x}$, $\mathbf{u}$, $\mathbf{I}$) to represent the random variables and the corresponding regular letters (e.g. x, u, I) to represent their realizations. We use subscripts to denote particular elements of a vector. We denote $\mathcal{B}^n$ to be the n-fold Cartesian product of the set $\{0,1\}$. That is $\mathcal{B}^n = \{0,1\} \times \cdots \times \{0,1\}$ n-times, with the interpretation that $\mathcal{B}^0 = \emptyset$.

2 The Hats Problem

Let us consider N players (prisoners) where player i is assigned a binary state variable (hat color) $\mathbf{x}_i$. $\mathbf{x}_1, \ldots, \mathbf{x}_N$ are i.i.d. Bernoulli random variables where $\text{Prob}(\mathbf{x}_i = 1) = p \leq \frac{1}{2}$. For $i = 1, \ldots, N$, player i observes $\mathbf{x}_{i+1}, \ldots, \mathbf{x}_N$. The action (guess) of player i is a binary variable $\mathbf{u}_i$ and is observed by all other players. The cost of player i's action is $\mathbf{1}_{\{\mathbf{u}_i \neq \mathbf{x}_i\}}$, where $\mathbf{1}_{\{A\}}$ is the indicator function of the event A. The objective is to find a decision strategy that minimizes the expected total cost

$$\mathbb{E}\left(\sum_{i=1}^{N} \mathbf{1}_{\{\mathbf{x}_i \neq \mathbf{u}_i\}} \right)$$

A *decision strategy* (denoted by s) consists of a permutation ρ^s and a decision rule μ^s. The permutation $\rho^s = \left(\rho_1^s, \cdots, \rho_N^s\right)$ is an order in which the players make decision. Thus $\rho_k^s = i$ implies that player i makes its decision at the k-th step. A player making a decision at the k-th step knows the previous actions $\mathbf{u}_{\rho_1^s}, \cdots, \mathbf{u}_{\rho_{k-1}^s}$ as well as the state of players $\mathbf{x}_{\rho_k^s+1}, \cdots, \mathbf{x}_N$. Let $\mathbf{I}_k^s$ be the information vector available to a player making decision at the k-th step under the decision strategy s. Then, we have

$$\mathbf{I}_k^s = \left(\mathbf{u}_{\rho_1^s}, \cdots \mathbf{u}_{\rho_{k-1}^s}, \mathbf{x}_{\rho_k^s+1} \cdots \mathbf{x}_N \right)$$

A decision rule $\mu^s = \left(\mu_1^s, \cdots, \mu_N^s\right)$ maps the information available to a player making decision at the k-th step to an action. That is,

$$\mu_k^s : \mathcal{B}^{k-1+N-\rho_k^s} \to \mathcal{B},$$

and the action $\mathbf{u}_{\rho_k^s} = \mu_k^s\left(\mathbf{I}_k^s\right)$. Given a strategy s, the cost function is given as

$$J_N^s = \mathbb{E}\left[\sum_{k=1}^{N} \mathbf{1}_{\left\{\mathbf{u}_{\rho_k^s} \neq \mathbf{x}_{\rho_k^s}\right\}} \right],$$

[1] An earlier version of this work was published as an extended abstract in the Proceedings of the 2009 Allerton Conference on Communications, Control and Computing [1].

where the expectation is with respect to the *state vector* $(\mathbf{x}_1, \dots, \mathbf{x}_N)$. Due to space limitations, in this paper we only consider the class of strategies with deterministic permutation and decision rule. However, using a similar approach as presented in this work, the results can be extended to the cases with randomized decision rules as well.

3 Binary Sum Coding Strategy

A classic solution to the hats problem is given as follows: the first prisoner, being able to see the hat color of all other prisoners, calculates the binary sum of their hat colors. He then reports the binary sum as his own hat color. The second prisoner hears the hat color of the first prisoner and also knows the hat color of all prisoners in front of him. By subtracting the binary sum of all hat colors he sees from the hat color of the first prisoner, the second prisoner can compute its own hat color correctly. Continuing in this manner, all prisoners from second prisoner onwards can correctly guess their hat color. This strategy, which we call as the *binary sum coding strategy*, ensures that all prisoners except the first one can correctly compute his hat color. As we show below, we cannot do better than this strategy. An intuitive explanation for that is that the color of the first prisoner's hat is not known to any one and hence there is always a possibility that the first prisoner will get his hat color wrong.

Let us formally define the binary sum coding strategy as follows:

Definition 1. *For the N player hats problem, a binary sum coding strategy (or just coding strategy, for short) consists of the permutation $\rho^{code} = \big(1, 2, \cdots, N\big)$ and $\mu^{code} = \big(\mu_1^{code}, \cdots, \mu_N^{code}\big)$, where*

$$\mu_k^{code}\left(\mathbf{u}_1, \cdots, \mathbf{u}_{k-1}, \mathbf{x}_{k+1}, \cdots, \mathbf{x}_k\right) = \mathbf{u}_1 \oplus \cdots \oplus \mathbf{u}_{k-1} \oplus \mathbf{x}_{k+1} \oplus \cdots \oplus \mathbf{x}_N.$$

The following proposition establishes that the binary sum coding strategy results in correct guesses for all players except possibly player 1.

Proposition 2. *Given any N, the binary sum coding strategy has the property that*

$$\mathbf{u}_k = \mathbf{x}_k$$

for all $k = 2, \cdots, N$.

Proof. We prove the proposition by induction on k. Note that in the binary sum coding strategy the decisions are made sequentially. Furthermore, $\mathbf{u}_1 = \mathbf{x}_2 \oplus \cdots \oplus \mathbf{x}_N$. For $k = 2$, we have

$$\begin{aligned}\mathbf{u}_2 &= \mathbf{u}_1 \oplus \mathbf{x}_3 \oplus \cdots \oplus \mathbf{x}_N \\ &= \mathbf{x}_2 \oplus \cdots \oplus \mathbf{x}_N \oplus \mathbf{x}_3 \oplus \cdots \oplus \mathbf{x}_N \\ &= \mathbf{x}_2.\end{aligned}$$

Let us assume that $\mathbf{u}_k = \mathbf{x}_k$ for all $k = 2, \cdots, n$. Then for $k = n + 1$, we have

$$\begin{aligned}\mathbf{u}_{n+1} &= \mathbf{u}_1 \oplus \cdots \oplus \mathbf{u}_n \oplus \mathbf{x}_{n+2} \oplus \cdots \oplus \mathbf{x}_N \\ &= \mathbf{x}_2 \oplus \cdots \oplus \mathbf{x}_{n+1} \oplus \cdots \oplus \mathbf{x}_N \oplus \mathbf{x}_2 \oplus \cdots \oplus \mathbf{x}_n \oplus \\ &\quad \mathbf{x}_{n+2} \oplus \cdots \oplus \mathbf{x}_N \\ &= \mathbf{x}_{n+1}.\end{aligned}$$

Here the second equality follows from the induction hypothesis as well as the definition of $\mathbf{u}_1$. This proves the proposition. ■

We will show in the next section that the binary sum coding strategy is the optimal solution to the hats problem stated in Section 2. Note that as stated above, the optimality of this scheme in the worst case scenario (minimizing the maximum cost) is trivial, however, this is not the case for our problem. In particular, if $0 < p < 1/2$, the binary sum as the first player's action increases its individual cost over taking the constant action $\mathbf{u}_1 = 0$. (Note that, since there is no knowledge about $\mathbf{x}_1$ other than the fact that it is more likely to be 0, the optimal action to minimize his cost is to choose $\mathbf{u}_1 = 0$ all the time.) This introduces a tension between coding and control. It turns out that in our scenario, coding wins in a sense that this sacrifice on the first player's part ensures the best expected outcome for the players as a whole.

4 Optimality of Binary Sum Coding Strategy

The following theorem is the main result in this section.

Theorem 3. *For the N player hats problem, the binary sum coding strategy given by $s^{code} = \{\rho^{code}, \mu^{code}\}$ is optimal and achieves an optimal cost of $J_N^* = \frac{1}{2} - \frac{1}{2}(1 - 2p)^N$.*

Before we present the proof, we make the following definition.

Definition 4. *For the N player hats problem and for any strategy s, the* perfect set *associated with that strategy (denoted by P_N^s) is given by*

$$P_N^s = \Big\{x \in \mathcal{B}^N \mid u_{\rho_k^s} = x_{\rho_k^s} \quad \textit{for all } k = 1, 2, \cdots, N\Big\},$$

where $u_{\rho_k^s} = \mu_k^s(I_k^s)$ and $I_k^s = \Big(u_{\rho_1^s}, \cdots u_{\rho_{k-1}^s}, x_{\rho_k^s+1} \cdots x_N\Big)$.

Note that ρ^s is a permutation of players associated with the strategy s. In other words, the perfect set associated with any strategy is a set of all possible initial hat configurations such that all resulting actions (guesses) match the states (hat colors). As a first step let us compute the perfect set associated with the binary sum coding strategy.

Lemma 5. *For the N player hats problem, the perfect set associated with the binary sum coding strategy (P_N^{code}) is given as:*

$$P_N^{code} = \left\{x \in \mathcal{B}^N \mid x_1 = x_2 \oplus \cdots \oplus x_N\right\}.$$

Proof. From Proposition 2, we know that for the binary sum coding strategy, we have $\mathbf{u}_k = \mathbf{x}_k$ for all $k \geq 2$. Furthermore, for any $x \in \mathcal{B}^N$, the first action is given as $u_1 = x_2 \oplus \cdots \oplus x_N$. Thus, the perfect set for the binary sum coding strategy is the set of all initial hat configurations where $x_1 = u_1$. This proves the lemma. ■

The next lemma gives the probability of the perfect set generated by the binary sum coding strategy.

Lemma 6. *The probability of the perfect set associated with the binary sum coding strategy is given by*

$$\text{Prob}\left(P_N^{code}\right) = \frac{1}{2} + \frac{1}{2}\left(1-2p\right)^N.$$

Proof. We prove the lemma by induction on N. For the case of $N = 2$, we have

$$\begin{aligned}
\text{Prob}\left(P_2^{code}\right) &= \text{Prob}\left\{\mathbf{x}_1 = \mathbf{x}_2\right\} \\
&= p^2 + (1-p)^2 \\
&= \frac{1}{2} + \frac{1}{2}\left(1-2p\right)^2.
\end{aligned}$$

Let us assume that the lemma holds for all $N = 2, \cdots, k$. From Lemma 5, we know that $\text{Prob}\left(P_{k+1}^{code}\right) = \text{Prob}\left\{\mathbf{x}_1 = \mathbf{x}_2 \oplus \cdots \oplus \mathbf{x}_{k+1}\right\}$. Thus, we have

$$\begin{aligned}
&\text{Prob}\left(P_{k+1}^{code}\right) = \\
&\quad \text{Prob}\left\{\mathbf{x}_{k+1} = 0\right\}.\text{Prob}\left\{\mathbf{x}_1 = \mathbf{x}_2 \oplus \cdots \oplus \mathbf{x}_k\right\} \\
&\quad + \text{Prob}\left\{\mathbf{x}_{k+1} = 1\right\}.\text{Prob}\left\{\mathbf{x}_1 \neq \mathbf{x}_2 \oplus \cdots \oplus \mathbf{x}_k\right\}
\end{aligned}$$

Using induction hypothesis, we get that

$$\begin{aligned}
\text{Prob}\left(P_{k+1}^{code}\right) &= (1-p)\,\text{Prob}\left(P_k^{code}\right) + p(1 - \text{Prob}\left(P_k^{code}\right)) \\
&= \frac{1}{2} + \frac{1}{2}\left(1-2p\right)^{k+1}
\end{aligned}$$

where the last equality follows by substituting the value of $\text{Prob}\left(P_k^{code}\right)$. This proves the lemma. ■

Having established the properties of the binary sum coding strategy, we now focus our attention on any arbitrary strategy. The following lemma gives a simple lower bound on the cost function associated with any strategy s in terms of the perfect set associated with that strategy.

Lemma 7. *For any given decision strategy s , we have $J_N^s \geq 1 - \text{Prob}(P_N^s)$ with equality if the strategy is the binary sum coding strategy.*

Proof. From the definition of the cost function, we have

$$\begin{aligned}
J_N^s &= \sum_{k=1}^{N} \text{Prob}\left\{\mathbf{u}_{\rho_k^s} \neq \mathbf{x}_{\rho_k^s}\right\} \\
&\geq \text{Prob}\left\{\mathbf{u}_{\rho_k^s} \neq \mathbf{x}_{\rho_k^s}, \text{for some } k\right\} \\
&= 1 - \text{Prob}\left\{\mathbf{u}_{\rho_k^s} = \mathbf{x}_{\rho_k^s} \text{for all } k\right\} \\
&= 1 - \text{Prob}\left(P_N^s\right).
\end{aligned}$$

Here the inequality follows from the union-of-events bound. This bound is tight for the binary sum coding strategy since according to the Proposition 2, only the first player ever makes a mistake. Therefore, the above inequality becomes an equality for the binary sum coding strategy and this completes the proof. ■

Corollary 8. *The cost associated with the binary sum coding strategy is given by*

$$J_N^{code} = \frac{1}{2} - \frac{1}{2}\left(1-2p\right)^N$$

Proof. Trivially follows from Lemmas 6 and 7. ■

Definition 9. *For any set $S \subseteq \mathcal{B}^n$ we define the internal Hamming distance of the set S (denoted by $d(S)$) as*

$$d(S) = \min_{\substack{a,b \in S,\\ a \neq b}} \sum_{i=1}^{n} \mathbf{1}_{\{a_i \neq b_i\}}.$$

The next lemma places a lower bound on the internal Hamming distance of a perfect set associated with any strategy.

Lemma 10. *For any strategy s, the internal Hamming distance associated with the perfect set P_N^s is at least 2. That is,*

$$d(P_N^s) \geq 2.$$

Proof. Let $x, x' \in P_N^s$ and $x \neq x'$. There must exist at least one index i such that $x_{\rho_i^s} \neq x'_{\rho_i^s}$. Let's consider such i. Then, we have

$$\begin{aligned}
x_{\rho_i^s} &= \mu_i^s\left(u_{\rho_1^s}, \cdots, u_{\rho_{i-1}^s}, x_{\rho_i^s+1}, \cdots, x_N\right) \\
&= \mu_i^s\left(x_{\rho_1^s}, \cdots, x_{\rho_{i-1}^s}, x_{\rho_i^s+1}, \cdots, x_N\right),
\end{aligned}$$

where the last equality follows from the fact that $x \in P_N^s$ and hence $u_{\rho_k^s} = x_{\rho_k^s}$ for all k. Similarly, we have $x'_{\rho_i^s} = \mu_i^s\left(x'_{\rho_1^s}, \cdots, x'_{\rho_{i-1}^s}, x'_{\rho_i^s+1}, \cdots, x'_N\right)$. Since, $x_{\rho_i^s} \neq x'_{\rho_i^s}$, it implies that $x_{\rho_k^s} \neq x'_{\rho_k^s}$ for some $k < i$ or $x_j \neq x'_j$ for some $j > \rho_i^s$. Thus, the Hamming distance between any two $x, x' \in P_N^s$ is at least two. This proves the lemma. ■

Definition 11. *For any $a \in \mathcal{B}^n$, define*

$$g_n(a) = \prod_{i=1}^{n} [pa_i + (1-p)(1-a_i)]$$

Furthermore, define $f_n : S \subseteq \mathcal{B}^n \to \mathbb{R}$ as

$$f_n(S) = \sum_{a \in S} g_k(a).$$

Note that $f_n(S)$ is the probability associated with the set $S \subseteq \mathcal{B}^n$.

Definition 12. *Let us define Θ_n as the maximum probability associated with a subset of $\mathcal{B}^n$ with internal Hamming distance of at least 2. That is,*

$$\Theta_n = \max_{\substack{S \subset \{0,1\}^n \\ d(S) \geq 2}} f_n(S).$$

The next lemma places an upper bound on the value of Θ_n.

Lemma 13. *For any $N \geq 2$, we have*

$$\Theta_N \leq \frac{1}{2} + \frac{1}{2}\big(1-2p\big)^N$$

Proof. It is easy to verify that the lemma holds for $N = 2$. Now let's assume that the lemma holds for all values of N less than n and consider the case where $N = n$. Consider $S \subseteq \mathcal{B}^n$ with $d(S) \geq 2$. We have

$$f_n(S) = f_n(S_0) + f_n(S_1),$$

where

$$S_0 = \{a \in S \mid a_n = 0\}$$
$$S_1 = \{a \in S \mid a_n = 1\}.$$

Using the definition of $f_n(S)$, we can thus write

$$f_n(S) = \sum_{a \in S_0} g_n(a) + \sum_{a \in S_1} g_n(a).$$

Define a truncation operator $T : \mathcal{B}^n \to \mathcal{B}^{n-1}$ as $T(a) = \big(a_1, \cdots, a_{n-1}\big)$ for all $a \in \mathcal{B}^n$. Note that for all $a \in S_0$, the last component of a is 0. Thus, for all $a \in S_0$, we have

$$\begin{aligned} g_n(a) &= (1-p) \prod_{i=1}^{n-1} [pa_i + (1-p)(1-a_i)] \\ &= (1-p) g_{n-1}(T(a)). \end{aligned}$$

Similarly, for all $a \in S_1$, we have $g_n(a) = p g_{n-1}(T(a))$. Let us define Q_0 as the image of the set S_0 under the truncation operator. That is, $Q_0 = T(S_0)$ and similarly define $Q_1 = T(S_1)$. We can then write

$$\begin{aligned} f_n(S) &= (1-p) \sum_{a \in Q_0} g_{n-1}(a) + p \sum_{a \in Q_1} g_{n-1}(a) \quad (1) \\ &= (1-p) f_{n-1}(Q_0) + p f_{n-1}(Q_1). \quad (2) \end{aligned}$$

Note that $Q_0 \cup Q_1 \subseteq \mathcal{B}^{n-1}$. Also the assumption of $d(S) \geq 2$ implies that $Q_0 \cap Q_1 = \phi$. Thus, we have that $f_{n-1}(Q_0 \cup Q_1) = f_{n-1}(Q_0) + f_{n-1}(Q_1) \leq 1$. This implies that $f_{n-1}(Q_1) \leq 1 - f_{n-1}(Q_0)$. Substituting this in equation (2), we get that

$$\begin{aligned} f_n(S) &\leq (1-p) f_{n-1}(Q_0) + p\big(1 - f_{n-1}(Q_0)\big) \quad (3) \\ &= \big(1-2p\big) f_{n-1}(Q_0) + p. \quad (4) \end{aligned}$$

Note that the set Q_0 is derived from the set S_0 by removing the last component. Furthermore, the internal Hamming distance of the set S_0 is at least 2. Since the last component of all elements in S_0 is 0, the internal Hamming distance is preserved by removing the last element. Thus, $d(Q_0) \geq 2$. Furthermore $Q_0 \subseteq \mathcal{B}^{n-1}$. This implies that

$$\begin{aligned} f_{n-1}(Q_0) &\leq \Theta_{n-1} \\ &\leq \frac{1}{2} + \frac{1}{2}(1-2p)^{n-1}, \end{aligned}$$

where the first inequality follows from Definition 12 and the second inequality is due to the inductive hypothesis. Substituting the above inequality in (4), we get that

$$\begin{aligned} f_n(S) &\leq \big(1-2p\big) \left(\frac{1}{2} + \frac{1}{2}\big(1-2p\big)^{n-1} \right) + p \\ &= \frac{1}{2} + \frac{1}{2}\big(1-2p\big)^n. \end{aligned}$$

Since S was an arbitrary subset of $\mathcal{B}^n$ with internal Hamming distance of at least 2, this proves the inductive step and hence the lemma. ■

Proof of Theorem 3. From Lemma 7, we have that $J_N^s \geq 1 - \text{Prob}\left(P_N^s\right)$. From Lemma 10, we know that for any strategy s, $d(P_N^s) \geq 2$ and from Definition 12, we have

$$\text{Prob}\left(P_N^s\right) \leq \Theta_N.$$

Thus, for any strategy s, we have

$$\begin{aligned} J_N^s &\geq 1 - \Theta_N \\ &\geq \frac{1}{2} - \frac{1}{2}\left(1-2p\right)^N \\ &= J_N^{code}. \end{aligned}$$

Here the second inequality follows from Lemma 13 and the last equality follows from Corollary 8. Thus, the binary sum coding strategy achieves the minimum cost. This proves the theorem. ■

5 Noisy Observations

In this section, we establish the fact that the optimality of binary sum coding relies heavily on the noiseless observation of the states. We consider the same information structure and cost function as in the previous sections but add noise to the observations made by each player. For simplicity, let us assume a 2-player scenario where player 2 does not observe any state variable but player 1 observes $\tilde{\mathbf{x}}_2$ such that

$$\tilde{\mathbf{x}}_2 = \begin{cases} \mathbf{x}_2 & \text{with probability } 1-\epsilon \\ 1-\mathbf{x}_2 & \text{with probability } \epsilon \end{cases}$$

where $0 < \epsilon < 1/2$ is the crossover probability of the binary symmetric channel through which the state of the second player is observed by the first player. Assuming $\text{Prob}\{\mathbf{x}_i = 1\} < 1/2$, we want to investigate whether the first player should still report what he observes or he should minimize his own cost by using the constant decision function $\mathbf{u}_1 = 0$. It is easy to check that for this scenario, the optimal decision order should be $\rho = (1, 2)$ since the action of player 2 cannot add to the first player's information. Also, we have

$$\begin{aligned} \text{Prob}\{\tilde{\mathbf{x}_2} = 0\} &= (1-\epsilon)\,\text{Prob}\{\mathbf{x}_2 = 0\} + \epsilon\,\text{Prob}\{\mathbf{x}_2 = 1\} \\ &= (1-p)(1-\epsilon) + p\epsilon \\ &> \frac{1}{2}. \end{aligned}$$

Therefore, if player 1 decides to signal what he observes, he should use the action $\mathbf{u}_1 = \tilde{\mathbf{x}_2}$ (as opposed to the other signaling option $\mathbf{u}_1 = 1 - \tilde{\mathbf{x}_2}$). In this case, the best action for the second player is $\mathbf{u}_2 = \mathbf{u}_1$. This will result in the total cost of

$$\begin{aligned} J^{code} &= \text{Prob}\{\mathbf{x}_1 \neq \tilde{\mathbf{x}_2}\} + \text{Prob}\{\mathbf{x}_2 \neq \tilde{\mathbf{x}_2}\} \\ &= p\,\text{Prob}\{\tilde{\mathbf{x}_2} = 0\} + (1-p)\,\text{Prob}\{\tilde{\mathbf{x}_2} = 1\} + \epsilon \\ &= 2\big(p(1-p)(1-2\epsilon) + \epsilon\big) \end{aligned}$$

On the other hand if the first player decides to minimize his own cost by choosing the constant policy $\mathbf{u}_1 = 0$, the second player should also do the same ($\mathbf{u}_2 = 0$) and the resulting cost will be

$$J^{no-code} = \text{Prob}\{\mathbf{x}_1 = 1\} + \text{Prob}\{\mathbf{x}_2 = 1\} = 2p.$$

Comparing J^{code} and $J^{no-code}$, we obtain a threshold

$$\epsilon^* = \frac{p^2}{1 - 2p(1-p)},$$

for which coding ($\mathbf{u}_1 = \tilde{\mathbf{x}_2}$) is optimal if and only if $\epsilon < \epsilon^*$. In the case of $\epsilon^* \leq \epsilon \leq 1/2$, it is optimal to use the constant non-coding strategy. As can be seen in the above figure, for $p < 1/2$, ϵ^* is also less than $1/2$ which means there are cases where player 1 has positive information (in terms of mutual information) about the second player's state but reporting that information is not optimal.

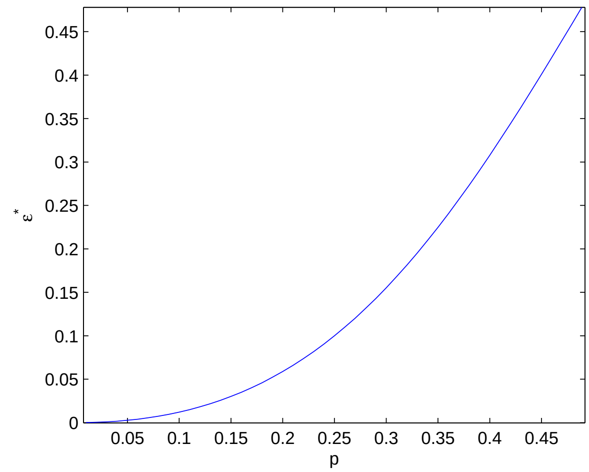

Figure 1: Crossover Threshold ϵ^*

6 Discussion and Conclusion

Decentralized control problems with non-classical information structures have been notoriously intractable (See [17] for some recent results). Witsenhausen [19] showed that for such problems, a non-linear controller based on signaling can outperform the best linear controller. Inspired by a mathematical puzzle called the hats problem, we present another class of decentralized control problems with non-classical information structures. For this particular class of problems, we show that an optimal solution can be obtained via coding.

The hats problem gives rise to several interesting questions. One question that arises is how much a player should signal? Imagine a scenario where the cardinality of the action set is different from the cardinality of the state space. Thus, a player could use a certain amount of control action to signal while using the remaining control action to lower its own cost. This tradeoff between signaling and minimizing a player's own cost is an interesting problem and is being currently investigated.

We believe that the problem studied in this work can be used as a starting point in characterizing a more general class of decentralized control problems which can be solved using coding mechanisms. The solution provided in this paper can be viewed as a special class of network codes. While the effectiveness of network coding in achieving the capacity of a network has been extensively explored from information theoretic point of view [2], this work introduces a new paradigm for effective use of network coding, namely decentralized team decision.

References

[1] S. Adlakha, R. Mirghaderi, S. Lall, and A. Goldsmith. Decentralized team decision via coding. *Allerton Conference on Communication, Control and Computing*, 2009.

[2] R. Ahlswede, N. Cai, S.-Y. R. Li, and R. W. Yeung. Network information flow. *IEEE Transaction on Information Theory*, 46:1204–1216, 2000.

[3] R. Bansal and T. Basar. Stochastic teams with non-classical information revisited: When is an affine law optimal? *IEEE Transactions on Automatic Control*, 32(6):554–559, 1987.

[4] J. Bismut. An example of interaction between information and control: The transparency of a game. *IEEE Transactions on Automatic Control*, 18(5):518–522, 1973.

[5] S. Butler, M. T. Hajiaghayi, R. D. Kleinberg, and T. Leighton. Hat guessing games. *SIAM Review*, 51(2):399–413, 2009.

[6] U. Feige. You can leave your hat on (if you guess its color). *Technical Report MCS04-03, Computer Science and Applied Mathematics, The Weizmann Institute of Science*, 2004.

[7] P. Grover and A. Sahai. Witsenhausen's counterexample as assisted interference suppression. To appear in a Special Issue on "Information Processing and Decision Making in Distributed Control Systems" in the International Journal of Systems, Control and Communications.

[8] Y. C. Ho and K. C. Chu. Team decision theory and information structures in optimal control problems – Part I. *IEEE Transactions on Automatic Control*, 17:15–22, 1972.

[9] Y.C. Ho, M. Kastner, and E. Wong. Teams, signaling, and information theory. *IEEE Transactions on Automatic Control*, 23(2):305–312, 1978.

[10] A. Liu. Two applications of a Hamming code. *The College Mathematics Journal*, 40(1), 2009.

[11] A. Mahajan, A. Nayyar, and D. Teneketzis. Identifying tractable decentralized problems on the basis of information structures. *Proceedings of the 46th annual Allerton conference on communication, control, and computing*, pages 1440–1449, 2008.

[12] J. Marschak. Elements for a Theory of Teams. *Management Science*, 1(2):127–137, 1955.

[13] N. C. Martins. Witsenhausens counter example holds in the presence of side information. *Proceedings of the 45th IEEE Conference on Decision and Control*, pages 1111–1116, 2006.

[14] P. Milgrom and J. Roberts. Predation, reputation, and entry deterrence. *Journal of economic theory*, 27(2):280–312, 1982.

[15] S. Mitter and A. Sahai. Information and control: Witsenhausen revisited. *Learning, control and hybrid systems*, pages 281–293, 2007.

[16] R. Radner. Team decision problems. *The Annals of Mathematical Statistics*, 33(3):857–881, 1962.

[17] M. Rotkowitz and S. Lall. Decentralized control information structures preserved under feedback. *Proceedings of IEEE Conference on Decision and Control*, pages 569–575, 2002.

[18] S. Tatikonda and S. Mitter. Control Under Communication Constraints. *IEEE Transactions on Automatic Control*, 49(7):1056–1068, 2004.

[19] H. S. Witsenhausen. A counterexample in stochastic optimum control. *SIAM Journal on Control*, 6(1), 1968.

[20] S. Yuksel. Decentralized computation and communication in stabilization of distributed control systems. *Proceedings of Information Theory and Applications Workshop, ITA'09*, 2009.