Analysis of Polynomial Systems With Time Delays via the Sum of Squares Decomposition

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IEEE Transactions on Automatic Control, Vol. 54, No. 5, pp. 1058–1064, May 2009

Abstract

We present a methodology for analyzing robust independent-of-delay and delay-dependent stability of equilibria of systems described by nonlinear Delay Differential Equations by algorithmically constructing appropriate Lyapunov-Krasovskii functionals using the sum of squares decomposition of multivariate polynomials and semidefinite programming. We illustrate the methodology using an example from population dynamics.

Index Terms: Linear matrix inequality (LMI), Lyapunov-Krasovskii, sum of squares (SOS), time delay.

1 Introduction

Delay differential equations (DDEs) are used to model systems that involve transport and propagation of data; examples include networked systems [1] and modeling maturation and growth in population dynamics [2]. The analysis and control of such systems is important [3, 4], as the presence of delays may induce performance degradation or even instabilities.

DDEs fall in the category of Functional Differential Equations (FDEs), which differ from Ordinary Differential Equations (ODEs) because the system state belongs to an infinite dimensional space. Assuming local existence and uniqueness of solutions, appropriate Lyapunov functions can be used for stability analysis. However, while for the case of ODEs these are functions, in the case of DDEs they are *functionals* as the state belongs in a function space itself.

For linear DDEs, the form of these functionals that is necessary and sufficient for Delay-Dependent (DD) and strong Independent-Of-Delay (IOD) stability is known [5, 6, 7], but these conditions are difficult to test algorithmically. Under restrictions on their structure, convex optimization was used to construct them with conservative results on the delay interval guaranteeing stability [8, 9]. This is because constructing the functional that is necessary and sufficient for stability amounts to parameterizing the set of positive operators on an infinite-dimensional space. Lyapunov functionals with piecewise-linear kernels can be constructed by solving a set of LMIs whose size depends on the discretization level [10], and as the discretization level is decreased, delay values closer to the boundary of stability can be tested. In [11] a new approach was proposed which uses an explicit parametrization of positive operators and uses the Sum of Squares (SOS) decomposition and semidefinite programming for computation.

As far as nonlinear time delay systems are concerned, the only methodologies centre on the construction of simple Lyapunov certificates for systems of low dimension through a judicious choice for a candidate Lyapunov function [2]. This is the case even for systems described by ODEs, where constructing Lyapunov functions is usually based on system structure and its properties (Volterra, gradient systems etc.). Recently, however, a computational methodology based on the SOS decomposition has been proposed [12, 13, 14].

In this paper we present an algorithmic methodology for constructing L-K functionals to assess IOD and DD stability for polynomial time delay systems. Preliminary results have been presented in [15, 16]. The present technical note offers significant improvements on the way these functionals can be constructed, as will be emphasized in the sequel. Applications of this approach to Internet congestion control problems have appeared in [17] and preliminary results on state feedback stabilization have appeared in [18]. The methodology unifies local and robust DD and IOD stability, but only the single-delay case is presented in this paper in order to simplify the exposition: the case of multiple, incommensurate delays can be treated in a unified way.

Section 2 outlines the proposed methodology and Section 3 shows how this can be used for the nominal, robust and local IOD and DD stability analysis of polynomial delayed systems, followed by an example from population dynamics.

Manuscript received July 31, 2007; revised March 14, 2008. Current version published May 13, 2009. This work was supported in part by the Engineering and Physical Sciences Research Council Grant EP/E05708X/1. Recommended by Guest Editors G. Chesi and D. Henrion.

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Digital Object Identifier 10.1109/TAC.2009.2017168

1.1 Notation

$\mathbb{R}$ denotes the reals and $\mathbb{R}^n$ the $n-$dimensional Euclidean space. For $x \in \mathbb{R}^n$, $\mathbb{R}[x]$ is the ring of polynomials in x with real coefficients and $Z_d[x]$ the vector of monomials in x of degree d or less. $C([-\tau, 0], \mathbb{R}^n)$ is the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^n$ with the topology of uniform convergence. The norm on C is $\|\phi\| = \sup_{-\tau \le \theta \le 0} |\phi(\theta)|$ where $|\cdot|$ is the infinity norm. Suppose $\sigma \in \mathbb{R}$, $\rho \ge 0$ and $x \in C([\sigma - \tau, \sigma + \rho], \mathbb{R}^n)$; then for any $t \in [\sigma, \sigma + \rho]$, define $x_t \in C$ by $x_t(\theta) = x(t + \theta), \theta \in [-\tau, 0]$. Symbolic independent variables will reference state and delayed state variables: $\hat{x}_l^{(i)}$ will reference $x_i(t - l\tau)$ where $l = 1, \ldots, L$ and $\hat{x}_l$ will denote the row vector of $\hat{x}_l^{(i)}$'s, $i = 1, \ldots, n$. Also, $\hat{y}_l^{(i)}$ will be used for $x_i(t + \theta - l\tau)$, $\hat{z}_l^{(i)}$ for $x_i(t + \xi - l\tau)$ and vectors $\hat{y}_l$ and $\hat{z}_l$ are similarly defined. Finally, we will use $\hat{X}_L = [\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_L]$ and $\hat{Y}_L = [\hat{y}_0, \hat{y}_1, \ldots, \hat{y}_L]$.

2 The Proposed Methodology

Background theory on stability and Lyapunov theory for DDEs can be found in [19]. Consider a polynomial time delay system with a single delay of the form

$$\dot{x}(t) = f(x(t), x(t - \tau)) \tag{1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ with $f(0, 0) = 0$ is such that a unique solution exists from an appropriate initial condition close to 0. The Lyapunov functional

$$V(\phi) = \overbrace{\int_{-\tau}^{0} v_1(\phi(0), \phi(\theta), \theta) d\theta}^{V_1} + \overbrace{\int_{-\tau}^{0}\int_{-\tau}^{0} v_2(\phi(\xi), \phi(\theta), \theta, \xi) d\theta d\xi}^{V_2}. \tag{2}$$

can be used to verify the DD stability of the zero steady-state; its derivative takes the form

$$\dot{V}(\phi) = \int_{-\tau}^{0} v_3(\phi(0), \phi(-\tau), \phi(\theta), \theta) d\theta - \int_{-\tau}^{0}\int_{-\tau}^{0} \left(\frac{\partial v_2}{\partial \theta} + \frac{\partial v_2}{\partial \xi}\right) d\theta d\xi, \tag{3}$$

where the map v_1, v_2 to v_3 will be presented in later sections.

Positivity of V_1 in (2) can be characterized using Theorem 1, the proof of which can be found in the Appendix.

Theorem 1. *Consider a continuous function $v(x, y(\theta), \theta)$ that satisfies the conditions of Lemma 14. Then*

$$\int_{-\tau}^{0} v(x, y(\theta), \theta) d\theta \ge 0 \textit{ for all } x \in \mathbb{R}^m \textit{ and } y \in C, \tag{4}$$

if and only if there exists a continuous function $r(x, \theta)$ such that for all $x \in \mathbb{R}^m$, $z \in \mathbb{R}^n$,

$$v(x, z, \theta) + r(x, \theta) \ge 0 \quad \textit{for all } \theta \in [-\tau, 0], \quad \int_{-\tau}^{0} r(x, \theta) d\theta = 0. \tag{5}$$

Non-negativity of V_2 can be guaranteed as follows:

Proposition 2. *Given a continuous function $v(y, z, \theta, \xi)$ suppose there exists a continuous vector-valued function $s : (y, \theta) \mapsto \mathbb{R}^d$ for some d so that $v(y, z, \theta, \xi) = (s(y, \theta))^T s(z, \xi)$. Then we have*

$$\int_{-\tau}^{0}\int_{-\tau}^{0} v(\phi(\xi), \phi(\theta), \theta, \xi) d\theta\, d\xi \ge 0.$$

Proof. Suppose that there exists such a decomposition for v. Then we have

$$\int_{-\tau}^{0}\int_{-\tau}^{0} v(\phi(\xi), \phi(\theta), \theta, \xi) d\theta d\xi = \left(\int_{-\tau}^{0} s(\phi(\theta), \theta) d\theta\right)^T \left(\int_{-\tau}^{0} s(\phi(\xi), \xi) d\xi\right) \ge 0.$$

Therefore the result follows. ■

Theorem 1 and Proposition 2 can reformulate the problem of testing positivity of an integral form to testing certain conditions on its kernel. However, even if we are given a function v_1, it may be hard to ensure that condition (5) is satisfied, let alone *construct* it. To allow the algorithmic construction of these functions, we make the following assumption:

Assumption 3. *We assume that v_1 and v_2 in (2) are* polynomials *in their arguments, and in particular that for some positive integer d,*

$$v_1(\hat{x}_0, \hat{y}_0, \theta) = Z_d^T[\hat{x}_0, \hat{y}_0] P(\theta) Z_d[\hat{x}_0, \hat{y}_0],$$
$$v_2(\hat{y}_0, \hat{z}_0, \theta, \xi) = Z_d^T[\hat{y}_0, \theta] R Z_d[\hat{z}_0, \xi].$$

where $P(\theta)$ is a polynomial matrix and R is a matrix of appropriate dimensions. We also assume that the vector field $f \in \mathbb{R}[\hat{x}_0, \hat{x}_1]$ as well as that $r(\hat{x}_0, \theta) \in \mathbb{R}[\hat{x}_0, \theta]$.

Even though the above assumption may seem restrictive, especially the restriction that the vector field be polynomial, it is true that in many cases polynomial vector fields arise in modeling of physical processes (e.g., using Volterra-type equations). In some other cases, one can employ a non-polynomial transformation to render a non-polynomial vector field polynomial. See [20, 21] for more details.

Under Assumption 3, all conditions in Theorem 1 and Proposition 2 are polynomial non-negativity conditions, which are in general difficult to test. In fact, it is known that testing non-negativity of polynomials of high degree (more than 4) is NP-hard. A sufficient condition for polynomial non-negativity which is worst-case polynomial-time verifiable by solving a Semidefinite Programme (SDP) is the existence of a sum of squares (SOS) decomposition. More details on positive polynomials and the SOS decomposition can be found in [12, 22, 23, 24, 25]. Here, we denote by Σ the SOS cone and by Σ_d the subset of Σ of polynomials of degree d of less.

If $a \in \mathbb{R}[x]$ is an SOS, then it is globally nonnegative. In order to ensure that it is positive *definite* and radially unbounded we can use a polynomial 'shaping' function $\varphi(x)$:

Proposition 4. *Given a polynomial $a(x)$ of degree $2d$, let*

$$\varphi(x) = \sum_{i=1}^{n}\sum_{j=1}^{d} \epsilon_{ij} x_i^{2j}, \quad \sum_{j=1}^{d} \epsilon_{ij} \geq \gamma \quad \forall\ i = 1, \ldots, n, \tag{6}$$

with γ a positive number, and $\epsilon_{ij} \geq 0$ for all i and j. Then the condition $a(x) - \varphi(x) \in \Sigma$ guarantees the positive definiteness of $a(x)$, i.e., $a(x) > 0, x \neq 0$. Moreover, $a(x)$ is radially unbounded.

Proof. The function $\varphi(x)$ is positive definite if ϵ_{ij}'s satisfy the above; $a(x) - \varphi(x)$ being SOS implies that $a(x) \geq \varphi(x)$, and therefore $a(x)$ is positive definite. Moreover it is radially unbounded since φ is radially unbounded – it is the positive sum of monomials in only one variable squared. ■

Testing non-negativity of a polynomial over a bounded domain instead of globally, (e.g., Condition (5) in Theorem 1, where polynomial non-negativity is required only for $\theta \in [-\tau, 0]$) are common. We will see similar conditions in the sequel: when studying local stability the non-negativity conditions will be required to hold only in some region of the state-space; when studying robust stability, these conditions will be required to hold for parameters inside a parameter set, etc. These conditional satisfiability conditions can be tested using a generalization to the S-procedure, which is based on Putinar's representation [26] in Real Algebraic Geometry.

Given $p \in \mathbb{R}[x]$, suppose we want to ensure that $p(x) > 0$ on the set $D = \{x \in \mathbb{R}^n : g_i(x) \leq 0, i = 1, \ldots, N_1\}$. Then one can search for Lagrange-type multipliers $\lambda_i \in \Sigma_k$ so that $p(x) + \sum_{i=1}^{N_1} \lambda_i(x) g_i(x) \in \Sigma$. Searching for $\lambda_i(x)$ of a fixed degree k so that the above expression is SOS is a semidefinite programme. Note that if p and g_i are quadratic forms and λ_i are constants, the above test is indeed the S-procedure, which can fail if $i \geq 2$. However, it has been shown in [26] that if D is compact and another mild condition holds on the $g_i(x)$, then there is a k for which the above test will succeed – it is indeed a necessary and sufficient condition. Other tests can also be formulated [27].

Note that $p(x) = Z(x)^T \left(Q_0 + \sum_{i=1}^{M} \lambda_i Q_i\right) Z(x)$ is SOS if and only if the LMI $Q_0 + \sum_{i=1}^{M} \lambda_i Q_i \succeq 0$ holding for some decision variables λ_i. Here $Z(x)$ is a vector of monomials and Q_i, $i = 0, \ldots, M$ are symmetric matrices so that $Z(x)^T Q_0 Z(x) = p(x)$ and $Z(x)^T Q_i Z(x) = 0$ for $i = 1, \ldots, M$. The size of the LMI (i.e., the length of $Z(x)$) is $\begin{pmatrix} n + m/2 \\ m/2 \end{pmatrix}$ if $p(x)$ is in n variables and of degree less than or equal to m, where m is even. However, the number of variables λ_i can be large.

3 Analysis of Polynomial Time Delay Systems

Testing whether a linear system with multiple delays is stable independent of the delay or stable for all $\tau_i \in [0, \overline{\tau}_i)$ with $\overline{\tau}_i \in \mathbb{R}_+,\ i = 1, \ldots, K$ given, is known to be NP-Hard [28]. Although obtaining an exact answer to these questions would probably be computationally intractable, sufficient stability tests that are algorithmically verifiable can still be formulated which may answer the stability question for some problem instances. In this case it is convenient to provide a nested family of such tests, each of which is at least as powerful as the previous one, but comes with an increased computational cost. This is the approach we follow here.

3.1 Independent-of-Delay (IOD) stability

A steady-state of a time delay system is IOD stable if it is stable for all fixed values of the delay. IOD stability conditions are used in controller synthesis when the size of the delay is unknown [29]. Given a linear time delay system of the form $\dot{x} = A_0 x(t) + A_1 x(t - \tau)$, "*strong IOD stability*" is equivalent to $\det(sI_n - A_0 - zA_1) \neq 0$ for all $s = j\omega$, and z in the closed unit disk [6]. This property is stronger than the standard notion of IOD stability, for which only $\det(sI_n - A_0 - e^{-s\tau} A_1) \neq 0$ for all $s = j\omega$ and $\tau \geq 0$ is required. The property is, however, robust to perturbations in A_0 and A_1. In [6] the class of Lyapunov functionals necessary and sufficient for strong IOD stability of linear systems has been characterized. The system $\dot{x} = A_0 x(t) + A_1 x(t - \tau)$ is strongly IOD stable if and only if it possesses, for a certain $L \in \mathbb{N}$, a Lyapunov functional of the form

$$\begin{aligned} V(\phi) = {} & v_0(\phi(0), \phi(-\tau), \ldots, \phi(-L\tau)) \\ & + \int_{-\tau}^{0} v_1(\phi(\theta), \phi(\theta - \tau), \ldots, \phi(\theta - L\tau))\, d\theta \end{aligned} \tag{7}$$

where v_0 and v_1 are *quadratic* polynomials. For general nonlinear time delay systems the appropriate structure of a Lyapunov functional is not known, and therefore here we generalize the above structure for polynomial time delay systems. This is the improvement from the results that appeared in [15], which used $L = 1$. We will assume that $f \in \mathbb{R}[\hat{x}_0, \hat{x}_1]$, that solutions exist at least in $[0, L\tau]$ and for the time being that 0 is the only steady-state of the system. We have the following conditions for IOD stability:

Proposition 5. *Consider the system described by (1). For a positive integer L, suppose there exist polynomials $v_0, v_1 : \mathbb{R}^{n(L+1)} \to \mathbb{R}$, a positive definite, radially unbounded polynomial $\varphi : \mathbb{R}^{n(L+1)} \to \mathbb{R}$ and a non-negative polynomial $\psi : \mathbb{R}^{n(L+1)} \to \mathbb{R}$ such that the following hold for all $\hat{X}_L, \hat{Y}_L \in \mathbb{R}^{n(L+1)}$, and $\hat{x}_{L+1} \in \mathbb{R}^n$:*

1) $v_0(\hat{X}_L) - \varphi(\hat{X}_L) \geq 0$,

2) $v_1(\hat{Y}_L) \geq 0$,

3) $$\sum_{l=0}^{L} \nabla_{\hat{x}_l} v_0(\hat{X}_L)^T f(\hat{x}_l, \hat{x}_{l+1}) + v_1(\hat{x}_0, \dots, \hat{x}_L). \\ -v_1(\hat{x}_1, \dots, \hat{x}_{L+1}) + \psi(\hat{X}_L) \leq 0$$

Then the 0 steady-state is globally IOD stable. If moreover ψ is positive definite, then the 0 steady-state is globally asymptotically IOD stable.

Proof. Consider the functional

$$V(\phi) = v_0(\phi(0), \dots, \phi(-L\tau)) \\ + \int_{-\tau}^{0} v_1(\phi(\theta), \dots, \phi(\theta - L\tau))\, d\theta.$$

The first two constraints impose that $V(\phi)$ is non-negative and radially unbounded. The derivative of V along the trajectories of system (1) is

$$\dot{V}(\phi) = \sum_{l=0}^{L} \nabla_{\phi(-l\tau)} v_0(\phi(0), \dots, \phi(-L\tau))^T \\ f(\phi(-l\tau), \phi(-(l+1)\tau)) \\ + v_1(\phi(0), \dots, \phi(-L\tau)) \\ - v_1(\phi(-\tau), \dots, \phi(-(L+1)\tau))$$

Under the third condition the above derivative is non-positive. Therefore if all three conditions are satisfied, then by the Lyapunov-Krasovskii Theorem [19] the steady-state of the system given by (1) is globally stable; since the delay size does not appear explicitly in the above conditions, then the zero steady-state is globally stable independent of delay. Moreover if $\psi > 0$, then the zero steady-state is globally asymptotically stable independent of delay. ■

If (1) were linear, we would recover the conditions given in [6] as the functional V is the functional given by (7) but we used it to analyze stability of polynomial systems. Compared to the Lyapunov functional given by (2) that was investigated in the previous section, only the first term of that expression is used, and the kernel $v_1(\phi(0), \phi(\theta), \theta)$ is only a function of $\phi(0)$ and $\phi(\theta)$ with no cross-terms. These restrictions ensure that the delay, τ, does not appear explicitly either in the Lyapunov positivity or the derivative non-positivity conditions.

Making use of Assumption 3, the conditions in the above Proposition can be tested algorithmically using the SOS decomposition and SOSTOOLS [30]. The functions φ and ψ can be constructed as per Equation (6). For illustration, the following is a simple example.

Example 6. *Consider the system*

$$\dot{x}_1(t) = -x_1(t) + x_2^2(t - \tau), \quad \dot{x}_2(t) = -x_2(t)$$

The equilibrium of this system is IOD stable. Consider a L-K functional $V(z_0)$ with v_0 and v_1 polynomials of bounded degree. When v_0 and v_1 are constrained to be second order polynomials, no certificate is found. However, when we search for fourth order polynomials, we obtain

$$V(x_t) = x_2^2(t) + \frac{3}{4}x_1^2(t) + (0.5x_1(t) + x_2^2(t))^2 \\ + \int_{-\tau}^{0} x_2^4(t+\theta) d\theta. \\ -\dot{V}(x_t) = (x_1(t) + x_2^2(t) - x_2^2(t-\tau))^2 + 2x_2^2(t) \\ + \frac{14}{16}x_1^2(t) + x_2^2(t)x_2^2(t-\tau) \\ + 2(x_2^2(t) + \frac{1}{4}x_1(t))^2.$$

We now concentrate on local and robust IOD stability analysis.

3.1.1 Local Stability

Nonlinear systems may have more than one equilibria, or the stability properties of a steady-state may not be global. In order to obtain a local result, we have to restrict our attention to a region $\Omega \subset C$ around the equilibrium of interest. In the sequel, we will be using the following form for the region Ω; other descriptions can also be accommodated:

$$\Omega = \left\{ x_t \in C : \|x_t\| = \sup_{-\tau \leq \theta \leq 0} |x(t+\theta)| \leq \gamma \right\}$$

Related to this is the semi-algebraic set $\{y \in \mathbb{R} : h(y) := (y - \gamma)(y + \gamma) \leq 0\}$, which we will use to describe (conservatively) the set Ω.

Proposition 7. *Let 0 be a steady-state of system (1) and given γ, let $h(y) = (y-\gamma)(y+\gamma)$. For an integer $L \geq 0$, let there exist functions $v_0, v_1 : \mathbb{R}^{n(L+1)} \to \mathbb{R}$, a positive definite function $\varphi : \mathbb{R}^{n(L+1)} \to \mathbb{R}$, non-negative functions $\psi, \check{p}_{ij} : \mathbb{R}^{n(L+1)} \to \mathbb{R}$, $i = 1, \dots, n$, $j = 0, \dots, L$ and non-negative functions $\check{q}_{ij} : \mathbb{R}^{n(L+2)} \to \mathbb{R}$ for $i = 1, \dots, n$, $j = 0, \dots, L+1$ such that the following hold for all $\hat{X}_L, \hat{Y}_L \in \mathbb{R}^{n(L+1)}$, and $\hat{x}_{L+1} \in \mathbb{R}^n$:*

1) $v_0(\hat{X}_L) - \varphi(\hat{X}_L) + \sum_{j=0}^{L}\sum_{i=1}^{n} \check{p}_{ij}(\hat{X}_L) h(\hat{x}_j^{(i)}) \geq 0$;

2) $v_1(\hat{Y}_L) \geq 0$;

3) $$\sum_{l=0}^{L} \nabla_{\hat{x}_l} v_0(\hat{X}_l)^T f(\hat{x}_l, \hat{x}_{l+1}) + v_1(\hat{x}_0, \dots, \hat{x}_l) - v_1(\hat{x}_1, \dots, \hat{x}_{L+1}) + \psi(\hat{X}_L) - \sum_{j=0}^{L+1} \sum_{i=1}^{n} \check{q}_{ij}(\hat{X}_{L+1}) h(\hat{x}_j^{(i)}) \leq 0$$

where $\hat{X}_{L+1} = [\hat{x}_0, \hat{x}_1, \dots, \hat{x}_{L+1}]$*. Then* 0 *is (locally) IOD stable. If moreover* $\psi(\hat{X}_L) > 0$*, then* 0 *is (locally) IOD asymptotically stable.*

Proof. Consider the functional

$$V(\phi) = v_0(\phi(0), \phi(-\tau), \dots, \phi(-L\tau)) + \int_{-\tau}^{0} v_1(\phi(\theta), \phi(\theta - \tau), \dots, \phi(\theta - L\tau)) d\theta.$$

When $h(\phi_i(-j\tau)) \leq 0$ and $\check{p}_{ij}(\phi(0), \phi(-\tau), \dots, \phi(-L\tau)) \geq 0$ for $i = 1, \dots, n$ and $j = 0, \dots, L$, we have from the first two conditions

$$V(\phi) \geq - \sum_{j=0}^{L} \sum_{i=1}^{n} \check{p}_{ij}(\phi(0), \phi(-\tau), \dots, \phi(-L\tau)) h(\phi_i(-j\tau)) + \varphi(\phi(0), \phi(-\tau), \dots, \phi(-L\tau)) > 0$$

and so the first Lyapunov condition is satisfied, i.e., $V > 0$ on Ω. The same is true for the derivative condition, given constraint (3) above, and so the zero steady-state of system (1) is locally IOD stable. If $\psi > 0$, then $-\frac{dV}{dt} > 0$ on Ω and so asymptotic stability independent of delay is concluded. ■

The conditions in the above Proposition can be tested algorithmically if we assume a polynomial structure of $v_0(\hat{X}_L)$, $v_1(\hat{Y}_L)$, $\check{p}_{ij}(\hat{X}_L)$ for $i = 1, \dots, n$, $j = 0, \dots, L$ and $\check{q}_{ij}(\hat{X}_{L+1})$ for $i = 1, \dots, n$, $j = 0, \dots, L+1$; construct $\varphi(\hat{X}_L)$ and possibly $\psi(\hat{X}_L)$ using (6); and replace non-negativity conditions by the existence of an SOS decomposition for them. SOSTOOLS can then be used to construct these polynomial functions algorithmically.

At this point we should note that having obtained a Lyapunov function that is valid locally and proves asymptotic stability, the domain of attraction of the steady-state can be estimated as the maximal level set of V that is contained in Ω^L. This can also be formulated as an SOS programme, but it is beyond the scope of this paper.

3.1.2 Robust Stability

Another important issue is robust stability under parametric uncertainty, which can be treated in a unified way as we will see in the sequel. Consider a time delay system of the form (1) with an uncertain parameter p

$$\dot{\tilde{x}}(t) = \tilde{f}(\tilde{x}(t), \tilde{x}(t - \tau), p), \qquad (8)$$

where $p \in P$, where P is a semi-algebraic set defined by

$$P = \{p \in \mathbb{R}^m \,|\, q_i(p) \leq 0, \ i = 1, \dots, N\} \qquad (9)$$

where $q_i \in \mathbb{R}[p]$. Define a new variable $x(t) := \tilde{x}(t) - \tilde{x}^*$, where $\tilde{x}^*$ is a steady-state of (8), satisfying $\tilde{f}(\tilde{x}^*, \tilde{x}^*, p) = 0$, and may change as the parameters $p \in P$ vary. Then we have

$$\dot{x}(t) = \tilde{f}(x(t) + \tilde{x}^*, x(t - \tau) + \tilde{x}^*, p) = f(x(t), x(t - \tau), p, \tilde{x}^*) \qquad (10)$$

$$0 = \tilde{f}(\tilde{x}^*, \tilde{x}^*, p) \qquad (11)$$

The transformed system has a steady-state at the origin. We assume for simplicity that there is a single steady-state for any given values of the parameters. Then the stability of this steady-state can be tested by constructing a *Parameter-Dependent* Lyapunov functional. See the remark at the end of this section for systems with multiple equilibria. The following Proposition constructs a Lyapunov functional which is only parameterized by p, even though the Lyapunov functional could also be parameterized by $\tilde{x}^*$.

Proposition 8. *Consider the system given by (10), where* $p \in P$ *as defined by (9). For a positive integer* L*, suppose that there exist functions* $v_0, v_1 : \mathbb{R}^{n(L+1)} \times \mathbb{R}^m \to \mathbb{R}$*, a positive definite radially unbounded function* $\varphi : \mathbb{R}^{n(L+1)} \to \mathbb{R}$*, non-negative functions* $\psi : \mathbb{R}^{n(L+1)} \to \mathbb{R}, \check{q}_i : \mathbb{R}^{n(L+1)} \times \mathbb{R}^m \to \mathbb{R}, \check{r}_i : \mathbb{R}^{n(L+1)} \times \mathbb{R}^m \to \mathbb{R}, \check{s}_i : \mathbb{R}^{n(L+2)} \times \mathbb{R}^m \to \mathbb{R}$*,* $i = 1, \dots, N$ *such that the following hold such that the following conditions hold for all* $\hat{X}_L, \hat{Y}_L \in \mathbb{R}^{n(L+1)}$ *and* $\hat{x}_{L+1} \in \mathbb{R}^n$

1) $v_0(\hat{X}_L, p) - \varphi(\hat{X}_L) + \sum_{i=1}^{N} \check{q}_i(\hat{X}_L, p) q_i(p) \geq 0$*;*

2) $v_1(\hat{Y}_L, p) + \sum_{i=1}^{N} \check{r}_i(\hat{Y}_L, p) q_i(p) \geq 0$*;*

3) $$\sum_{l=0}^{L} \nabla_{\hat{x}_l} v_0(\hat{X}_L, p)^T f(\hat{x}_l, \hat{x}_{l+1}, p, \tilde{x}^*) + v_1(\hat{x}_0, \dots, \hat{x}_L, p) - v_1(\hat{x}_1, \dots, \hat{x}_{L+1}, p) + \sum_{i=1}^{N} \check{s}_i(\hat{X}_{L+1}, p) q_i(p) + \psi(\hat{X}_L) \leq 0,$$

   when (11) is satisfied.

Then the steady-state 0 *of the system given by (10–11) is robustly globally IOD stable for all* $p \in P$*. Moreover, if* $\psi(\hat{x}_L) > 0$*,* 0 *is IOD robustly globally asymptotically stable for all* $p \in P$*.*

The proof is based on the fact that the following functional

$$V(\phi) = v_0(\phi(0), \phi(-\tau), \dots, \phi(-L\tau), p) + \int_{-\tau}^{0} v_1(\phi(\theta), \phi(\theta - \tau), \dots, \phi(\theta - L\tau), p) d\theta \qquad (12)$$

is a L-K functional, and is omitted for brevity. Polynomial multipliers can be used to adjoin condition (11) to the third constraint.

Remark 9. *Many times there are multiple steady-states in (8) that move as p is allowed to vary in P; in this case we seek a local result, and the parameter set P should be extended to include the 'motion' of the steady-state $\tilde{x}^*$ and the region Ω has to be sufficiently small so that no other equilibria cross into Ω as the parameters change within P. See Section 4.*

3.2 Delay-Dependent (DD) stability

When the stability properties of the steady-state change as the delay size, seen as a static parameter, changes, the stability is termed delay-dependent. In this case, a different type of Lyapunov functional has to be used to allow for the delay size to appear explicitly in the Lyapunov conditions. The structure of the L-K functional we will be considering will take the form (2). This functional reduces to the complete Lyapunov functional used for linear time delay systems when v_1 and v_2 are quadratic in ϕ [7]. We have the following result:

Proposition 10. *Let* 0 *be a steady-state for the system given by (1) with a polynomial vector field. Given $Z_d[\hat{y}_0, \theta]$, let $Q \succeq 0$ be of appropriate dimensions and define*

$$v_2(\hat{y}_0, \hat{z}_0, \theta, \xi) = Z_d[\hat{y}_0, \theta]^T Q Z_d[\hat{z}_0, \xi]. \tag{13}$$

Suppose also that there exists a function $v_1 : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, a radially unbounded positive definite function $\varphi : \mathbb{R}^n \to \mathbb{R}$, as well as a non-negative function $\psi : \mathbb{R}^n \to \mathbb{R}$, and functions $r_1 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and $r_2 : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, all polynomials, that satisfy the following conditions for all $\hat{x}_0, \hat{x}_1, \hat{y}_0, \hat{z}_0 \in \mathbb{R}^n$:

1) $v_1(\hat{x}_0, \hat{y}_0, \theta) + r_1(\hat{x}_0, \theta) - \varphi(\hat{x}_0) \geq 0$, *for* $\theta \in [-\tau, 0]$;

2) $$\begin{aligned}\nabla_{\hat{x}_0} v_1(\hat{x}_0, \hat{y}_0, \theta)^T f(\hat{x}_0, \hat{x}_1) - \frac{\partial v_1(\hat{x}_0, \hat{y}_0, \theta)}{\partial \theta} \\ + \frac{1}{\tau} v_1(\hat{x}_0, \hat{x}_0, 0) - \frac{1}{\tau} v_1(\hat{x}_0, \hat{x}_1, -\tau) \\ + 2v_2(\hat{y}_0, \hat{x}_0, \theta, 0) - 2v_2(\hat{y}_0, \hat{x}_1, \theta, -\tau) \\ + r_2(\hat{x}_0, \hat{x}_1, \theta) + \psi(\hat{x}_0) \leq 0\end{aligned}$$
for $\theta \in [-\tau, 0]$;

3) $\displaystyle\int_{-\tau}^{0} r_1(\hat{x}_0, \theta)\, d\theta = 0$;

4) $\displaystyle\int_{-\tau}^{0} r_2(\hat{x}_0, \hat{x}_1, \theta)\, d\theta = 0.$

5) *There exists a $R \succeq 0$ so that* $\frac{\partial v_2}{\partial \theta}(\hat{y}_0, \hat{z}_0, \theta, \xi) + \frac{\partial v_2}{\partial \xi}(\hat{y}_0, \hat{z}_0, \theta, \xi) = Z_d[\hat{y}_0, \theta]^T R Z_d[\hat{z}_0, \xi]$.

Then the steady-state 0 *of the system given by (1) is* globally stable *for delay size τ. Moreover, if $\psi(\hat{x}_0) > 0$, then* 0 *is* globally asymptotically stable *for delay size τ.*

Proof. Consider (2) where v_2 is given by (13). The first term is positive definite owing to conditions 1) and 3) in the above Proposition, and the fact that $\varphi(\phi(0)) > 0$. The second term is positive semidefinite by construction, as $Q \succeq 0$. Therefore the first Lyapunov condition is satisfied.

The time derivative of $V(\phi)$ is given by (14)

$$\begin{aligned}\dot{V}(\phi) = \int_{-\tau}^{0} \big(&\nabla_{\phi(0)} v_1(\phi(0), \phi(\theta), \theta)^T f(\phi(0), \phi(-\tau)) \\ &- \frac{\partial v_1}{\partial \theta}(\phi(0), \phi(\theta), \theta) + \frac{1}{\tau} v_1(\phi(0), \phi(0), 0) \\ &- \frac{1}{\tau} v_1(\phi(0), \phi(-\tau), -\tau) + 2v_2(\phi(\theta), \phi(0), \theta, 0) \\ &- 2v_2(\phi(\theta), \phi(-\tau), \theta, -\tau)\big)\, d\theta \\ &- \int_{-\tau}^{0} \int_{-\tau}^{0} \bigg(\frac{\partial v_2}{\partial \theta}(\phi(\xi), \phi(\theta), \theta, \xi) \\ &+ \frac{\partial v_2}{\partial \xi}(\phi(\xi), \phi(\theta), \theta, \xi)\bigg)\, d\theta\, d\xi.\end{aligned} \tag{14}$$

This derivative condition has a form similar to (3). Conditions 2) and 4) in the Proposition guarantee that the first term of $\dot{V}$ is non-positive for $\theta \in [-\tau, 0]$. Condition 5) guarantees that the second term in $\dot{V}$ is non-positive. So (2) is a L-K functional, and the zero steady-state is *stable*. Since φ is radially unbounded, the result holds globally. Moreover if $\psi > 0$, then the steady-state is globally asymptotically stable for delay τ. ∎

Different L-K structures may be considered which may have better properties and similar conditions can be derived, e.g., v_1 may not be a function of θ (see the example in Section 4). For the case in which v_1 is a function of θ, a significant improvement from the result in [15] is the introduction of the functions r_i.

Example 11. *The delayed logistic (Hutchinson's) equation $\dot{x}(t) = \alpha x(t-1)[1 + x(t)]$ models single species growth with delay. The stability condition $\alpha < \frac{37}{24}$ for $\min_{-\tau \leq \theta \leq 0} |x_t(\theta)| > -1$ was obtained using the solution properties of the DDE [2]. The Proposition above can be used to construct algorithmically a L-K functional to verify the (local) asymptotic stability of the zero equilibrium. For example, for $\|x_t\| < 0.2$ and v_1, v_2 of order 2 we get a stability condition for $\alpha = 1.23$, while if we set $r_i = 0$ then we can only ensure stability for $\alpha = 1.11$. This shows the improvement of using the functions r_i.*

An important issue unique to the case of DD stability, is to ensure that the stability properties hold for a *delay interval* rather than for a specific value of the delay. For this, one can consider τ in the conditions in Proposition 10 as a static parameter, itself being allowed to vary within the interval. One can then view this as a robustness problem and construct a (possibly parameterized) L-K functional that guarantees DD stability for the whole interval. Similar arguments allow the construction of L-K functionals for *local* DD stability and robust DD stability under parameter variations.

3.3 Computational Considerations

The largest LMI constraint in the SDP corresponding to the above Propositions is the one related to the derivative condition, which has lower order 2. Let the vector field be of order k and the candidate $v_i's$ of order m.

In the derivative condition of Proposition 5 there are $n(L+1)$ variables and the highest degree is $k+(m-1)$. Therefore the size of the LMI is

$$\begin{pmatrix} n(L+1)+(k+m-1)/2 \\ (k+m-1)/2 \end{pmatrix} - n(L+1) - 1$$

to account for monomials of degree 0 and 1. For example, if $L=1$, $k=3$ and $n=4$ and $m=4$ this gives an LMI of size 156. If n is increased to 6, this gives 442, while if m is increased to 6 this gives an LMI of size 486. All these are sizes that current semidefinite programming solvers can handle well if the number of variables is not large.

For the case of delay-dependent stability (Proposition 10), in the derivative condition there are $3n+1$ variables and the largest degree is $k+m-1$. Therefore the size of the LMI is $\begin{pmatrix} 3n+1+(k+m-1)/2 \\ (k+m-1)/2 \end{pmatrix} - 3n$. If $k=3$ and $n=4$, then a Lyapunov function of order 4 already gives an LMI of size 548. Hence one can see that the delay-dependent conditions are more computationally expensive to test.

The simplest Lyapunov functional structures should be considered first, before increasing the order of the functional or using more complicated structures. Also, the above considerations do not take into account sparsity, which could reduce the LMI size considerably, or even symmetry reduction which would result in better conditioned semidefinite programmes.

4 Example: A population dynamics model

A realistic predator-prey model which takes into account *maturation* of the predator population takes the form [31]

$$\dot{x}(t) = x(t)[b - ax(t) - k_1 y(t)], \tag{15}$$

$$\dot{y}(t) = -\sigma y(t) + k_2 x(t-\tau) y(t-\tau) \tag{16}$$

where $-ax(t)^2$ limits the growth of the prey, and $\tau \geq 0$ is a constant capturing the average period between death of prey and birth of a subsequent number of predators. Here x and y are the prey and predator populations, b is the rate of increase of prey, k_1 and k_2 are the coefficients of the effect of predation on x and y and σ is the death rate of y. We assume that a, b, k_1, k_2 and σ are positive. The equilibria (x^*, y^*) of the above system are $(x^*, y^*) = (0,0), (x^*, y^*) = (b/a, 0)$ and

$$(x^*, y^*) = \left(\frac{\sigma}{k_2}, \frac{bk_2 - a\sigma}{k_1 k_2} \right), \tag{17}$$

the last of which is the equilibrium of interest. We further assume that $(bk_2 - a\sigma) > 0$ which ensures that (17) is in the first quadrant. Linearization of the system about this equilibrium and subsequent analysis gives the following result, which is proven in [15]:

Proposition 12. *Consider the linearization of system (15–16) about the equilibrium point (17). Then if* $(bk_2 - 3a\sigma) < 0$ *the zero equilibrium is stable independent of the delay. If* $(bk_2 - 3a\sigma) > 0$ *the zero equilibrium is stable if the delay satisfies* $\tau < \tau^*$ *and is unstable otherwise, where* τ^* *is given by*

$$\tau^* = \frac{1}{\omega} atan \left[\omega \frac{(a\sigma^2 - \omega^2 k_2)k_2 - \sigma(2a\sigma - bk_2)(k_2 + a)}{k_2 \sigma \omega^2 (k_2 + a) + (2a\sigma - bk_2)(a\sigma^2 - \omega^2 k_2)} \right]$$

and ω *solves* $\omega^4 + \frac{a^2\sigma^2}{k_2^2}\omega^2 + \frac{\sigma^2}{k_2^2}(bk_2 - a\sigma)(3a\sigma - bk_2) = 0$.

Consider now (15), (16) with parameters $\sigma = 10$, $a = 1$, $k_1 = 1$, and $k_2 = 3$. Denote $x_1 = x - x^*$ and $x_2 = y - y^*$.

4.0.1 Delay-independent stability analysis

The system (15), (16) has many equilibria, and so we need to define a region around the steady-state of interest to obtain a stability condition. We let

$$\|x_{1_t}\| \leq \gamma_1 x^*, \quad \|x_{2_t}\| \leq \gamma_2 y^* \tag{18}$$

where the steady-state (x^*, y^*) is given by (17). We consider b to be a parameter in the problem. From Proposition 12, the linear version of this system is delay-independent stable when $\frac{a\sigma}{k_2} < b < \frac{3a\sigma}{k_2}$. For the given values of a, σ and k_2, the system is delay-independent stable for $10/3 < b < 10$. For the purpose of calculating (x^*, y^*) we use a value of $b = 20/3$. The steady-state (x^*, y^*) of system (15–16) does not move as b changes, however the other two equilibria cross through the region defined by (18). If we choose $\gamma_1 = \gamma_2 = 0.1$, then no other steady-state enters this region for $11/3 < b < 10$.

We consider the following Lyapunov structure:

$$V(x_t) = v_0(x_1(t), x_2(t), b) + \int_{-\tau}^{0} v_1(x_1(t+\theta), x_2(t+\theta), b) d\theta.$$

We use Proposition 8 to obtain parameter regions for which robust delay-independent stability of the origin can be proven. When v_0 is second order and v_1 is 4th order, we can construct $V(x_t)$ for $4.56 \leq b \leq 7.11$. When they are 4th and 6th order respectively, then this region becomes $3.67 \leq b \leq 9.95$, which is essentially the full interval. The total size of the LMI in the former case is 173, while in the latter it is 662.

4.0.2 Delay-dependent stability analysis

We now use the same parameters as before and fix $b = 15$, which gives $\tau^* = 0.0541$. The system has several equilibria and so we use the same constraints on x_1 and x_2 on the state-space given by (18) with $\gamma_1 = \gamma_2 = 0.1$. We can construct the Lyapunov functional $V(x_t)$ given by (2) with v_1 zeroth order with respect to θ and 2nd order with respect to the rest of the variables for $\tau = 0.04$. When v_1 is quartic with respect to all variables but θ (which is kept at 0 order) then we can construct this $V(x_t)$ for $\tau = 0.053$. Here instead of the term V_2 in (2) we use

$$V_2 = \int_{-\tau}^{0} \int_{t+\theta}^{t} v_2(\phi(\zeta)) d\zeta d\theta.$$

Terms like these were used in [8] and are useful when v_1 is zeroth order in θ. The corresponding SDP is bigger as the functional is more complicated, but we can see that values of the delay closer to the stability boundary can be tested. We have also tested the case $b = 10.5$, for which $\tau^* = 0.307$ and found that the above functional can show stability for $\tau = 0.292$, which is again very close to the stability boundary.

Functionals of the form (2) can also be used at an increased computational cost; however the simple functional shown above can be used to test stability close to the largest delay bound, emphasizing the fact that an appropriate choice of Lyapunov functional structure can lead to a successful stability test at a lower computational cost.

A Proof of Theorem 1

First, we develop a lemma that will be used in the proof of Theorem 1.

Definition 13. *We say that a function $f(y)$ has a unique global minimum, if there exists a y^* such that for any $\epsilon > 0$, there exists a $\beta > 0$ such that $f(x) \geq f(y^*) + \beta$ for all x such that $\|x - y\| \geq \epsilon$.*

Lemma 14. *Suppose there exists $\alpha > 0$ such that $\alpha\|y\|^2 \leq f(t, y)$, where $f(t, y)$ is continuous in y and piecewise continuous in t. Suppose f has a unique global minimum for every $t \in [-\tau, 0]$. Then there exists a piecewise continuous function z, such that $\min_y f(t, y) = f(t, z(t))$ for all $t \in [-\tau, 0]$.*

Proof. Let $z(t) = \arg\min_{y\in\mathbb{R}^n} f(t, y)$. We now demonstrate that z is piecewise continuous. Suppose $f(t, y)$ is continuous on a closed interval $t \in I$. We first show that $z(t)$ is bounded on I. Choose an arbitrary $z_0 \in \mathbb{R}^n$ and let $B = \sup_{t\in I} f(t, z_0)$. B is finite because f is continuous in t on the compact interval I. Now choose $r > B/\alpha$. Then if $\|z(t_1)\| \geq r$ for some $t_1 \in I$, we have that

$$f(t_1, z(t_1)) \geq \alpha\|z(t_1)\| > B,$$

while $f(t_1, z_0) \leq B$, which contradicts the definition of $z(t_1)$. Thus we have that $z(t) \leq r$ for all $t \in I$.

Suppose we are given some arbitrary $\epsilon > 0$ and a point t in the interior of I. To show that z is continuous at t, we must find a $\gamma > 0$ such that $|t-s| \leq \gamma$ implies $\|z(t) - z(s)\| \leq \epsilon$. Since $z(t)$ is the unique global minimizer of f at t, there exists a $\beta > 0$ such that

$$f(t, z(t)) \leq f(t, y) - \beta \text{ for any } y \text{ such that } \|y - z(t)\| \geq \epsilon$$

By continuity of f at t, there exists a $\delta > 0$ such that for any $s \in I$ with $|t-s| \leq \delta$, we have $\|f(t, y) - f(s, y)\| \leq \frac{\beta}{4}$ for all $\|y\| \leq r$. Now choose $\gamma < \delta$ such that $s \in I$ for all $|s - t| \leq \gamma$. Suppose there exists an s, with $|t - s| \leq \gamma$ and $|z(t) - z(s)| > \epsilon$. Then

$$\begin{aligned} f(s, z(s)) &\geq f(t, z(s)) - \beta/4 \geq f(t, z(t)) + \beta - \beta/4 \\ &\geq f(s, z(t)) + \beta - \beta/2 = f(s, z(t)) + \beta/2. \end{aligned}$$

This contradicts the definition of $z(s)$ as being the minimizer of f at s. Therefore, $\|z(t) - z(s)\| \leq \epsilon$ for all $|s - t| \leq \gamma$, as desired and z is continuous at t. Since $f(t, y)$ is piecewise continuous in t, if f is continuous at a point t, then it is continuous in an open neighborhood of t. Since we have shown that this implies z is also continuous at point t, we have that z is continuous at every point for which $f(t, z)$ is continuous. Since f is piecewise continuous, z is piecewise continuous. ∎

Recall also the following Theorem from [32].

Theorem 15. *Suppose $f : [-\tau, 0] \times \mathbb{R}^n \to \mathbb{R}$ is continuous on $[-\tau, 0] \times \mathbb{R}^n$ and suppose there exists a bounded function $z : [-\tau, 0] \to \mathbb{R}$, continuous on $[-\tau, 0]$, such that for all $t \in [-\tau, 0]$, $f(t, z(t)) = \inf_x f(t, x)$. Further suppose that for each bounded set $X \subset \mathbb{R}^n$ the set $\{f(t, x) | x \in X, t \in [-\tau, 0]\}$ is bounded. Then the following are equivalent:*

(i) For all $y \in C[-\tau, 0]$, $\int_{-\tau}^{0} f(t, y(t))dt \geq 0$.

(ii) There exists $g : [-\tau, 0] \to \mathbb{R}$ which is piecewise continuous and satisfies

$$f(t, z) + g(t) \geq 0 \text{ for all } t, z, \qquad \int_{-\tau}^{0} g(t)dt = 0.$$

Proof. (Of Theorem 1) That (5) implies (4) follows from a simple integration argument. The converse follows by combining the results in Theorem 15 and Lemma 14. ∎

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