A Graph-Theoretic Approach to Distributed Control over Networks

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Abstract

We consider a network of control systems connected over a graph. Considering the graph structure as constraints on the set of permissible controllers, we show that such systems are simply constrained by a certain sparsity pattern. We provide conditions for which such systems are well-posed, and, under the appropriate assumptions, we show that such systems are quadratically invariant. This allows for efficient solution via convex programming, and we provide a construction for the optimal controllers.

1 Introduction

We consider optimization problems of the form

$$\begin{aligned} \text{minimize} \quad & \|P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}\| \\ \text{subject to} \quad & K \text{ is stabilizing} \\ & K \in S \end{aligned}$$

Here, P is a given matrix transfer function representing the dynamics of the plant, and S is a subset of all possible controllers. It was shown by Witsenhausen that the constraint $K \in S$, no matter how simple, can make the problem intractable [12]. Moreover, the objective function is not generally convex in K. Despite these difficulties, several special cases of this problem have been solved [5, 6]. More recently, it was shown that sets S which are *quadratically invariant* under the plant P allow us to use a clever Youla parametrization in order to make the objective function convex [10]. With the advent of efficient convex programming algorithms [3], quadratic invariance has proven a powerful tool for solving many previously intractable problems.

We consider sets S defined by an underlying sparsity pattern. These sets correspond to individual plants and controllers that are connected according to some underlying graph structures. Unfortunately, not every sparsity set S is quadratically invariant under a general set of P matrices. However, under certain conditions, we show that quadratic invariance holds for some sparsity structures, even in cases when the sparsity structures of the plant and controller differ. In particular, in this paper, we provide a specific analytical relationship between quadratic invariance and adjacency matrices. As a result, every such distributed control optimization can be efficiently solved.

Much work has been done on the problem of distributed control over networks [1, 7, 9]. In this paper, we take a graph-theoretic approach to the distributed control problem. Whereas [9, 10] addressed the problem from a global controller perspective, this work addresses the distributed control problem from the local controller perspective, as a system of sub-controllers arranged on an arbitrary graph. This approach allows us to consider issues not addressed in those papers. Among these issues is the concept of well-posedness for these systems. Also, in finding an optimal global controller, it is crucial that there exists a set of individual sub-controllers that satisfy the sparsity constraints of the problem and produce the optimal global behavior. We address these issues in this paper. Although there is significant overlap between this paper and [9], the proof techniques used here are very different and provide substantial and important insight.

This paper is organized as follows. In Section 2, we introduce the graph-theoretic notation used throughout the paper. Since sparsity sets are the cornerstone of this paper, Section 3 provides some fundamental results for the algebra of such sets. In Section 4, we provide the conditions under which the sparsity sets describing the global controller are quadratically invariant with respect to the plant. We also show that these conditions do not necessarily require the plant and controller structures to be identical.

In Section 5, we detail the results needed to construct the sparsity structures of the overall feedback system, and we provide the conditions under which the system is well-posed. Lastly, Section 6 provides a method to construct the optimal individual controllers.

2 Preliminaries

Graph notation. We represent a directed graph $\mathcal{G}$ by the set of vertices $\mathcal{V} = \{v_1, \ldots, v_N\}$ and the set of edges $\mathcal{E} = \{e_1, \ldots, e_M\} \subseteq \mathcal{V} \times \mathcal{V}$. There is an edge from vertex

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v_i to vertex v_j if $(v_i, v_j) \in \mathcal{E}$. We say $\mathcal{G}$ has no self-loops if $(v_i, v_i) \notin \mathcal{E}$ for all vertices $v_i \in \mathcal{V}$, and will assume this. For all $e = (v_i, v_j) \in \mathcal{E}$, we define $\text{init}(e) = v_i$ and $\text{ter}(e) = v_j$. We represent the set of incoming and outgoing edges to a vertex $v_i \in \mathcal{V}$ by the sets $\mathcal{I}_i = \{e \in \mathcal{E} \mid \text{ter}(e) = v_i\}$ and $\mathcal{O}_i = \{e \in \mathcal{E} \mid \text{init}(e) = v_i\}$, respectively.

We define the following binary matrices. For a directed graph $\mathcal{G}$, we define the *vertex adjacency matrix* $A^{\mathcal{V}}_{ij} = 1$ iff $(v_j, v_i) \in \mathcal{E}$. Similarly we define the *edge adjacency matrix* $A^{\mathcal{E}}_{ij} = 1$ iff $\text{ter}(e_j) = \text{init}(e_i)$. We define the *input incidence matrix* $B^{\mathcal{I}}_{ij} = 1$ iff $v_i = \text{ter}(e_j)$, and the *output incidence matrix* $B^{\mathcal{O}}_{ij} = 1$ iff $v_i = \text{init}(e_j)$.

Networked System. Each vertex in $\mathcal{V}$ represents an independent plant with corresponding controller, as shown in Figure 1.

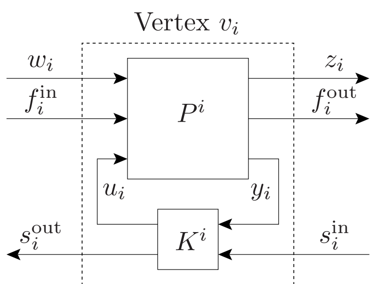

Figure 1: Plant and Controller for Each Vertex

From Figure 1, the plant P^i is affected by the control input $u_i \in U_i$, the influence of other plants connected to it $f_i^{\text{in}} \in F_i^{\text{in}}$, and noise $w_i \in W_i$. The measured output of the plant is $y_i \in Y_i$, and the plant influences other plants to which it is connected via $f_i^{\text{out}} \in F_i^{\text{out}}$. The controller K^i takes the measurement y_i and the signals sent from other subsystems' controllers $s_i^{\text{in}} \in S_i^{\text{in}}$, and produces the action u_i and $s_i^{\text{out}} \in S_i^{\text{out}}$, the signals sent to other subsystems' controllers.

Our goal is to express the overall system in the general feedback structure shown in Figure 2.

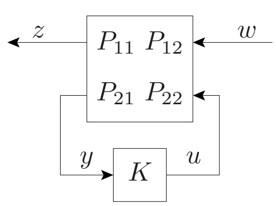

Figure 2: General Feedback System

Here, we have defined $U = \prod_i U_i$, and similarly for W, Y, and Z, so that $z \in Z$, $u \in U$, $y \in Y$, and $w \in W$. Then, P is the mapping $W \times U \to Z \times Y$. In this general feedback structure, the global plant P is constructed from the local plants $P^1, \ldots, P^N$. The global controller K is similarly constructed from the local controllers $K^1, \ldots, K^N$. We will assume, for the time being, that such constructions are possible. In Section 5, we will formalize this construction and provide conditions under which it is possible.

3 Sparsity Structures

Since we will be dealing with systems represented by directed graphs, we will need to deal with matrices that satisfy certain sparsity constraints. For any $m \times n$ matrix A, we define the set of matrices with similar sparsity structure by

$$\text{Sparse}(A) = \{B \in \mathbb{C}^{m\times n} \mid B_{ij} = 0 \text{ if } A_{ij} = 0\}$$

We define the function $\text{Bin} : \mathbb{C}^{m\times n} \to \mathbb{C}^{m\times n}$, such that

$$\text{Bin}(A)_{ij} = \begin{cases} 1 & A_{ij} \neq 0 \\ 0 & A_{ij} = 0 \end{cases}$$

Then $\text{Sparse}(A) = \text{Sparse}(\text{Bin}(A))$. We have the following straightforward properties:

$$\begin{aligned} \text{Sparse}(A)+\text{Sparse}(B) &= \text{Sparse}(\text{Bin}(A) + \text{Bin}(B)) \\ \text{Sparse}(A + B) &\subseteq \text{Sparse}(A) + \text{Sparse}(B) \\ \text{Sparse}(AB) &\subseteq \text{Sparse}(\text{Bin}(A)\,\text{Bin}(B)) \end{aligned} \tag{1}$$

We also have the following lemma; see the Appendix for the proof.

Lemma 1. *For any matrix* $A \in \mathbb{C}^{n\times n}$*, if* $\det(I-A) \neq 0$*, then* $\text{Sparse}(I - A)^{-1} \subseteq \text{Sparse}(I + \text{Bin}(A))^{n-1}$.

4 Graphs and Quadratic Invariance

Main Results Given the graphs specifying the interconnections of the plants and controllers, one may determine the sparsity and well-posedness of the overall plant and controller maps. In Section 5, it will be shown that this sparsity structure is given by $(I + A^{\mathcal{V}})^{N-1}$. This motivates one of the main results.

Theorem 2. *Suppose $\mathcal{G}$ is a directed graph. Let S be the subspace $S = \text{Sparse}(I + A^{\mathcal{V}})^{N-1}$, and suppose $P \in \mathbb{C}^{N\times N}$. Then S is quadratically invariant under P if and only if $P \in S$.*

Proof. To see the *if* direction, suppose $K, P \in S$. Using the proof of Lemma 1 and properties (1), we have

$$\begin{aligned} \text{Sparse}(KPK) &\subseteq \text{Sparse}(\text{Bin}(K)\,\text{Bin}(P)\,\text{Bin}(K)) \\ &\subseteq \text{Sparse}(I + A)^{3(N-1)} \\ &= \text{Sparse}(I + A)^{N-1} \end{aligned}$$

from which the result follows. Conversely, we show that if $P \notin S$ then there exists $K \in S$ such that $KPK \notin S$.

One such K is the identity matrix, which completes the proof. ■

Applying Theorem 2 pointwise in frequency, and using the results from [10], we can transform the problem

$$\begin{aligned} \text{minimize} \quad & \|P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}\| \\ \text{subject to} \quad & K \text{ stabilizes } P \\ & K \in \text{Sparse}(I + A^{\mathcal{V}})^{N-1} \end{aligned}$$

to a convex problem of the form

$$\begin{aligned} \text{minimize} \quad & \|T_{11} + T_{12}QT_{21}\| \\ \text{subject to} \quad & Q \in \text{Sparse}(I + A^{\mathcal{V}})^{N-1} \\ & Q \text{ is stable} \end{aligned}$$

Solving this optimization problem for Q, quadratic invariance implies that there exists a bijective mapping $Q \leftrightarrow K$ which preserves the sparsity structure. Namely, in the case when the plant is stable, $K = (I + QP_{22})^{-1}Q \in \text{Sparse}(I + A^{\mathcal{V}})^{N-1}$.

4.1 Multiple Graph Structures

The converse of Theorem 2 provides a very simple condition for quadratic invariance of distributed systems when the plant and controller structures are not the same. It tells us that the system is quadratically invariant, and thus amenable to solution by convex optimization, if and only if the plant dynamics have a sparsity structure whose transitive closure is a subset of that of the controller.

Interestingly, Theorem 2 does not require that the plant edges be a subset of the controller edges for quadratic invariance to hold. The condition required for quadratic invariance is $P_{22} \in S$.

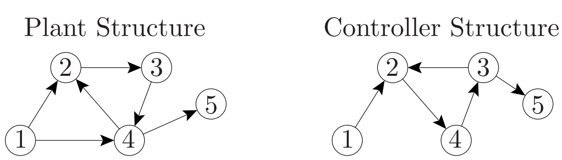

Figure 3: Multiple Graphs Example

Example 3. *Consider Figure 3, which shows a plant graph structure which is very different from the controller graph structure. Let* $A^{\mathcal{V}^P}$ *be the vertex-adjacency matrix of the plant, and* $A^{\mathcal{V}^K}$ *be that for the controller. Then*

$$A^{\mathcal{V}^P} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad A^{\mathcal{V}^K} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

However, we see that

$$(I + A^{\mathcal{V}^P})^{N-1} = (I + A^{\mathcal{V}^K})^{N-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Since $\text{Sparse}(I + A^{\mathcal{V}^P})^{N-1} \subseteq \text{Sparse}(I + A^{\mathcal{V}^K})^{N-1}$, *from Theorem 2, the set* $\text{Sparse}(I + A^{\mathcal{V}^K})^{N-1}$ *is quadratically invariant under the system* $P_{22} \in \text{Sparse}(I + A^{\mathcal{V}^P})^{N-1}$. *Hence, we can find an optimal controller for the system shown in Figure 3.*

Contrast Example 3 with the following fully decentralized problem.

Example 4. *Suppose the global controller is fully decentralized, so that* $S = \text{Sparse}(I)$. *Using Theorem 2, we have that* S *is quadratically invariant under* P_{22} *if and only if* $P_{22} \in \text{Sparse}(I)$. *In other words, this system is quadratically invariant if and only if the dynamics of all of the subsystems are completely decoupled. This result supports the well-known conclusion that such decentralized problems under more general plant structures are intractable.*

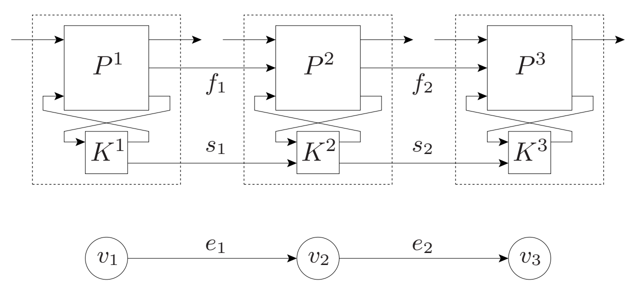

Figure 4: Connected Subsystems

5 Feedback Structure

Having established quadratic invariance of sparsity sets, we must now return to establish some of our underlying assumptions. To this end, some additional notation is needed. Figure 4 shows the interconnection of subsystems for a simple system. Our goal for this section is to express this system as seen in Figure 2.

Theorem 5. *Suppose* $\mathcal{G}$ *is a directed graph, and for each* $e_j \in \mathcal{E}$ *let* S_j *be a vector space, and* $S = \prod_j S_j$. *For each* $v_i \in \mathcal{V}$, *let* U_i *and* Y_i *be vector spaces, and* $U = \prod_i U_i$ *and similarly for* Y, *and define*

$$S_i^{in} = \prod_{j | e_j \in \mathcal{I}_i} S_j \qquad S_i^{out} = \prod_{j | e_j \in \mathcal{O}_i} S_j$$

For each $v_i \in \mathcal{V}$, *define* $N_i^{in} : S \to S_i^{in}$ *and* $N_i^{out} : S \to S_i^{out}$ *to be the natural projections.*

Suppose for each $v_i \in \mathcal{V}$ we have $K^i : Y_i \times S_i^{in} \to U_i \times S_i^{out}$. Then there exists $\hat{K} : Y \times S \to U \times S$ such that

$$\begin{bmatrix} \hat{K}_{11} & \hat{K}_{12} \\ \hat{K}_{21} & \hat{K}_{22} \end{bmatrix} \in \text{Sparse} \begin{bmatrix} I & B^{\mathcal{I}} \\ B^{\mathcal{O}T} & A^{\mathcal{E}} \end{bmatrix}$$

and $u \in U$, $y \in Y$, and $s \in S$ satisfy the local equations

$$\begin{bmatrix} u_i \\ N_i^{out} s \end{bmatrix} = K^i \begin{bmatrix} y_i \\ N_i^{in} s \end{bmatrix} \quad \textit{for all } i \tag{2}$$

if and only if

$$\begin{bmatrix} u \\ s \end{bmatrix} = \begin{bmatrix} \hat{K}_{11} & \hat{K}_{12} \\ \hat{K}_{21} & \hat{K}_{22} \end{bmatrix} \begin{bmatrix} y \\ s \end{bmatrix} \tag{3}$$

Proof. Here, $s_i \in S_i$ is the value of the signal on edge e_i. As shown in Figure 1, the controller at vertex v_i maps $K^i : Y_i \times S_i^{\text{in}} \to U_i \times S_i^{\text{out}}$, according to

$$u_i = K_{11}^i y_i + K_{12}^i s_i^{\text{in}}$$
$$s_i^{\text{out}} = K_{21}^i y_i + K_{22}^i s_i^{\text{in}}$$

The projection N_i^{in} is such that, if $s \in S$, then $s_i^{\text{in}} = N_i^{\text{in}} s$. To start, we partition the blocks of K^i, so that

$$\left[\begin{array}{c|c} K_{11}^i & K_{12}^i \\ \hline K_{21}^i & K_{22}^i \end{array}\right] = \left[\begin{array}{c|ccc} K_{11}^i & K_{12}^{ik_1} & \cdots & K_{12}^{ik_p} \\ \hline K_{21}^{j_1 i} & K_{22}^{j_1 ik_1} & \cdots & K_{22}^{j_1 ik_p} \\ \vdots & \vdots & \ddots & \vdots \\ K_{21}^{j_n i} & K_{22}^{j_n ik_1} & \cdots & K_{22}^{j_n ik_p} \end{array}\right] \tag{4}$$

where $\{j_1, \ldots, j_n\} = \{j \,|\, e_j \in \mathcal{O}_i\}$ and $\{k_1, \ldots, k_p\} = \{k \,|\, e_k \in \mathcal{I}_i\}$.

We now construct the map $\hat{K}$. It is clear from (2) and (4) that

$$\begin{aligned} u_i &= K_{11}^i y_i + K_{12}^i N_i^{\text{in}} s \\ &= K_{11}^i y_i + \sum_{\substack{k \\ e_k \in \mathcal{I}_i}} K_{12}^{ik} s_k \\ &= K_{11}^i y_i + \sum_{k=1}^{M} B_{ik}^{\mathcal{I}} (\hat{K}_{12})_{ik} s_k \end{aligned}$$

Hence, we construct $\hat{K}_{11} = \text{diag}(K_{11}^1, \ldots, K_{11}^N)$. Thus, $\hat{K}_{11} \in \text{Sparse}(I)$. Moreover, we see that $\hat{K}_{12}$ has the same sparsity structure as $B^{\mathcal{I}}$, and the non-zero entries of $\hat{K}_{12}$ correspond to those K_{12}^{ij} defined by (4). Hence, $\hat{K}_{12} \in \text{Sparse}(B^{\mathcal{I}})$ and

$$(\hat{K}_{12})_{ij} = \begin{cases} K_{12}^{ij} & B_{ij}^{\mathcal{I}} \neq 0 \\ 0 & B_{ij}^{\mathcal{I}} = 0 \end{cases} \tag{5}$$

Similarly, letting $K_{22}^{ij:}$ denote the i'th row of K_{22}^j, we have

$$\begin{aligned} s_i &= \sum_{\substack{j \\ e_i \in \mathcal{O}_j}} \left(K_{21}^{ij} y_j + K_{22}^{ij:} N_j^{\text{in}} s \right) \\ &= \sum_{\substack{j \\ e_i \in \mathcal{O}_j}} K_{21}^{ij} y_j + \sum_{\substack{j,k \\ e_k \in \mathcal{I}_j \\ e_i \in \mathcal{O}_j}} K_{22}^{ijk} s_k \\ &= \sum_{j=1}^{N} B_{ji}^{\mathcal{O}} (\hat{K}_{21})_{ij} y_j + \sum_{k=1}^{M} A_{ik}^{\mathcal{E}} (\hat{K}_{22})_{ik} s_k \end{aligned}$$

We note that the summation over j in the last expression has been eliminated since there exists only one vertex v_j satisfying $\text{ter}(e_k) = v_j = \text{init}(e_i)$. Thus, we see that $\hat{K}_{21}$ has the same sparsity structure as $(B^{\mathcal{O}})^T$ and $\hat{K}_{22}$ has the same sparsity structure as $A^{\mathcal{E}}$. Consequently, we have

$$(\hat{K}_{21})_{ij} = \begin{cases} K_{21}^{ij} & B_{ji}^{\mathcal{O}} \neq 0 \\ 0 & B_{ji}^{\mathcal{O}} = 0 \end{cases} \tag{6}$$

$$(\hat{K}_{22})_{ij} = \begin{cases} K_{22}^{ikj} & A_{ij}^{\mathcal{E}} \neq 0 \\ 0 & A_{ij}^{\mathcal{E}} = 0 \end{cases} \tag{7}$$

Again, we note that $(\hat{K}_{22})_{ij}$ is well-defined in (7) since K_{22}^{ikj} exists for exactly one k, whenever $A_{ij}^{\mathcal{E}} = 1$. Then, $\hat{K}_{21} \in \text{Sparse}(B^{\mathcal{O}})^T$ and $\hat{K}_{22} \in \text{Sparse}(A^{\mathcal{E}})$. ■

Straightforward modifications may be made to Theorem 5 in the cases when S_i^{in} or S_i^{out} are empty sets. Using this construction of $\hat{K}_{11}$, $\hat{K}_{12}$, $\hat{K}_{21}$, and $\hat{K}_{22}$, we can now precisely state when our system is well-posed.

Corollary 6. *Suppose $\mathcal{G}$ is a directed graph, and for each $v_i \in \mathcal{V}$ we have $K^i : Y_i \times S_i^{in} \to U_i \times S_i^{out}$. Let $\hat{K}$ satisfy the conditions of Theorem 5. Then for all $y \in Y$ there exists a unique $u \in U$ and $s \in S$ such that*

$$\begin{bmatrix} u_i \\ N_i^{out} s \end{bmatrix} = K^i \begin{bmatrix} y_i \\ N_i^{in} s \end{bmatrix} \quad \textit{for all } i$$

if and only if $I - \hat{K}_{22}$ is invertible.

If $K^1, \ldots, K^N$ satisfies the conditions of Corollary 6, we say that it is *well-posed*. For any controller which is well-posed, we see that the map $K : Y \to U$ is

$$K = \hat{K}_{11} + \hat{K}_{12}(I - \hat{K}_{22})^{-1} \hat{K}_{21} \tag{8}$$

We can show that every acyclic graph produces a well-posed controller, but we first need the following lemma.

Lemma 7. *Suppose $\mathcal{G}$ is a directed acyclic graph. Then, there exists a partition $\mathcal{V}_1, \ldots, \mathcal{V}_k$ of the vertex set $\mathcal{V}$ which satisfies*

i) $\mathcal{V} = \mathcal{V}_1 \cup \ldots \cup \mathcal{V}_k$

ii) $\mathcal{V}_i \cap \mathcal{V}_j = \emptyset$ *for all* $i \neq j$

iii) For all $e \in \mathcal{E}$*, if* $\text{ter}(e) \in \mathcal{V}_j$*, then* $\text{init}(e) \in \bigcup_{i=1}^{j-1} \mathcal{V}_i$

Proof. Due to space constraints, we omit the details of the proof here, but an outline is as follows. We must show that there exists a partition of $\mathcal{V}$, such that every parent vertex of a vertex $v \in \mathcal{V}_j$ is an element of one of the sets $\mathcal{V}_1, \ldots, \mathcal{V}_{j-1}$. This follows from the fact that we can group the parent-less vertices in the graph, remove them from the graph, and recursively perform this operation on the new acyclic subgraph. ■

With this lemma, we can prove the following theorem.

Theorem 8. *Suppose* $\mathcal{G}$ *is a directed graph, and for each* $v_i \in \mathcal{V}$ *we have* $K^i : Y_i \times S_i^{in} \to U_i \times S_i^{out}$*. If* $\mathcal{G}$ *is acyclic then* $K^1, \ldots, K^N$ *is well-posed.*

Proof. Since $\mathcal{G}$ is acyclic, we can partition the vertex set $\mathcal{V}$ into the sets $\mathcal{V}_1, \ldots, \mathcal{V}_k$ as in Lemma 7. Using this partition, let us order the edges $e_1, \ldots, e_M$ such that for all $e_i, e_j \in \mathcal{E}$, we have $i < j$ whenever $\text{init}(e_i) \in \mathcal{V}_a$ and $\text{init}(e_j) \in \mathcal{V}_b$ and $a < b$. In other words, we apply the hierarchical structure of our partitioned vertex set to edges in $\mathcal{E}$, so that edges with initial vertices in $\mathcal{V}_i$ are numbered ahead of edges whose initial vertices are in $\mathcal{V}_{i+1}$.

As a result of this ordering of the edges, we see that the edge adjacency matrix $A^{\mathcal{E}}$ is strictly lower triangular since every $e_i, e_j \in \mathcal{E}$ which satisfies $\text{ter}(e_j) = \text{init}(e_i)$ must have $i > j$. Since $\hat{K}_{22} \in \text{Sparse}(A^{\mathcal{E}})$, we see that $(I - \hat{K}_{22})$ is invertible, regardless of the value of the elements of $\hat{K}_{22}$. Hence, any controller is well-posed. ■

Example 3 demonstrates that the sparsity structure of K in (8) is not any of the adjacency or incidence matrices themselves, but is most likely some combination of them.

Lemma 9. *Consider a directed graph* $\mathcal{G}$ *and suppose the controller is well-posed. Then,*

$$\text{Sparse}(I + A^{\mathcal{V}})^{N-1} = \text{Sparse}\big(I + B^{\mathcal{I}}(I + A^{\mathcal{E}})^{M-1}(B^{\mathcal{O}})^T\big)$$

Proof. Since there are M total edges, $((I + A^{\mathcal{E}})^{M-1})_{ij} \neq 0$ if and only if there is a path connecting e_j to e_i or $i = j$. Similarly, it is straightforward to see that $(B^{\mathcal{I}}(I+A^{\mathcal{E}})^{M-1}(B^{\mathcal{O}})^T)_{ij} \neq 0$ if and only if there exists a path connecting vertex v_j to vertex v_i. Lastly, it is clear that $(I + B^{\mathcal{I}}(I + A^{\mathcal{E}})^{M-1}(B^{\mathcal{O}})^T)_{ij} \neq 0$ if and only if there exists a path connecting v_j to v_i or $i = j$. Similarly, $((I + A^{\mathcal{V}})^{N-1})_{ij} \neq 0$ if and only if there exists a path connecting v_j to v_i or $i = j$, giving the desired result. ■

Theorem 10. *Suppose* $\mathcal{G}$ *is a directed graph,*

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \in \text{Sparse} \begin{bmatrix} I & B^{\mathcal{I}} \\ B^{\mathcal{O}T} & A^{\mathcal{E}} \end{bmatrix}$$

and $I - K_{22}$ *is invertible, and let* $K = K_{11} + K_{12}(I - K_{22})^{-1}K_{21}$*. Then*

$$K \in \text{Sparse}(I + A^{\mathcal{V}})^{N-1}$$

Proof. We have $K \in \text{Sparse}(K_{11}+K_{12}(I-K_{22})^{-1}K_{21})$. Using Lemma 1, Lemma 9 and (1), we see that

$$\begin{aligned} &\text{Sparse}(K_{11} + K_{12}(I - K_{22})^{-1}K_{21}) \\ &\quad \subseteq \text{Sparse}(I + \text{Bin}(K_{12})\,\text{Bin}((I - K_{22})^{-1})\,\text{Bin}(K_{21})) \\ &\quad \subseteq \text{Sparse}(I + B^{\mathcal{I}}(I + A^{\mathcal{E}})^{M-1}(B^{\mathcal{O}})^T) \\ &\quad = \text{Sparse}(I + A^{\mathcal{V}})^{N-1} \end{aligned}$$

Hence, $K \in \text{Sparse}(I + A^{\mathcal{V}})^{N-1}$. ■

Theorem 10 is important since it provides a simple means to find the global interconnections of the subsystems given the local interconnections described by $\mathcal{G}$. It is straightforward to show that the results of this section apply equally to constructing P_{22} and the well-posedness of the plant.

6 Reconstructing the Local Controllers

Our last issue to consider before we can solve this problem is finding the individual controller matrices $K^1, \ldots, K^N$ from the global controller $K \in \text{Sparse}(I + A^{\mathcal{V}})^{N-1}$.

Theorem 11. *Suppose* $\mathcal{G}$ *is a directed graph, and* S*,* U *and* Y *are as in Theorem 5. Suppose*

$$K \in \text{Sparse}(I + A^{\mathcal{V}})^{N-1}$$

Then there exist $K^i : Y_i \times S_i^{in} \to U_i \times S_i^{out}$ *such that* $u \in U$*,* $y \in Y$*, and* $s \in S$ *satisfy the local equations*

$$\begin{bmatrix} u_i \\ N_i^{out} s \end{bmatrix} = K^i \begin{bmatrix} y_i \\ N_i^{in} s \end{bmatrix} \quad \text{for all } i \tag{9}$$

if and only if $u = Ky$*.*

Proof. We explicitly construct such a set $K^1, \ldots, K^N$. However, due to space constraints, we only provide an outline to this construction. Since K is simply a mapping of $Y \to U$, we have freedom in choosing the signals $s_1, \ldots, s_M$. For convenience, let $U = (I + A^{\mathcal{V}})^{N-1}$. We construct the signals to simply pass observations between vertices. In other words, every signal is a subset of $\{y_1, \ldots, y_N\}$. To this end, for each vertex $v_j \in \mathcal{V}$, we find a subgraph $H_j \subset \mathcal{G}$ which is a maximally connected directed acyclic tree (with no undirected cycles) with v_j as the only root. As a result, if $U_{ij} = 1$, there exists a unique path in H_j from v_j to v_i. Thus, if $e_k \in \mathcal{E}(H_j)$, we let $y_j \in s_k$. This construction specifies K_{21}^j and K_{22}^j for each $v_j \in \mathcal{V}$.

To construct K_{11}^j and K_{12}^j, since the signals are constructed such that $s_j^{\text{in}} = \{y_k \,|\, U_{jk} = 1 \text{ and } j \neq k\}$, we let $K_{11}^j = K_{jj}$ and $K_{12}^j = \begin{bmatrix} K_{jk_1} & \cdots & K_{jk_n} \end{bmatrix}$, where $\{k_1, \ldots, k_n\} = \{k \,|\, U_{jk} = 1 \text{ and } j \neq k\}$. As a result, we have constructed K^j satisfying (9), for all $v_j \in \mathcal{V}$. Moreover, since we explicitly constructed each signal s_k

to be a given subset of Y, it is clear that s_k is uniquely determined for every $y \in Y$. Hence, our construction for the local controllers $K^1, \ldots, K^N$ produces a well-posed controller. ∎

7 Conclusion

In this paper, we showed that the interconnection of individual systems according to an underlying graph structure produced constraints on the set of acceptable controllers which were simply sparsity constraints. We provided the conditions for such systems to be well-posed. Also, provided that the graph structures of the individual plants and controllers satisfy certain conditions, we demonstrated that the resulting sparsity constraints were quadratically invariant. Interestingly, this result allowed for systems where the plant and controller graph structures are different. This result implies that existing convex programming methods can be used to solve such problems.

Lastly, we provided a construction for the optimal individual controllers. However, it is certainly possible that other constructions may exist which reduce the size of messages being passed among the controllers.

Appendix

Proof of Lemma 1. In this proof, we make use of Boolean algebra, so that addition corresponds to logical OR. Consider the spectral radius $\rho(A)$, and suppose $0 < k < \frac{1}{\rho(A)}$ (if $\rho(A) = 0$, suppose $k > 0$). Then, we know $(I - kA)^{-1} = I + kA + k^2A^2 + \ldots$. As a result, from properties (1), we know

$$\text{Sparse}(I - kA)^{-1} \subseteq \text{Sparse}\left(\sum_{j=0}^{\infty} \text{Bin}(A)^j\right)$$

Because we are using Boolean algebra, the series converges. Now, we will show that we can truncate this series to the first n terms. For any $m \in \mathbb{Z}^+$, $(\text{Bin}(A)^m)_{ij} = 1$ if and only if there exist $k_1, k_2, \ldots, k_{m-1} \in \{1, \ldots, n\}$ such that $\text{Bin}(A)_{ik_1} = \text{Bin}(A)_{k_1k_2} = \ldots = \text{Bin}(A)_{k_{m-1}j} = 1$.

Since each element in the sequence $i, k_1, \ldots, j$ is an element of $\{1, \ldots, n\}$, then longest sequence of unique elements is of length n. As a result, if there exists a sequence of length $m > n$, then there must also exist a sequence of length less than or equal to n, which has unique elements. In other words, for any $m > n-1$, $(\text{Bin}(A)^m)_{ij} = 1$ only if $(\text{Bin}(A)^q)_{ij} = 1$ for some $q \in \{0, \ldots, n-1\}$. This implies that $(\text{Bin}(A)^m)_{ij} = 1$ only if $\sum_{q=0}^{n-1}(\text{Bin}(A)^q)_{ij} = 1$. Thus, $\sum_{q=0}^{n-1} \text{Bin}(A)^q = \sum_{q=0}^{\infty} \text{Bin}(A)^q$.

Now, consider the zero elements of $\sum_{q=0}^{n-1} \text{Bin}(A)^q$. Suppose $\left(\sum_{q=0}^{n-1} \text{Bin}(A)^q\right)_{ij} = 0$. From above, we must have $((I - kA)^{-1})_{ij} = 0$. However, we also know that $((I - kA)^{-1})_{ij} = (\det(I - kA))^{-1}(\text{adj}(I - kA))_{ij}$, which is a rational expression in k. We know from complex analysis that a rational expression which is not identically zero can be equal to zero at only a finite number of points. Since $((I - kA)^{-1})_{ij} = 0$ for all $0 < k < \frac{1}{\rho(A)}$, then $((I - kA)^{-1})_{ij}$ must be identically zero for all k, including when $k = 1$. As a result, since $(I - A)^{-1}$ exists, $\text{Sparse}((I - A)^{-1}) \subseteq \text{Sparse}(\sum_{q=0}^{n-1} \text{Bin}(A)^q)$. Lastly, since $(I + \text{Bin}(A))^{n-1} = \sum_{q=0}^{n-1} \text{Bin}(A)^q$, we have $\text{Sparse}((I - A)^{-1}) \subseteq \text{Sparse}(I + \text{Bin}(A))^{n-1}$. ∎

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