Proceedings of the IFAC Symposium on Nonlinear Control Systems (NOLCOS), 2004

CONSTRUCTING LYAPUNOV FUNCTIONS FOR NONLINEAR DELAY-DIFFERENTIAL EQUATIONS USING SEMIDEFINITE PROGRAMMING

Matthew Peet [*,1] **Sanjay Lall** [**]

[*] *Department of Aeronautics and Astronautics 4035, Stanford University, Stanford CA 94305-4035, U.S.A.*
mmpeet@stanford.edu
[**] *Department of Aeronautics and Astronautics 4035, Stanford University, Stanford CA 94305-4035, U.S.A.*
lall@stanford.edu

Abstract: The search for a polynomial Lyapunov function proving delay-independent stability of multivariate nonlinear polynomial delay differential equations is approached using semidefinite programming. The functional non-negativity constraints are tightened to be sum of squares constraints, a condition which is computationally feasible to check. The algorithm uses recent advances in computational semi-algebraic geometry.

Keywords: Algebraic, Algorithm, Convex, Delay, Lyapunov, Nonlinear, Stability

1. INTRODUCTION

The purpose of this paper is develop a computational procedure using optimization tools to determine stability of nonlinear delay differential equations, of the form

$$\dot{x}(t) = f(x(t), x(t-\tau))$$

In this paper, we consider the case of delay-independent stability. We would like to determine whether stability holds for arbitrary values of $\tau > 0$. Here f is a possibly nonlinear polynomial and $x \in \mathbb{R}^n$. The case where f is linear has been thoroughly researched using both time and frequency domain techniques. See, for example, the survey paper Kharitonov (1999) or the comprehensive works by Niculescu (2001) and Kolmanovskii and Myshkis (1999) for an overview of the subject. It is interesting to note, however, that if one adds multiple non-commensurate delays, a tractable nonconservative algorithm is unlikely to exist since this more general problem has been shown to be NP-hard in Toker and Ozbay (1996).

In the case where f is nonlinear with no special structure, there are effectively no reasonably nonconservative, computationally tractable procedures for determining stability. However, various computational tests do exist for cases when f has special structure. These results are mostly based on Razumikin theory, Lyapunov functionals or modeling the nonlinearity as uncertainty. See, e.g. Bliman (2002) and Verriest and Aggoune (1998) as well as the references cited above. In this paper, we require only that the delay system be described by a polynomial functional.

The approach of using Lyapunov functionals for infinite dimensional systems was pioneered by Krasovskii (1963). This report details a new algorithm for constructing such functionals to

[1] Both authors were partially supported by the Stanford URI *Architectures for Secure and Robust Distributed Infrastructures*, AFOSR DoD award number 49620-01-1-0365.

prove delay-independent stability when f is an arbitrary polynomial. The algorithm searches for a valid polynomial Lyapunov functional by tightening polynomial non-negativity constraints to be sum of squares constraints. We use recent work by Parrilo (2000) to show that a search for a standard form of polynomial Lyapunov functional of bounded order that proves delay-independent stability of a delay-differential equation is equivalent to testing feasibility of a semidefinite program.

In Section 2.2, we present some background on delay-differential equations. In Section 2.3 we show how Lyapunov theory is extended via Lyapunov functionals to delay-differential equations. In Section 2.4 we discuss sum-of-squares programming. In Section 3 we present our algorithm. In Section 4 we compute a simple numerical example. Finally, in Section 5 we talk about possible generalizations and extensions of this result not explicitly presented in this paper.

2. BACKGROUND

2.1 Notation

Let $\mathbb{R}_+ = \{\, x \in \mathbb{R} \mid x \geq 0 \,\}$, and define the set of continuous functions

$$\mathcal{D} = \{\, u : \mathbb{R}_+ \to \mathbb{R}^n \mid u \text{ is continuous} \,\}$$

We also define the norm

$$\|u\| = \sup_{t \in \mathbb{R}_+} \|u(t)\|_2$$

and the associated Banach space $\mathcal{C}$

$$\mathcal{C} = \{\, u \in \mathcal{D} \mid \|u\| \text{ is finite} \,\}$$

We also use $\mathcal{C}_\tau$ to denote the Banach space of continuous functions $u : [-\tau, 0] \to \mathbb{R}^n$ with the same norm. We will also make use of the Banach space

$$\mathcal{D}_0 = \{\, u \in \mathcal{C} \mid \lim_{t \to \infty} u(t) = 0 \,\}$$

For a polynomial f, the degree of f is denoted by $\deg(f)$.

2.2 Delay-Differential Equations

We consider delay-differential equations of the following form

$$\dot{x}(t) = f\big(x(t), x(t-\tau)\big) \tag{1}$$

where $f(0,0) = 0$. We assume that for each $y \in \mathcal{C}_\tau$ there exists a unique $x \in \mathcal{D}$ such that x satisfies (1) and

$$x(t) = y(t - \tau) \qquad \text{for all } t \in [0, \tau] \tag{2}$$

That is, the system (1) has a unique solution $x \in \mathcal{D}$ for all initial conditions $y \in \mathcal{C}_\tau$. In this case the system defines a map

$$\Phi : \mathcal{C}_\tau \to \mathcal{D}$$

where $x = \Phi y$ if and only if x and y satisfy (1) and (2). The function $x \in \mathcal{C}$ is called a ***solution*** of (1) with initial condition $y \in \mathcal{C}_\tau$.

One can associate with solution x and time t an element $x_t \in \mathcal{C}_\tau$, where $x_t(\theta) = x(t + \theta)$ for all $\theta \in [-\tau, 0]$. This element is called the ***state*** of the system at time t. Furthermore, define the ***flow map*** $\Gamma : \mathcal{C}_\tau \times \mathbb{R}_+ \to \mathcal{C}_\tau$ by $x = \Gamma(y, t)$ if

$$x(s) = (\Phi y)(s + t + \tau) \quad \text{for all } s \in [-\tau, 0]$$

which maps the state at time t_0 to the state at time $t_0 + t$ for any $t_0 \geq \tau$.

The system of differential equations in (1) is infinite dimensional, and hence the flow map operates on the infinite-dimensional state-space $\mathcal{C}_\tau$, as illustrated in Figure 1. In this paper, we assume f is a polynomial. This does not guarantee global existence of solutions, however existence and uniqueness are guaranteed over some interval since f is locally Lipschitz continuous; see Hale and Lunel (1993); Krasovskii (1963); Kolmanovskii and Nosov (1986).

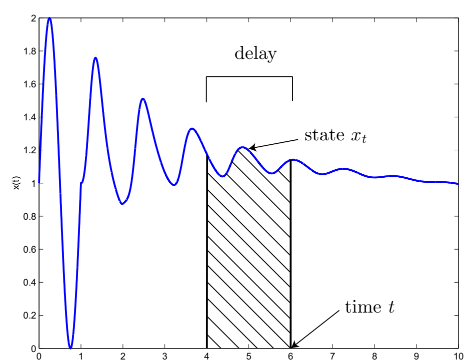

Fig. 1. The state x_t of the system at time t

Definition 1. The system (1) defined by f is ***globally stable*** if

(i) The map $\Phi : \mathcal{C}_\tau \to \mathcal{C}$
(ii) Φ is continuous at 0 with respect to the norms on $\mathcal{C}$ and $\mathcal{C}_\tau$.

This is the usual notion of Lyapunov stability, which states that for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\|y\| < \delta$ implies $\|\Phi y\| < \varepsilon$.

Definition 2. The system (1) defined by f is ***globally asymptotically stable*** if it is globally stable and $\Phi : \mathcal{C}_\tau \to \mathcal{D}_0$.

2.3 Lyapunov Functionals

Since the state of system (1) is infinite dimensional, finite-dimensional Lyapunov functions $V : \mathbb{R}^n \to \mathbb{R}$ can only account for a small amount of the energy in the system. Functions which only consider $x(t)$ fail to account for energy that may remain in the system but has been temporarily hidden by the delay. An infinite dimensional approach to Lyapunov stability is as follows.

Suppose $V : \mathcal{C}_\tau \to \mathbb{R}$. Define the ***right upper Lie derivative*** $D^+V : \mathcal{C}_\tau \to \mathbb{R}$ by

$$D^+V(\phi) = \limsup_{h\to 0^+} \frac{1}{h}\Big(V(\Gamma(\phi,h)) - V(\phi)\Big)$$

The following theorem is from Hale and Lunel (1993).

Theorem 3. Suppose f is continuous. Let $a : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous non-decreasing function such that $a(0) = 0$, $a(t) > 0$ for $t > 0$, and $a(t) \to \infty$ as $t \to \infty$. Also let $b : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous non-decreasing function such that $b(0) = 0$ and $b(t) > 0$ for $t > 0$. Similarly let $c : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous non-decreasing function.

If $V : \mathcal{C}_\tau \to \mathbb{R}_+$ is continuous, and satisfies

(i) $a\big(\|\phi(0)\|\big) \le V(\phi) \le b\big(\|\phi\|\big)$ for all $\phi \in \mathcal{C}_\tau$;
(iii) $D^+V(\phi) \le -c\big(\|\phi(0)\|\big)$ for all $\phi \in \mathcal{C}_\tau$;

then the system (1) is globally stable. Further, if $c(t) > 0$ for $t > 0$ then the system is globally asymptotically stable.

2.4 Sum of Squares Programming

A polynomial, $f(x)$ can be expressed as the linear combination of monomials

$$f(x) = \sum_{i=1}^{m} c_i x_1^{\gamma_{i,1}} \cdots x_n^{\gamma_{i,n}} \quad \gamma_{i,j} \in \mathbb{Z}^+ \qquad (3)$$

The question of whether a polynomial is globally non-negative, that is, $f(x) \ge 0$ for all $x \in \mathbb{R}^n$, is generally hard to answer. However, a sufficient condition for global non-negativity of f is the existence of a *sum of squares* (SOS) representation. The polynomial f is globally non-negative if, for some polynomials g_i, f can be written as

$$f(x) = \sum_i g_i(x)^2$$

Clearly, if f is SOS that f is globally non-negative. In the case of 1 or 2 variables, quartic forms in three variables and all quadratic forms, global non-negativity is actually equivalent to existence of a SOS representation. In Parrilo (2000), it has been shown that the existence of a bounded degree SOS representation of a polynomial is equivalent to a semidefinite program with equality constraints expressed in terms of the monomial coefficients in (3).

More specifically, if $\deg(f)$ is even, let z_1 be the vector of n_1 monomials of degree less than or equal to $\deg(f)$ and z_2 be the vector of n_2 monomials of degree less than or equal to $\deg(f)/2$. Define c to be the vector such that $c^T z_1 = f(x)$. Then f is SOS if and only if there exists a $Q \ge 0$ such that $f(x) = c^T z_1 = z_2^T Q z_2$. Equating coefficients gives a set of affine constraints on Q. Thus f is SOS if and only if there exists a $Q \ge 0$ satisfying these affine constraints.

In this manner, polynomial non-negativity constraints can be tightened to SOS constraints, which can be tested using semidefinite programming. Furthermore, because the equality constraints are linear in the coefficients of f and Q, one can optimize over all polynomials of bounded degree with the constraint that the polynomial be globally non-negative.

Parrilo (2000) shows that if f is a polynomial, there is a simple, computationally tractable sufficient condition for the existence of a polynomial Lyapunov function to prove stability of solutions of

$$\dot{x}(t) = f(x(t))$$

The condition is the existence of polynomials $p(x), q(x)$ and constant $\beta_1 > 0$ such that $p(0) = 0$, $p(x) - \beta_1 x^T x$ is SOS and

$$q(x) = -\nabla p(x)^T f(x) \quad \text{is SOS}$$

If such polynomials exist, then $p(x)$ is a Lyapunov function proving stability of the system. If $q(x) - \beta_2 x^T x$ is SOS for some $\beta_2 > 0$, then we have asymptotic stability.

3. ALGORITHM

For polynomials p_1, p_2, consider the following candidate *Lyapunov-Krasovskii functional*

$$V(\phi) = p_1(\phi(0)) + \int_{-\tau}^{0} p_2(\phi(\theta))\, d\theta$$

Using Leibniz's rule, we obtain the derivative

$$\dot{V}(\phi) = q\big(\phi(0), \phi(-\tau)\big)$$

where q is given by

$$q(a,b) = \nabla p_1(a)^T f(a,b) + p_2(a) - p_2(b)$$

This particular form of functional is a generalization of a standard functional used to prove delay-independent stability for linear delay systems. This functional is useful for proving delay-independent stability because q does not depend on τ. To arrive at a computationally feasible condition, we simply tighten the constraint that the derivative is negative for all $\phi(t)$ and $\phi(t-\tau)$ such that ϕ is a solution of (1) to be that the q is negative for all $a, b \in \mathbb{R}^n$.

Lemma 4. Suppose there exist $\alpha_1 > 0$ and polynomials p_1, p_2, and q such that $p_1(0) = p_2(0) = 0$. Furthermore, suppose $p_1(x) - \alpha_1 x^T x$ is SOS and

$$q(x, y) = -(\nabla p_1(x)^T f(x, y) + p_2(x) - p_2(y))$$

is SOS. Then equation (1) is stable. Furthermore, if in addition $q(x, y) - \alpha_2 x^T x$ is SOS for some $\alpha_2 > 0$, then the system is asymptotically stable.

Proof: Since p_1 and p_2 are SOS, they are globally non-negative with $p_1(0) = p_2(0) = 0$. Furthermore $p_1(x) \geq \alpha_1 x^T x$ for some $\alpha_1 > 0$. Now since for the given polynomial $f(x, y)$,

$$q(x, y) = -(\nabla p_1(x)^T f(x, y) + p_2(x) - p_2(y))$$

Then $q(\phi(0), \phi(-\tau)) = -\dot{V}(\phi)$. By Theorem 3, if q is globally non-negative, as implied by the SOS condition, then the system (1) is stable. Furthermore, if $q(x, y) - \alpha_2 x^T x$ is globally non-negative for some $\alpha_2 > 0$, then the system is asymptotically stable. □

Because the coefficients of the polynomials p_1, p_2 enter linearly into the expression for q, we can express the above stability condition as a set of semidefinite constraints. Therefore, this stability condition is readily implementable as a semidefinite program either directly using interior point solvers such as Sedumi as documented in Sturm (1999) or using SOS optimization toolboxes such as SOStools as described by Parrilo (2004).

4. NUMERICAL EXAMPLE

This paper shows that Lyapunov functionals proving delay-independent stability of a system of equations may be computed algorithmically. In the numerical example presented here, we consider a simple model for disease pathology developed by Mackey and Glass (1977) and presented by Kuang (1993). Here, β, γ, θ and n are positive constants and $p(t)$ is positive and represents the number of mature blood cells in circulation, while τ represents the delay between between cell production and cell maturation.

$$\dot{p}(t) = \frac{\beta \theta^n p(t-\tau)}{\theta^n + p^n(t-\tau)} - \gamma p(t)$$

For simplicity, we consider the case $\theta = 1$. Using the Positivstellensatz, given an n, we construct for a parameter varying Lyapunov functional which proves stability for positive initial conditions and for all positive β, γ, τ such that $\gamma > \beta + .01$. We find that the corresponding SDP is feasible for at least $n \leq 6$. This proves global stability for all such γ. The resulting Lyapunov functional is omitted due to space constraints. In the specific case when $\beta = .5$ and $\gamma = .6$, this functional is

$$\begin{aligned} V(\phi) &= 14.6\phi(0)^2 + 3.68\phi(0)^3 + 5.66\phi(0)^4 \\ &\quad + 1.42\phi(0)^5 + 1.95\phi(0)^6 \\ &\quad + \int_{-\tau}^{0} 8.32\phi(\theta)^2 + 2.57\phi(\theta)^3 + 1.26\phi(\theta)^4 \, d\theta \end{aligned}$$

This functional is plotted in the following figures along a trajectory of the system for a delay of 1 second.

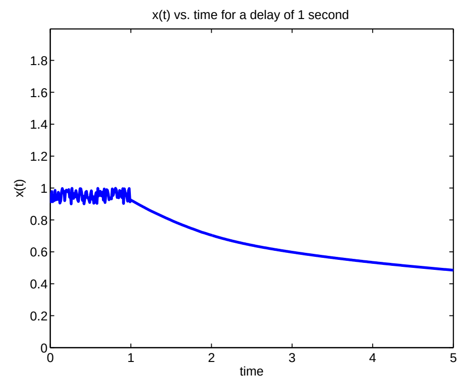

Fig. 2. Trajectory of $x(t)$

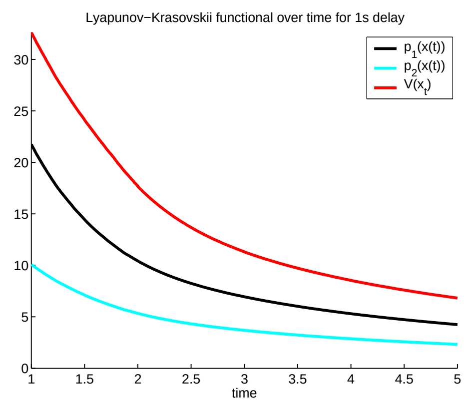

Fig. 3. Trajectory of $V(x_t)$

5. CONCLUSION

This paper provides a polynomial-time algorithm for constructing Lyapunov functionals of bounded degree that prove delay-independent stability of time-delay systems whose dynamics are given by polynomials. As illustrated in the example, this method is easily extended to parameter varying Lyapunov functionals and the case when dynamics are given by rational functions. Also illustrated in the example is the restriction of domain of attractive to an invariant region, namely $p(t) \geq 0$. As an extension, one may also use more complicated candidate functionals to derive delay-dependent stability conditions. Another extension is to compute Lyapunov functionals for more general forms of functional differential equations, such as the case of continuous dependence and neutral delay differential equations. One may also use the Positivstellensatz as discussed in Parrilo and Lall (2003) to reduce conservatism of the SOS approximation to non-negativity, and to include non-polynomial terms which can be parameterized using polynomials.

REFERENCES

P.-A. Bliman. Stability of non-linear delay systems: delay-independent small-gain theorem and frequency domain interpretation of the Lyapunov-Krasovskii method. *International Journal of Control*, 75(4):265–274, 2002.

J. K. Hale and S. M. Lunel. *Introduction to Functional Differential Equations*, volume 99 of *Applied Mathematical Sciences*. Springer-Verlag, 1993.

V. L. Kharitonov. Robust stability analysis of time delay systems: a survey. *Annual Reviews in Control*, 23:185–196, 1999.

V. Kolmanovskii and A. Myshkis. *Introduction to the Theory and Applications of Functional Differential Equations*. Kluwer Academic Publishers, 1999.

V.B. Kolmanovskii and V.R. Nosov. *Stability of Functional Differential Equations*, volume 180 of *Mathematics in Science and Engineering*. Academic Press, 1986.

N. N. Krasovskii. *Stability of Motion*. Stanford University Press, 1963.

Y. Kuang. *Delay Differential Equations with Applications in Population Dynamics*, volume 191 of *Mathematics in Science and Engineering*. Academic Press, 1993.

M. C. Mackey and L. Glass. Oscillations and chaos in physiological control systems. *Science*, 197: 287–289, 1977.

S.-I. Niculescu. *Delay Effects on Stability: A Robust Control Approach*, volume 269 of *Lecture Notes in Control and Information Science*. Springer-Verlag, May 2001.

P. A. Parrilo. *Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization*. PhD thesis, California Institute of Technology, 2000.

P. A. Parrilo. Web site for SOSTOOLS. `http://control.ee.ethz.ch/~parrilo/sostools/index.html`, 2004.

P. A. Parrilo and S. Lall. Semidefinite programming relaxations and algebraic optimization in control. *European Journal of Control*, 9(2-3): 307–321, 2003.

J. F. Sturm. Using SeDuMi 1.02, a matlab toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11-12:625–653, 1999. Version 1.05 available at `http://fewcal.kub.nl/sturm/software/sedumi.html`.

O. Toker and H. Ozbay. Complexity issues in robust stability of linear delay-differential systems. *Mathematics of Control Signals and Systems*, 9:386–400, 1996.

E. I. Verriest and W. Aggoune. Stability of nonlinear differential delay systems. *Mathematics and Computers in Simulation*, 45:257–267, 1998.