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Convexification of Optimal Decentralized Control Without a Stabilizing Controller

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Abstract

The problem of finding an optimal decentralized controller is considered, where both the plant and the controllers under consideration are rational. It has been shown that a condition called quadratic invariance, which relates the plant and the constraints imposed on the desired controller, allows the optimal decentralized control problem to be cast as a convex optimization problem, provided that a controller is given which is both stable and stabilizing. This paper shows how, even when such a controller is not provided, the optimal decentralized control problem may still be cast as a convex optimization problem, albeit a more complicated one. The solution of the resulting convex problem is then discussed.

The result that quadratic invariance gives convexity is thus extended to all finite-dimensional linear problems. In particular, this result may now be used for plants which are not strongly stabilizable, or for which a stabilizing controller is simply difficult to find. The results hold in continuous-time or discrete-time.

1 Introduction

The problem of finding an optimal decentralized controller is considered, where both the plant and the controllers under consideration are rational. It has been shown that a condition called quadratic invariance [4], which relates the plant and the constraints imposed on the desired controller, allows the optimal decentralized control problem to be cast as a convex optimization problem.

When the plant is unstable, these results rely upon the existence of a nominal controller which is both stable and stabilizing. However, finding such a controller may be a difficult task, or in some cases, one may not exist at all. This paper shows how, even when such a controller is not provided, the optimal decentralized control problem may still be cast as a convex optimization problem, albeit a more complicated one. The solution of the resulting convex problem is then discussed.

The result that quadratic invariance gives convexity is thus extended to all finite-dimensional linear problems. We further see that the techniques in this paper result in a standard unconstrained control problem whose solution yields stabilizing decentralized controllers for this large class of problems.

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2 Preliminaries

We consider a generalized plant $P \in \mathcal{R}_p^{(n_z+n_y)\times(n_w+n_u)}$, partitioned as in Figure 1. Note that we refer to the 2-2 block simply as G.

For all $K \in \mathcal{R}_p$, we define the ***closed-loop map*** $f(P,K) \in \mathcal{R}_p^{n_z\times n_w}$ as

$$f(P,K) = P_{11} + P_{12}K(I - GK)^{-1}P_{21}$$

and this is also called the (lower) ***linear fractional transformation*** (LFT) of P and K.

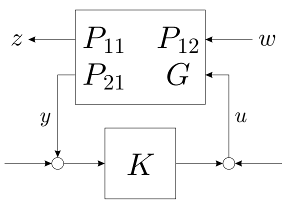

Figure 1: Linear fractional interconnection of P and K

2.1 Stabilization

We say that K ***stabilizes*** P if in Figure 1 the nine transfer matrices from (w, v_1, v_2) to (z, u, y) belong to $\mathcal{RH}_\infty$. We say that K ***stabilizes*** G if in the figure the four transfer matrices from (v_1, v_2) to (u, y) belong to $\mathcal{RH}_\infty$. P is called ***stabilizable*** if there exists $K \in \mathcal{R}_p^{n_u\times n_y}$ such that K stabilizes P, and it is called ***strongly stabilizable*** if there exists $K \in \mathcal{RH}_\infty^{n_u\times n_y}$ such that K stabilizes P. We denote by $C_{\text{stab}} \subseteq \mathcal{R}_p^{n_u\times n_y}$ the set of controllers $K \in \mathcal{R}_p^{n_u\times n_y}$

which stabilize P. The following standard result relates stabilization of P with stabilization of G.

Theorem 1. *Suppose $G \in \mathcal{R}_{sp}^{n_y \times n_u}$ and $P \in \mathcal{R}_p^{(n_z+n_y)\times(n_w+n_u)}$, and suppose P is stabilizable. Then K stabilizes P if and only if K stabilizes G.*

Proof. See, for example, Chapter 4 of [1]. ∎

2.2 Kronecker products

Given $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{s\times q}$ let the *Kronecker product* of A and B be denoted by $A \otimes B$ and given by

$$A \otimes B = \begin{bmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & & \vdots \\ A_{m1}B & \cdots & A_{mn}B \end{bmatrix} \in \mathbb{C}^{ms\times nq}$$

Given $A \in \mathbb{C}^{m\times n}$, we may write A in term of its columns as

$$A = \begin{bmatrix} a_1 & \ldots & a_n \end{bmatrix}$$

and then associate a vector $\mathrm{vec}(A) \in \mathbb{C}^{mn}$ defined by

$$\mathrm{vec}(A) = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

Lemma 2. *Let $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{s\times q}, X \in \mathbb{C}^{n\times s}$. Then*

$$vec(AXB) = (B^T \otimes A)vec(X)$$

Proof. See, for example, [2]. ∎

2.3 Problem Formulation

The optimization problem we address is as follows. Given $P \in \mathcal{R}_p^{(n_z+n_y)\times(n_w+n_u)}$, and a subspace of admissible controllers $S \subseteq \mathcal{R}_p^{n_u\times n_y}$, we would like to solve:

$$\begin{aligned} \text{minimize} \quad & \|f(P,K)\| \\ \text{subject to} \quad & K \text{ stabilizes } P \\ & K \in S \end{aligned} \tag{1}$$

Here $\|\cdot\|$ is any norm on $\mathcal{R}_p^{n_z\times n_w}$, chosen to encapsulate the control performance objectives, and S is a subspace of admissible controllers which encapsulates the decentralized nature of the system. All of the results regarding convexification in this paper apply for arbitrary norm, but we will limit ourselves to the $\mathcal{H}_2$-norm when we discuss further reduction of the problem to unconstrained problems.

Most decentralized control problems may be formulated in this manner, with the subspace S typically being defined by sparsity constraints or delay constraints. We often refer to S as the ***information constraint***.

The cost function $\|f(P,K)\|$ is in general a nonconvex function of K, and no computationally tractable approach is known for solving this problem for arbitrary P and S.

2.4 Feedback Map

We define the map $h : \mathcal{R}_{sp} \times \mathcal{R}_p \to \mathcal{R}_p$ by

$$h(G,K) = -K(I - GK)^{-1}$$

We will also make use of the notation $h_G(K) = h(G,K)$. Given $G \in \mathcal{R}_{sp}$, we note that h_G is an involution on $\mathcal{R}_p$, as a straightforward calculation shows that $h_G(h_G(K)) = K$.

2.5 Quadratic Invariance

In this subsection we define quadratic invariance, and give a brief overview of results regarding this condition, in particular, that it renders the information constraint invariant under a feedback map, and that it allows for convex synthesis of optimal decentralized controllers when a stable and stabilizing controller is provided.

Definition 3. *The set S is called* ***quadratically invariant*** *under G if*

$$KGK \in S \qquad \text{for all } K \in S$$

The following is a special case of the main theorem of [4] and was first proved with this level of generality in [3]. It states that quadratic invariance of the constraint set is necessary and sufficient for the set to be invariant under the LFT defined by h_G.

Theorem 4. *Suppose $G \in \mathcal{R}_{sp}^{n_y\times n_u}$ and $S \subseteq \mathcal{R}_p^{n_u\times n_y}$ is a closed subspace. Then*

S is quadratically invariant under G

$$\iff h_G(S) = S$$

Proof. See [3, 4]. ∎

From this, and from a specific Youla parameterization, it ultimately follows that if S is a closed subspace, S is quadratically invariant under G, and $K_{\text{nom}} \in \mathcal{RH}_\infty \cap S$ is a stabilizing controller, then K is optimal for problem (1) if and only if $K = K_{\text{nom}} - h\big(h(K_{\text{nom}}, G), Q\big)$ and Q is optimal for following optimization problem

$$\begin{aligned} \text{minimize} \quad & \|T_1 - T_2QT_3\| \\ \text{subject to} \quad & Q \in \mathcal{RH}_\infty \\ & Q \in S \end{aligned} \tag{2}$$

where $T_1, T_2, T_3 \in \mathcal{RH}_\infty$.

This is a convex optimization problem. We may solve it to find the optimal Q, and then recover the optimal K for our original problem.

If the norm of interest is the $\mathcal{H}_2$-norm, it was shown in [4] that vectorization can be used to further reduce the problem to an unconstrained optimal control problem which may then be solved with standard software, as follows.

Theorem 5. *Suppose x is an optimal solution to*

$$\begin{aligned} &\textit{minimize} && \|b + Ax\|_2 \\ &\textit{subject to} && x \in \mathcal{RH}_\infty \end{aligned} \tag{3}$$

where $D \in \mathbb{R}^{n_u n_y \times a}$ is a matrix whose columns form an orthonormal basis for $vec(S)$, and

$$b = vec(T_1), \qquad A = -(T_3^T \otimes T_2)D.$$

Then $Q = vec^{-1}(Dx)$ is optimal for (2) and the optimal values are equivalent.

Proof. See [4]. ∎

3 Convexity without a Stabilizing Controller

Suppose that one cannot find a $K_{\text{nom}} \in C_{\text{stab}} \cap \mathcal{RH}_\infty \cap S$; that is, a controller with the admissible structure which is both stable and stabilizing. This may occur either because the plant is not strongly stabilizable, or simply because it is difficult to find. In this section we will show that problem (1) can still be reduced to a convex optimization problem, albeit one which is less straightforward to solve.

We will achieve this by bypassing the Youla parameterization, and using the change of variables typically associated with stable or bounded plants

$$R \;=\; h_G(K) \;=\; -K(I - GK)^{-1}$$

where R will be used instead of Q to elucidate that this *is not* a Youla parameter. The key observation is that internal stabilization is equivalent to an affine constraint in this parameter.

The constraint that K stabilize G, which is equivalent to the constraint that K stabilize P when the standard conditions of Theorem 1 hold, is defined as requiring that the maps from (v_1, v_2) to (u, y) in Figure 1 belong to $\mathcal{RH}_\infty$. This can be stated explicitly as

$$\begin{bmatrix} (I - KG)^{-1} & (I - KG)^{-1}K \\ G(I - KG)^{-1} & G(I - KG)^{-1}K \end{bmatrix} \in \mathcal{RH}_\infty$$

Making use of the relations

$$\begin{aligned} (I - GK)^{-1}G \;&=\; G(I - KG)^{-1} \\ (I - KG)^{-1} \;&=\; I + K(I - GK)^{-1}G \end{aligned}$$

we find that K stabilizes G if and only if

$$\begin{bmatrix} RG & R \\ G - GRG & GR \end{bmatrix} \in \mathcal{RH}_\infty \tag{4}$$

Suppose $G \in \mathcal{R}_{sp}^{n_y \times n_u}$ and $S \subseteq \mathcal{R}_p^{n_u \times n_y}$ is a quadratically invariant closed subspace. We may then use this result to transform the stabilization constraint of problem (1) and use Theorem 4 to transform the information constraint to obtain the following equivalent problem. K is optimal for problem (1) if and only if $K = h_G(R)$ and R is optimal for

$$\begin{aligned} &\text{minimize} && \|P_{11} - P_{12}RP_{21}\| \\ &\text{subject to} && \begin{bmatrix} RG & R \\ G - GRG & GR \end{bmatrix} \in \mathcal{RH}_\infty \\ &&& R \in S \end{aligned} \tag{5}$$

This is a convex optimization problem.

4 Solution without a Stabilizing Controller

We show in this section that vectorization can similarly be used to eliminate the information constraint when a nominal stable and stabilizing controller can not be found. The resulting problem is not immediately amenable to standard software, as is problem 3, but methods for obtaining its solution are discussed.

Let $D \in \mathbb{R}^{n_u n_y \times a}$ be a matrix whose columns form an orthonormal basis for $\text{vec}(S)$, and now let

$$f = \text{vec}(P_{11}), \qquad E = -(P_{21}^{\text{T}} \otimes P_{12})D,$$

$$d = \begin{bmatrix} 0 \\ \text{vec}(G) \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} (G^T \otimes I)D \\ -(G^T \otimes G)D \\ (I \otimes G)D \end{bmatrix}$$

We then may solve the following equivalent problem. Suppose x is an optimal solution to

$$\begin{aligned} &\text{minimize} && \|f + Ex\|_2 \\ &\text{subject to} && d + Cx \in \mathcal{RH}_\infty \\ &&& x \in \mathcal{RH}_\infty \end{aligned} \tag{6}$$

Then $R = \text{vec}^{-1}(Dx)$ is optimal for (5) and the optimal values are equivalent. The optimal K for problem (1) could then be recovered as $K = h_G(R)$.

Remark 6. *While A, b of problem (3) are stable, C, d, E, f of problem (6) may very well be unstable. Notice also that $G \in \mathcal{R}_{sp}$ implies $C, d \in \mathcal{R}_{sp}$.*

Remark 7. *The last constraint comes from the upper right-hand block of Condition (4), and the others come from the rest of that condition.*

Remark 8. *The relaxed problem*

$$\begin{aligned} &\textit{minimize} && \|f + Ex\|_2 \\ &\textit{subject to} && x \in \mathcal{RH}_\infty \end{aligned} \tag{7}$$

can be solved with standard software in the same manner as problem (3), *and gives a lower bound on the solution. If the result is such that the entire constraint of problem* (6) *is satisfied, then the optimal value has been achieved.*

Remark 9. *For any* $\mu > 0$ *the following problem may be solved in the same standard manner*

$$\begin{aligned} &\textit{minimize} && \left\| \begin{bmatrix} f \\ \mu d \end{bmatrix} + \begin{bmatrix} E \\ \mu C \end{bmatrix} x \right\|_2 \\ &\textit{subject to} && x \in \mathcal{RH}_\infty \end{aligned} \tag{8}$$

and then the optimal value of x *as well as the optimal value of the objective function will approach those of problem* (6) *as* μ *approaches* 0 *from above.*

A reasonable solution procedure for problem (6) would then be to first solve the relaxed problem of Remark 8, and test whether $d + Cx \in \mathcal{RH}_\infty$ for the optimal value. If so, we are done and can recover the optimal K. If not, then solve problem (8) for values of μ which decrease and approach 0. This procedure in no way requires a controller that is both stable and stabilizing, so it is most useful when the plant is actually not strongly stabilizable, and thus no such controller exists.

Alternatively, as long as P is stabilizable by some $K \in S$, the solution to problem (8) for any $\mu > 0$ results in an x such that $\|d + Cx\|_2$ is finite. Thus $R = \text{vec}^{-1}(Dx)$ satisfies Condition (4), and $K = h_G(R)$ is both stabilizing and lies in S. If it is also stable, we have then found a $K_{\text{nom}} \in C_{\text{stab}} \cap \mathcal{RH}_\infty \cap S$, and the procedures from [4] may be used to find the optimal decentralized controller. This is ideal for the case where the plant is strongly stabilizable, but a stabilizing controller is difficult to find with other methods.

The techniques discussed here involve not only finding optimal decentralized controllers, but also develop explicit procedures for first finding a stabilizing decentralized controller when one is not available otherwise. As there are no known systematic methods of finding stabilizing controllers for most quadratically invariant problems, this is an extremely important development, and an exciting avenue for future research.

5 Conclusions

We showed that an optimal decentralized control problem can be reduced to a convex optimization problem if the information constraint is quadratically invariant, even if a stabilizing controller is not provided. The key was using a change of variables in which stabilization is an affine constraint, and in which the parameter takes on the same constraints as the controller. We discussed further reduction of this problem to centralized control problems so that standard techniques could be used for computation. In our final remark, we showed how regularization can then be used to formulate a centralized problem which yields a stabilizing controller for the original problem.

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