Inverse Filtering for Compressible Flows

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The instantaneous equations are :

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0 \tag{1}$$

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) + \frac{\partial p}{\partial x_i} = \frac{\partial}{\partial x_j}(\sigma_{ij}) \tag{2}$$

$$\frac{\partial}{\partial t}(\rho E) + \frac{\partial}{\partial x_j}(\rho u_j H) = \frac{\partial}{\partial x_j}(u_i \sigma_{ij} + k\frac{\partial T}{\partial x_j}) \tag{3}$$

Let P be an invertible filter with inverse Q such that,

Commute with derivatives,

$$P\frac{\partial}{\partial x_i} = \frac{\partial P}{\partial x_i}, \quad Q\frac{\partial}{\partial x_i} = \frac{\partial Q}{\partial x_i} \tag{4}$$

Note that P is not a projector such that,

$$P^2 = P$$

If this were so, then

$$P^2 Q = PQ = I$$

and

$$P^2 Q = P(PQ) = P$$

so,

$$P = I$$

Define filtered quantities,

$$\bar{\rho} = P\rho, \rho = Q\bar{\rho}, \bar{p} = Pp, p = Q\bar{p} \tag{5}$$

and mass weighted filtered velocity

$$\tilde{u_i} = \frac{P(\rho u_i)}{P\rho} = \frac{P(\rho u_i)}{\bar{\rho}} \tag{6}$$

$$\bar{\rho}\tilde{u_i} = P(\rho u_i), \rho u_i = Q(\bar{\rho}\tilde{u_i}) \tag{7}$$

Then substituting for $\rho$and ρu_i, the mass conservation equation becomes,

$$\frac{\partial}{\partial t}Q\bar{\rho} + \frac{\partial}{\partial x_j}Q(\bar{\rho}\tilde{u_i}) = 0 \tag{8}$$

and multiplying by P $= Q^{-1}$,

$$\frac{\partial\bar{\rho}}{\partial t} + \frac{\partial}{\partial x_j}(\bar{\rho}\tilde{u_i}) = 0 \tag{9}$$

The filtered Momentum equations are,

$$\frac{\partial}{\partial t}P(\rho u_i) + \frac{\partial}{\partial x_j}P(\rho u_i u_j) + \frac{\partial}{\partial x_i}Pp = \frac{\partial}{\partial x_j}P(\sigma_{ij}) \tag{10}$$

substituting for $\rho u_i$and p,

$$\frac{\partial}{\partial t}(\bar{\rho}\tilde{u_i}) + \frac{\partial}{\partial x_j}P(\rho u_i u_j) + \frac{\partial}{\partial x_i}\bar{p} = \frac{\partial}{\partial x_j}P(\bar{\sigma_{ij}}) \tag{11}$$

where $\bar{\sigma_j}$ is evaluated from $\bar{u} =$ Pu_i.

To reduce this to the usual form,

$$\rho u_i u_j = \frac{Q(\bar{\rho}\tilde{u_i})Q(\bar{\rho}\tilde{u_j})}{Q\bar{\rho}} \tag{12}$$

so,

$$\frac{\partial}{\partial x_j}P(\rho u_i u_j) = \frac{\partial}{\partial x_j}P(\bar{\rho}\tilde{u_i}\tilde{u_j}) + \frac{\partial}{\partial x_j}\tilde{\tau_{ij}} \tag{13}$$

where,

$$\tilde{\tau_{ij}} = Q^{-1}\Big[\frac{Q(\bar{\rho}\tilde{u_i})Q(\bar{\rho}\tilde{u_j})}{Q\bar{\rho}} - Q(\bar{\rho}\tilde{u_i}\tilde{u_j})\Big] \tag{14}$$

Also,

$$\frac{\partial}{\partial x_j}\bar{\sigma_{ij}} = \frac{\partial}{\partial x_j}(\tilde{\sigma_{ij}} + \eta_{ij}) \tag{15}$$

where,

$$\tilde{\sigma_{ij}} = \mu\Big(\frac{\partial\tilde{u_i}}{\partial x_j} + \frac{\partial\tilde{u_j}}{\partial x_i}\Big) + \lambda\delta_{ij}\frac{\partial\tilde{u_k}}{\partial x_k} \tag{16}$$

and η_{ij} is the same quantity evaluated with,

$$\tilde{u_i} - \bar{u_i} = \frac{\bar{m_i}}{\bar{\rho}} - \bar{u_i} \tag{17}$$

where,

$$m_i = \rho u_i, \quad \bar{m_i} = P(\rho u_i) \tag{18}$$

then,

$$\bar{u_i} = P\frac{m_i}{\rho} = Q^{-1}\Big(\frac{Q\bar{m_i}}{Q\bar{\rho}}\Big) \tag{19}$$

so,

$$\tilde{u_i} - \bar{u_i} = Q^{-1}\Big[Q\Big(\frac{\bar{m_i}}{\bar{\rho}}\Big) - \frac{Q\bar{m_i}}{Q\bar{\rho}}\Big] \tag{20}$$

Finally, the momentum equation is,

$$\frac{\partial}{\partial t}(\bar{\rho}\tilde{u_i}) + \frac{\partial}{\partial x_j}(\bar{\rho}\tilde{u_i}\tilde{u_j}) + \frac{\partial \bar{p}}{\partial x_i} = \frac{\partial}{\partial x_j}\tilde{\sigma_{ij}} - \frac{\partial}{\partial x_j}(\tilde{\tau_{ij}} + \eta_{ij}) \tag{21}$$

Here,

$$\tilde{\tau_{ij}} = Q^{-1}\Big[\frac{Q(\bar{\rho}\tilde{u_i})Q(\bar{\rho}\tilde{u_j})}{Q\bar{\rho}} - Q(\bar{\rho}\tilde{u_i}\tilde{u_j})\Big] \tag{22}$$

$$\tilde{\tau_{ij}} = Q^{-1}\Big[\frac{Q\bar{m_i}Q\bar{m_j}}{Q\bar{\rho}} - Q\Big(\frac{\bar{m_i}\bar{m_j}}{\bar{\rho}}\Big)\Big] \tag{23}$$

The filtered energy equations are,

$$\frac{\partial}{\partial t}P(\rho E) + \frac{\partial}{\partial x_j}\big[P(\rho u_j H)\big] = \frac{\partial}{\partial x_j}\big[P(u_i\sigma_{ij} + k\frac{\partial T}{\partial x_j})\big] \tag{24}$$

Define,

$$\tilde{E} = \frac{P(\rho E)}{P\rho} = \frac{P(\rho E)}{\bar{\rho}} \tag{25}$$

$$\bar{\rho}\tilde{E} = P(\rho E), \quad \rho E = Q(\bar{\rho}\tilde{E}) \tag{26}$$

$$\bar{\rho}\tilde{H} = \bar{\rho}\tilde{E} + \bar{p}, \quad Q(\bar{\rho}\tilde{H}) = \rho E + p \tag{27}$$

Then,

$$\rho u_j H = \frac{\rho u_j \rho H}{\rho} = \frac{Q(\bar{\rho}\tilde{u_i})Q(\bar{\rho}\tilde{H})}{Q\bar{\rho}} \tag{28}$$

$$u_i\sigma_{ij} = \frac{\rho u_i \sigma ij}{\rho} = \frac{Q(\bar{\rho}\tilde{u_i})Q\tilde{\sigma_{ij}}}{Q\bar{\rho}} = \frac{Q(\bar{\rho}\tilde{u_i})Q(\tilde{\sigma_{ij}} + \eta_{ij})}{Q\bar{\rho}} \tag{29}$$

$$P(\rho u_j H) = Q^{-1}\Big[\frac{Q(\bar{\rho}\tilde{u_j})Q(\bar{\rho}\tilde{H})}{Q\bar{\rho}}\Big] \tag{30}$$

$$P(u_i\sigma_{ij}) = Q^{-1}\Big[\frac{Q(\bar{\rho}\tilde{u_i})Q(\tilde{\sigma_{ij}}+\eta_{ij})}{Q\bar{\rho}}\Big] \tag{31}$$

Thus, the energy equations can be written as,

$$\frac{\partial}{\partial t}(\bar{\rho}\tilde{E}) + \frac{\partial}{\partial x_j}(\bar{\rho}\tilde{u_j}\tilde{H}) = \frac{\partial}{\partial x_j}(\tilde{u_i}\tilde{\sigma_{ij}} + k\frac{\partial \bar{T}}{\partial x_j}) - \frac{\partial}{\partial x_j}(\alpha_{ij}+\beta_{ij}) \tag{32}$$

Where,

$$\alpha_{ij} = Q^{-1}\Big[\frac{Q(\bar{\rho}\tilde{u_j})Q(\bar{\rho}\tilde{H})}{Q\bar{\rho}} - Q(\bar{\rho}\tilde{u_j}\tilde{H})\Big] \tag{33}$$

and

$$\beta_{ij} = Q^{-1}\Big[\frac{Q(\bar{\rho}\tilde{u_i})Q(\tilde{\sigma_{ij}}+\eta_{ij})}{Q\bar{\rho}} - Q(\tilde{u_i}\tilde{\sigma_{ij}})\Big] \tag{34}$$