Filtering the Navier Stokes equations with an invertible filter

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Consider the incompressible Navier–Stokes equations

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial p}{\partial x_i} = \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \tag{1}$$

where

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{2}$$

In large eddy simulation (LES) the solution is filtered to remove the small scales. Typically one sets

$$\bar{u}_i(x) = \int G(x - x')u(x')dx'$$

where the kernel G is concentrated in a band defined by the filter width. Then the filtered equations contain the extra virtual stress

$$\tau_{ij} = \overline{u_i u_j} - \bar{u}_i \bar{u}_j \tag{3}$$

because the filtered value of a product is not equal to the product of the filtered values. This stress has to be modeled

A filter which completely cuts off the small scales or the high frequency components is not invertible. The use, on the other hand, of an invertible filter would allow equation (1) to be directly expressed in terms of the filtered quantities. Thus one can identify desirable properties of a filter as

1. attenuation of small scales
2. commutativity with the differential operator $\frac{\partial}{\partial x_i}$
3. invertibility

Suppose the filter has the form

$$\bar{u}_i = P u_i \tag{4}$$

which can be inverted as

$$Q \bar{u}_i = u_i \tag{5}$$

where $Q = P^{-1}$. Moreover Q should be coercive, so that

$$||Qu|| > c||u|| \tag{6}$$

for some positive constant c. Note that if Q commutes with $\frac{\partial}{\partial x_i}$ then so does Q^{-1}, since for any quantity f which is sufficiently differentiable

$$\begin{aligned} \frac{\partial}{\partial x_i}(Q^{-1}f) &= Q^{-1}Q\frac{\partial}{\partial x_i}(Q^{-1}f) \\ &= Q^{-1}\frac{\partial}{\partial x_i}(QQ^{-1}f) \\ &= Q^{-1}\frac{\partial}{\partial x_i}f \end{aligned}$$

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Also since Q commutes with $\frac{\partial}{\partial x_i}$,

$$\frac{\partial \bar{u}}{\partial x_i} = 0 \tag{7}$$

As an example P can be the inverse Helmholtz operator, so that one can write

$$Q\bar{u}_i = \left(1 - \alpha^2 \frac{\partial^2}{\partial x_k \partial x_k}\right) \bar{u}_i = u_i \tag{8}$$

where α is a length scale proportional to the largest scales to be retained. One may also introduce a filtered pressure p, satisfying the equation

$$Q\bar{p} = \left(1 - \alpha^2 \frac{\partial^2}{\partial x_k \partial x_k}\right) \bar{p} = p \tag{9}$$

Now one can substitute equation (8) and (9) for u_i and p in equation (1) to get

$$\rho \frac{\partial}{\partial t}\left(1 - \alpha^2 \frac{\partial^2}{\partial x_k \partial x_k}\right) \bar{u}_i + \rho \left(1 - \alpha^2 \frac{\partial^2}{\partial x_k \partial x_k}\right) \bar{u}_j \frac{\partial}{\partial x_j}\left(1 - \alpha^2 \frac{\partial^2}{\partial x_l \partial x_l}\right) \bar{u}_i + \frac{\partial}{\partial x_i}\left(1 - \alpha^2 \frac{\partial^2}{\partial x_k \partial x_k}\right) \bar{p}$$

$$= \mu \frac{\partial^2}{\partial x_j \partial x_j}\left(1 - \alpha^2 \frac{\partial^2}{\partial x_k \partial x_k}\right) \bar{u}_i$$

Because the order of the differentiations can be interchanged and the Helmholtz operator satisfies condition (6), it can be removed. The product term can be written as

$$\begin{aligned}
& \rho \frac{\partial}{\partial x_j}\left\{\left(1 - \alpha^2 \frac{\partial^2}{\partial x_k \partial x_k}\right) \bar{u}_i \left(1 - \alpha^2 \frac{\partial^2}{\partial x_l \partial x_l}\right) \bar{u}_j\right\} \\
= \; & \rho \frac{\partial}{\partial x_j}\left\{\bar{u}_i \bar{u}_j - \alpha^2 \bar{u}_i \frac{\partial^2 \bar{u}_j}{\partial x_k \partial x_k} - \alpha^2 \bar{u}_j \frac{\partial^2 \bar{u}_i}{\partial x_k \partial x_k} + \alpha^4 \frac{\partial^2 \bar{u}_i}{\partial x_k \partial x_k} \frac{\partial^2 \bar{u}_j}{\partial x_l \partial x_l}\right\} \\
= \; & \rho \frac{\partial}{\partial x_j}\left\{\bar{u}_i \bar{u}_j - \alpha^2 \frac{\partial^2}{\partial x_k \partial x_k}(\bar{u}_i \bar{u}_j) + 2\alpha^2 \frac{\partial \bar{u}_i}{\partial x_k} \frac{\partial \bar{u}_j}{\partial x_k} + \alpha^4 \frac{\partial^2 \bar{u}_i}{\partial x_k \partial x_k} \frac{\partial^2 \bar{u}_j}{\partial x_l \partial x_l}\right\} \\
= \; & \rho Q \frac{\partial}{\partial x_j}\left\{\bar{u}_i \bar{u}_j + \alpha^2 Q^{-1}\left(2 \frac{\partial \bar{u}_i}{\partial x_k} \frac{\partial \bar{u}_j}{\partial x_k} + \alpha^2 \frac{\partial^2 \bar{u}_i}{\partial x_k \partial x_k} \frac{\partial^2 \bar{u}_j}{\partial x_l \partial x_l}\right)\right\}
\end{aligned}$$

According to condition (6), if $Qf = 0$ for any sufficiently differentiable quantity f, then $f = 0$. Thus the filtered equation finally reduces to

$$\rho \frac{\partial \bar{u}_i}{\partial t} + \rho \frac{\partial}{\partial x_j}(\bar{u}_i \bar{u}_j) + \frac{\partial \bar{p}}{\partial x_i} = \mu \frac{\partial^2 \bar{u}_i}{\partial x_k \partial x_k} - \rho \frac{\partial}{\partial x_j} \tau_{ij} \tag{10}$$

with the virtual stress

$$\tau_{ij} = \alpha^2 Q^{-1}\left(2 \frac{\partial \bar{u}_i}{\partial x_k} \frac{\partial \bar{u}_j}{\partial x_k} + \alpha^2 \frac{\partial^2 \bar{u}_i}{\partial x_k \partial x_k} \frac{\partial^2 \bar{u}_j}{\partial x_l \partial x_l}\right) \tag{11}$$

□

The virtual stress may be calculated by solving

$$\left(1 - \alpha^2 \frac{\partial^2}{\partial x_k \partial x_k}\right) \tau_{ij} = \alpha^2 \left(2 \frac{\partial \bar{u}_i}{\partial x_k} \frac{\partial \bar{u}_j}{\partial x_k} + \alpha^2 \frac{\partial^2 \bar{u}_i}{\partial x_k \partial x_k} \frac{\partial^2 \bar{u}_j}{\partial x_l \partial x_l}\right) \tag{12}$$

Taking the divergence of equation (10), it also follows that $\bar{p}$ satisfies the Poisson equation

$$\frac{\partial^2 \bar{p}}{\partial x_i \partial x_i} + \rho \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}(\bar{u}_i \bar{u}_j) + \rho \frac{\partial^2}{\partial x_i \partial x_j} \tau_{ij} = 0 \tag{13}$$

With appropriate boundary conditions, equations (7) and (10)–(13) are closed, if possibly intractable. In a discrete solution scales smaller than the mesh width would not be resolved, amounting to an implicit cut off. There is the possibility of introducing an explicit cut off in τ_{ij}. Also one could use equation (8) to restore an estimate of the unfiltered velocity.

In order to avoid solving the Helmholtz equation (12), the inverse Helmholtz operator could be expanded formally as

$$\left(1-\alpha^2\Delta\right)^{-1} = 1 + \alpha^2\Delta + \alpha^4\Delta^2 + \dots$$

where Δ denotes the Laplacian $\frac{\partial^2}{\partial x_k \partial x_k}$. Now retaining terms up to the fourth power of α, the approximate virtual stress tensor assumes the form

$$\tau_{ij} = 2\alpha^2 \frac{\partial \bar{u}_i}{\partial x_k}\frac{\partial \bar{u}_j}{\partial x_k} + \alpha^4 \left[2\Delta\left(\frac{\partial \bar{u}_i}{\partial x_k}\frac{\partial \bar{u}_j}{\partial x_k}\right) + \Delta\bar{u}_i\Delta\bar{u}_j\right] \tag{14}$$

One may regard the forms (11) or (14) as prototypes for subgrid scale (SGS) models.

The inverse Helmholtz operator cuts off the smaller scales quite gradually. One could design filters with a sharper cut off by shaping their frequency response. Denote the Fourier transform of f as

$$\hat{f} = Ff$$

where (in one space dimension)

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk$$

Then the general form of an invertible filter is

$$\begin{aligned} F\widehat{Pf} &= S(k)\hat{f}(k) \\ F\widehat{Qf} &= \frac{1}{S(k)}\hat{f}(k) \end{aligned}$$

where $S(k)$ should decrease rapidly beyond a cut off wave number inversely proportional to a length scale α. Since

$$\widehat{\frac{\partial f}{\partial x}} = ik\hat{f}$$

it can easily be verified that filters of this form commute with $\frac{\partial}{\partial x}$. In the case of a general filter with inverse Q, the virtual stress follows from the relation

$$Q\overline{u_i u_j} = u_i u_j = Q\bar{u}_i Q\bar{u}_j$$

Then

$$\begin{aligned} \tau_{ij} &= \overline{u_i u_j} - \bar{u}_i\bar{u}_j \\ &= Q^{-1}\left(Q\bar{u}_i Q\bar{u}_j - Q(\bar{u}_i\bar{u}_j)\right) \end{aligned}$$

This formula provides the form for a family of subgrid-scale models.