Performance of TOA and TDOA in a Non-homogeneous Transmitter Network Combining GPS and Terrestrial Signals

Ju-Yong Do, Matthew Rabinowitz, and Per Enge, *Stanford University*

BIOGRAPHY

Ju-Yong Do is a Ph.D. candidate in Electrical Engineering at Stanford University. He received B.S. degree from Yonsei University in Korea in 1998 and M.S. degree from Stanford University in 2003 in Electrical Engineering. He was with Motorola between 1998 and 2000 as a communication hardware engineer in the CDMA division. Major interests are positioning technology, wireless communication, and estimation theory.

Dr. Matthew Rabinowitz is a consulting Professor of Aeronautics and Astronautics at Stanford University. He received his B.S. in Physics and his M.S. and Ph.D., both in Electrical Engineering, from Stanford University in 1996 and 2000. He is the co-founder and the CTO of Rosum corp. Professor Rabinowitz's research focuses on the design of the terrestrial signal based positioning system and the urban/indoor positioning technology.

Dr. Per Enge is a Professor of Aeronautics and Astronautics at Stanford University. He received his B.S. in Electrical Engineering from the University of Massachusetts at Amherst in 1975, and his M.S. and Ph.D., both in Electrical Engineering, from the University of Illinois at Urbana-Champaign in 1979 and 1983. Professor Enge's research focuses on the design of navigation systems which satisfy stringent requirements with respect to accuracy, integrity (truthfulness), time availability, and continuity.

ABSTRACT

The time of arrival (TOA) and the time difference of arrival (TDOA) measurements are transformable to each other without a loss of information regarding positioning and thus the position estimations based on them should be theoretically equivalent. It is proved by Shin [1] in case of homogeneous and uncorrelated ranging sources with the weighted least square (WLS) method. In this paper, their equivalence is proved in a general case and examined under various practical scenarios including non-homogeneous and correlated ranging sources and suboptimal weighting methods such as the least square (LS) by simulation.

I. INTRODUCTION

The Global Positioning System (GPS) satellites are under constant monitoring and calibration and thus contain relatively high homogeneity in the transmitter errors. But terrestrial communication systems such as television stations and WiFi transmitters considered as possible ranging sources lack such sophisticated supporting platforms and experience higher variation in their noise distributions. Therefore the integration of the GPS signals and the terrestrial signals for positioning becomes challenging because the transmitter errors are no longer homogeneous.

Pseudorange measurement in the time difference of arrival (TDOA) is often employed to use the terrestrial communication signals as ranging signals and its performance was shown to be equivalent to that of the time of arrival (TOA) when the Weighted Least Square (WLS) method is applied given the noise covariance under the assumption of homogeneous and uncorrelated noises [1]. It is also expected to be true for any noise distribution since either measurement format contains same position information. However, if a suboptimal weighting scheme such as the least square method (LS) is applied either for less complexity or due to inacurrate knowledge of the noise distribution, there exists a discrepancy in the performance of the TOA and the TDOA. Thus, in practical situation, they may not be considered equivalently. The difference between their performances are to be studied in this paper in various noise distributions considering homogeneity and correlation.

The weighted dilution of precision (WDOP) is re-defined for a general case of a noise distribution and used for comparison between the TOA and the TDOA in Section II. The TOA and the TDOA are compared with various weighting schemes in Section III and their performance is simulated in Section IV and the conclusion is given in Section V.

II. DILUTION OF PRECISION

The original dilution of precision (DOP) was derived assuming that the measurement noises are uncorrelated to one another and identically distributed with the noise covariance $\boldsymbol{\Sigma_v} = \sigma_v^2\mathbf{I}$. Thus either the LS or the WLS performs equivalently and the DOP depends only on the geometry of the transmitters represented by $\boldsymbol{G}$ [3], [4].

$$\text{DOP} = \sqrt{tr[(\boldsymbol{G}^T\boldsymbol{G})^{-1}]} \tag{1}$$

$tr(\cdot)$ is the trace of a given matrix. But when other types of ranging sources are used together with the GPS satellites, there are significant variations in the noise distributions and the DOP is no longer a proper measure of the user position domain variance. To address this problem, the KDOP, the weighted DOP (WDOP), and other similar types of measures were proposed [6]-[10]. The KDOP is a variant of the DOP for the TOA solution with the LS and the WDOP is for the TOA and the WLS.

$$\text{KDOP} = \sqrt{tr[\boldsymbol{G}^{\dagger}\boldsymbol{\Sigma_v}(\boldsymbol{G}^{\dagger})^T]} \quad (2)$$

$$\text{WDOP} = \sqrt{tr[(\boldsymbol{G}^T\boldsymbol{\Sigma_v}^{-1}\boldsymbol{G})^{-1}]} \quad (3)$$

The KDOP and the WDOP serve well the individual scenarios but not all general cases. Therefore, instead of providing a separate expression for each case, a single governing definition is desirable to combine and link them together. Because the purpose of the DOP is to find a translation metric between range domain errors and position domain errors, a subtle extension of the original DOP definition suits well. A ratio between the total variance of user variables which usually are position variables, $tr(\boldsymbol{\Sigma}_{\hat{\boldsymbol{\theta}}})$, and the average range domain variance , $tr(\boldsymbol{\Sigma_v})/n = \sigma_{v,\text{RMS}}^2$, is a direct extension of the DOP and introduced as the extended DOP (XDOP). In other words, the XDOP is a position variance normalized by a range variance.

$$\begin{aligned}\text{XDOP} &= \sqrt{\frac{\text{sum of user variable variances}}{\text{average of pseudorange variances}}} \\ &= \sqrt{tr(\boldsymbol{\Sigma}_{\hat{\boldsymbol{\theta}}})}/\sigma_{v,\text{RMS}} \quad (4)\end{aligned}$$

$\boldsymbol{\theta}$ is a vector of variables in interest and $\hat{\boldsymbol{\theta}}$ is the estimation of $\boldsymbol{\theta}$. $\sigma_{v,\text{RMS}}$ is a root mean square (RMS) average of the standard deviation of range errors. For the geometric DOP (GDOP), $\boldsymbol{\theta} = \delta\boldsymbol{x}$ and the position DOP (PDOP), the horizontal DOP (HDOP), the vertical DOP (VDOP), and the time DOP (TDOP) also can be defined by selecting $\boldsymbol{\theta}$ accordingly.

Other variants of the DOP can be expressed as the XDOP by evaluating $\boldsymbol{\Sigma}_{\hat{\boldsymbol{\theta}}}$ and $\sigma_{v,\text{RMS}}$ for each case.

$$\text{XDOP}_{\boldsymbol{\Sigma_v}=\sigma_v^2\mathbf{I}} = \text{DOP} \quad (5)$$

$$\text{XDOP}_{\text{TOA/LS}} = \text{KDOP}/\sigma_{v,\text{RMS}} \quad (6)$$

$$\text{XDOP}_{\text{TOA/WLS}} = \text{WDOP}/\sigma_{v,\text{RMS}} \quad (7)$$

The XDOP is to be used as a standard metric in the comparison between the TOA and the TDOA with the LS and the WLS.

III. TOA AND TDOA

A set of pseudorange measurements can be used either in the TOA format containing an unknown common receiver clock bias b as they are in (8) or can be transformed into the TDOA format by differencing between them and removing the clock bias b in (9).

$$\boldsymbol{W}\delta\boldsymbol{\rho} = \boldsymbol{W}\boldsymbol{G}\delta\boldsymbol{x} + \boldsymbol{W}\boldsymbol{v} \quad (8)$$

$$\boldsymbol{W}_{\text{D}}\boldsymbol{D}\delta\boldsymbol{\rho} = \boldsymbol{W}_{\text{D}}\boldsymbol{D}\boldsymbol{G}_{\text{D}}\delta\boldsymbol{u} + \boldsymbol{W}_{\text{D}}\boldsymbol{D}\boldsymbol{v} \quad (9)$$

where $\boldsymbol{W}$ is the $n \times n$ weighting matrix and $\boldsymbol{W}_{\text{D}}$ is the $(n-1) \times (n-1)$ weighting matrix for the TDOA. $\boldsymbol{D} = [\boldsymbol{I}_{(n-1)\times(n-1)}, -\boldsymbol{1}_{(n-1)\times 1}]$ is the $(n-1) \times n$ differencing matrix for the TDOA assuming that the last pseudorange $\delta\rho_n$ is with least variance without loss of generality. $\boldsymbol{G}$ is the $n \times 4$ geometry matrix for the TOA and $\boldsymbol{G} = [\boldsymbol{G}_{\text{D}}, \boldsymbol{1}_{n\times 1}]$ where $\boldsymbol{G}_{\text{D}}$ is the $n \times 3$ geometry matrix for the TDOA. $\delta\boldsymbol{\rho}$ is the $n \times 1$ vector of the differentiated pseudorange measurements assuming n transmitters and $\delta\boldsymbol{x}$ is the 4×1 vector of the differentiated user variables assuming 3 dimensional positioning and $\delta\boldsymbol{x} = [\delta\boldsymbol{u}^T, \delta b]^T$ where $\boldsymbol{u}$ is the user position variables. $\boldsymbol{v}$ is the $n \times 1$ residual measurement noise vector after the differentiation.

There are two contradicting arguments regarding to the TOA and the TDOA. The first argument is that pseudoranges in the TOA format can be transformed into ones in the TDOA format without loss of information related to the position estimation and vice verse. It is because the pseudoranges in the TOA format have the unknown common variable b and thus only the relative values between them contain the positioning information which either the TOA format or the TDOA format carries. Hence, the choice of the format should not make any difference in the final result as long as proper due processes to the specific format are done. The ith pseudorange measurement ρ_i in the TOA and the TDOA format can be represetned as follows.

$$\rho_i = r_i + b + \epsilon_i \quad (10)$$

$$\Delta\rho_{i,n} = (r_i - r_n) + (\epsilon_i - \epsilon_n) \quad (11)$$

where r_i is the true range and ϵ_i is the measurement error. The transformation from the TOA to the TDOA is straightforword by differencing each measurement by the nth measurement. The transformation from the TDOA to the TOA is possible if the clock bias b is not needed to be estimated.

$$\tilde{\rho}_i = \Delta\rho_{i,n} = r_i + b + \epsilon_i - \rho_n = r_i + \tilde{b} + \epsilon_i \quad (12)$$

$$\tilde{\rho}_n = 0 = r_n + b + \epsilon_n - \rho_n = r_n + \tilde{b} + \epsilon_n \quad (13)$$

where $\tilde{b} = b - \rho_n$. $\tilde{\rho}_i$ transformed from the TDOA format is exactly same with the original TOA format measurement ρ_i except the new clock bias $\tilde{b}$. And both b and $\tilde{b}$ are unknown variables to their TOA equation and thus, if b is not important to know, the TDOA is shown to be transformable to the TOA without any loss of information regarding user position.

The second argument is that the estimation error variance of the TDOA is bigger than that of the TOA. By differencing, the residual measurement noise terms ϵ_i are subtracted to one another $\Delta\epsilon_{i,j} = \epsilon_i - \epsilon_j$ and if uncorrelated the resulting variance becomes the sum of the two and is larger than the individuals $\sigma_{\Delta\epsilon_{i,j}}^2 = \sigma_{\epsilon_i}^2 + \sigma_{\epsilon_j}^2 \geq \max(\sigma_{\epsilon_i}^2, \sigma_{\epsilon_j}^2)$. For example, if $\sigma_{\epsilon_i}^2 = \sigma_{\epsilon_j}^2\ \forall i, j$, the TDOA format carries twice larger noise variances than the TOA format resulting in larger estimation error variance.

$$\begin{bmatrix} \sigma^2 & 0 & \dots & 0 \\ 0 & \sigma^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma^2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 2\sigma^2 & \sigma^2 & \dots & \sigma^2 \\ \sigma^2 & 2\sigma^2 & \dots & \sigma^2 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma^2 & \sigma^2 & \dots & 2\sigma^2 \end{bmatrix} \quad (14)$$

The answer to these contradicting arguments comes from distinction on the LS and the WLS. The 2nd argument is based on the LS solution and predicts the superiority of the TOA/LS over the TDOA/LS but the TOA and the TDOA with the WLS are not governed by it because the WLS decorrelates both inherent correlation and artificial correlation created by differencing and effectively diminishes the variance. The 1st argument supposes the best efforts processing which is the WLS and thus predicts the equivalence of the TOA/WLS and the TDOA/WLS.

The LS and the WLS solutions for (8) and (9) are given as follows supposing the measurement noise $\boldsymbol{v}$ with zero mean and the known covariance $\boldsymbol{\Sigma_v}$. The statistics of the user variables which are the location and the clock bias are supposed to be unknown.

$$\begin{aligned} \hat{\boldsymbol{\theta}}_{\text{TOA/LS}} =& \boldsymbol{G}^{\dagger}\delta\boldsymbol{\rho} && (15)\\ \hat{\boldsymbol{\theta}}_{\text{TOA/WLS}} =& (\boldsymbol{W}\boldsymbol{G})^{\dagger}\boldsymbol{W}\delta\boldsymbol{\rho} \\ =& (\boldsymbol{G}^T\boldsymbol{\Sigma_v}^{-1}\boldsymbol{G})^{-1}\boldsymbol{G}^T\boldsymbol{\Sigma_v}^{-1}\delta\boldsymbol{\rho} && (16)\\ \hat{\boldsymbol{\theta}}_{\text{D,TDOA/LS}} =& (\boldsymbol{D}\boldsymbol{G}_{\text{D}})^{\dagger}\boldsymbol{D}\delta\boldsymbol{\rho} && (17)\\ \hat{\boldsymbol{\theta}}_{\text{D,TDOA/WLS}} =& (\boldsymbol{W}_{\text{D}}\boldsymbol{D}\boldsymbol{G}_{\text{D}})^{\dagger}\boldsymbol{W}_{\text{D}}\boldsymbol{D}\delta\boldsymbol{\rho} \\ =& [\boldsymbol{G}_{\text{D}}^T\boldsymbol{D}^T(\boldsymbol{D}\boldsymbol{\Sigma_v}\boldsymbol{D}^T)^{-1}\boldsymbol{D}\boldsymbol{G}_{\text{D}}]^{-1} \\ & \times \boldsymbol{G}_{\text{D}}^T\boldsymbol{D}^T(\boldsymbol{D}\boldsymbol{\Sigma_v}\boldsymbol{D}^T)^{-1}\boldsymbol{D}\delta\boldsymbol{\rho} && (18)\end{aligned}$$

where $\boldsymbol{\theta}_{\text{D}} = \delta\boldsymbol{u}$ and $\boldsymbol{\Sigma_v}$, $\boldsymbol{G}$, and $\boldsymbol{G}_{\text{D}}$ are assumed to be full rank and $(\cdot)^{\dagger}$ is the Moore-Penrose pseudoinverse of a matrix. The optimal weighting matrices for the TOA and the TDOA are

$$\boldsymbol{W}^* = \boldsymbol{\Sigma_v}^{-1/2} \quad (19)$$

$$\boldsymbol{W}_{\text{D}}^* = (\boldsymbol{D}\boldsymbol{\Sigma_v}\boldsymbol{D}^T)^{-1/2} \quad (20)$$

The WLS solution is equivalent to the best linear unbiased estimator (BLUE). Although the linear minimum mean square error estimator (LMMSE) performs better than the BLUE, it is only applicable when the statistics of the user variables are known. The variance of the estimated user variables can be calculated accordingly.

$$\begin{aligned} \boldsymbol{\Sigma}_{\hat{\boldsymbol{\theta}},\text{TOA/LS}} &= \boldsymbol{G}^{\dagger}\boldsymbol{\Sigma_v}(\boldsymbol{G}^{\dagger})^T && (21)\\ \boldsymbol{\Sigma}_{\hat{\boldsymbol{\theta}},\text{TOA/WLS}} &= (\boldsymbol{G}^T\boldsymbol{\Sigma_v}^{-1}\boldsymbol{G})^{-1} && (22)\\ \boldsymbol{\Sigma}_{\hat{\boldsymbol{\theta}}_{\text{D}},\text{TDOA/LS}} &= (\boldsymbol{D}\boldsymbol{G}_{\text{D}})^{\dagger}\boldsymbol{D}\boldsymbol{\Sigma_v}\boldsymbol{D}^T[(\boldsymbol{D}\boldsymbol{G}_{\text{D}})^{\dagger}]^T && (23)\\ \boldsymbol{\Sigma}_{\hat{\boldsymbol{\theta}}_{\text{D}},\text{TDOA/WLS}} &= [\boldsymbol{G}_{\text{D}}^T\boldsymbol{D}^T(\boldsymbol{D}\boldsymbol{\Sigma_v}\boldsymbol{D}^T)^{-1}\boldsymbol{D}\boldsymbol{G}_{\text{D}}]^{-1} && (24)\end{aligned}$$

Because of the optimality of the WLS, the estimation error variances of the TOA/WLS and the TDOA/WLS are less than those of the TOA/LS and the TDOA/LS respectively [5]. From the 2nd argument in this section the TOA/LS is supposed to be superior to the TDOA/LS due to the increase in the noise variance and it is shown to be generally true by simulation in Section IV in average of multiple random geometry trials. For $\boldsymbol{\Sigma_v} = \sigma^2\boldsymbol{I}$, the optimal schemes which are the TOA/WLS and the TDOA/WLS were proved to be equivalent in their position variance

$$\boxed{\boldsymbol{\Sigma}_{\hat{\boldsymbol{\theta}}_{\text{D}},\text{TOA/WLS}} \equiv \boldsymbol{\Sigma}_{\hat{\boldsymbol{\theta}}_{\text{D}},\text{TDOA/WLS}}}$$

as well as their position solutions [1], [2].

$$\boxed{\hat{\boldsymbol{\theta}}_{\text{D,TOA/WLS}} \equiv \hat{\boldsymbol{\theta}}_{\text{D,TDOA/WLS}}}$$

Here it is to be proved that the equivalence holds for any noise distributions $\boldsymbol{\Sigma_v}$.

Proof: First, the equivalence of the covariance matrices of the TOA and the TDOA are to be proved from (22) and (24). For fair comparison, the covariance matrix for only position variables $\boldsymbol{\Sigma}_{\hat{\boldsymbol{\theta}}_{\text{D}}}$ needs to be obtained excluding the terms for the clock bias.

$$\begin{aligned} \boldsymbol{\Sigma}_{\hat{\boldsymbol{\theta}},\text{TOA/WLS}} &= (\boldsymbol{G}^T\boldsymbol{\Sigma_v}^{-1}\boldsymbol{G})^{-1} \\ &= \begin{bmatrix} \boldsymbol{G}_{\text{D}}^T\boldsymbol{\Sigma_v}^{-1}\boldsymbol{G}_{\text{D}} & \boldsymbol{G}_{\text{D}}^T\boldsymbol{\Sigma_v}^{-1}\boldsymbol{1} \\ \boldsymbol{1}^T\boldsymbol{\Sigma_v}^{-1}\boldsymbol{G}_{\text{D}} & \boldsymbol{1}^T\boldsymbol{\Sigma_v}^{-1}\boldsymbol{1}\end{bmatrix}^{-1} \\ &= \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22}\end{bmatrix} && (25)\end{aligned}$$

where $\boldsymbol{1}$ is a $n \times 1$ vector of one's and the submatrices of $\boldsymbol{\Sigma}_{\hat{\boldsymbol{\theta}},\text{TOA/WLS}}$ are

$$\begin{aligned} \boldsymbol{\Sigma}_{11} &= \left(\boldsymbol{G}_{\text{D}}^T\boldsymbol{\Sigma_v}^{-1}\boldsymbol{G}_{\text{D}} - \frac{\boldsymbol{G}_{\text{D}}^T\boldsymbol{\Sigma_v}^{-1}\boldsymbol{1}\boldsymbol{1}^T\boldsymbol{\Sigma_v}^{-1}\boldsymbol{G}_{\text{D}}}{\boldsymbol{1}^T\boldsymbol{\Sigma_v}^{-1}\boldsymbol{1}}\right)^{-1} \\ &= \left[\boldsymbol{G}_{\text{D}}^T\left(\boldsymbol{\Sigma_v}^{-1} - \frac{\boldsymbol{\Sigma_v}^{-1}\boldsymbol{1}\boldsymbol{1}^T\boldsymbol{\Sigma_v}^{-1}}{\boldsymbol{1}^T\boldsymbol{\Sigma_v}^{-1}\boldsymbol{1}}\right)\boldsymbol{G}_{\text{D}}\right]^{-1} \\ &= (\boldsymbol{G}_{\text{D}}^T\boldsymbol{P}\boldsymbol{G}_{\text{D}})^{-1} && (26)\\ \boldsymbol{\Sigma}_{12} &= -\frac{(\boldsymbol{G}_{\text{D}}^T\boldsymbol{P}\boldsymbol{G}_{\text{D}})^{-1}\boldsymbol{G}_{\text{D}}^T\boldsymbol{\Sigma_v}^{-1}\boldsymbol{1}}{\boldsymbol{1}^T\boldsymbol{\Sigma_v}^{-1}\boldsymbol{1}} = \boldsymbol{\Sigma}_{21}^T && (27)\\ \boldsymbol{\Sigma}_{22} &= \frac{1}{\boldsymbol{1}^T\boldsymbol{\Sigma_v}^{-1}\boldsymbol{1}} \\ &\quad + \frac{\boldsymbol{1}^T\boldsymbol{\Sigma_v}^{-1}\boldsymbol{G}_{\text{D}}(\boldsymbol{G}_{\text{D}}^T\boldsymbol{P}\boldsymbol{G}_{\text{D}})^{-1}\boldsymbol{G}_{\text{D}}^T\boldsymbol{\Sigma_v}^{-1}\boldsymbol{1}}{(\boldsymbol{1}^T\boldsymbol{\Sigma_v}^{-1}\boldsymbol{1})^2} && (28)\end{aligned}$$

and $\boldsymbol{P} = \boldsymbol{\Sigma_v}^{-1} - \boldsymbol{\Sigma_v}^{-1}\boldsymbol{1}\boldsymbol{1}^T\boldsymbol{\Sigma_v}^{-1}/(\boldsymbol{1}^T\boldsymbol{\Sigma_v}^{-1}\boldsymbol{1})$. $\boldsymbol{\Sigma}_{11}$ is the error covariance matrix corresponding to $\boldsymbol{\theta}_{\text{D}}$. In other words, $\boldsymbol{\Sigma}_{11} = \boldsymbol{\Sigma}_{\hat{\boldsymbol{\theta}}_{\text{D}},\text{TOA/WLS}}$ and is equal to $\boldsymbol{\Sigma}_{\hat{\boldsymbol{\theta}}_{\text{D}},\text{TDOA/WLS}}$.

$$\begin{aligned} \boldsymbol{\Sigma}_{\hat{\boldsymbol{\theta}}_{\text{D}},\text{TDOA/WLS}} &= \left[\boldsymbol{G}_{\text{D}}^T\boldsymbol{D}^T(\boldsymbol{D}\boldsymbol{\Sigma_v}\boldsymbol{D}^T)^{-1}\boldsymbol{D}\boldsymbol{G}_{\text{D}}\right]^{-1} \\ &= \left[\tilde{\boldsymbol{G}}_{\text{D}}^T\tilde{\boldsymbol{D}}^T(\tilde{\boldsymbol{D}}\tilde{\boldsymbol{D}}^T)^{-1}\tilde{\boldsymbol{D}}\tilde{\boldsymbol{G}}_{\text{D}}\right]^{-1} \\ &= \left[\tilde{\boldsymbol{G}}_{\text{D}}^T\left(\boldsymbol{I} - \frac{\tilde{\boldsymbol{1}}\tilde{\boldsymbol{1}}^T}{\tilde{\boldsymbol{1}}^T\tilde{\boldsymbol{1}}}\right)\tilde{\boldsymbol{G}}_{\text{D}}\right]^{-1} \\ &= \left[\boldsymbol{G}_{\text{D}}^T\left(\boldsymbol{\Sigma_v}^{-1} - \frac{\boldsymbol{\Sigma_v}^{-1}\boldsymbol{1}\boldsymbol{1}^T\boldsymbol{\Sigma_v}^{-1}}{\boldsymbol{1}^T\boldsymbol{\Sigma_v}^{-1}\boldsymbol{1}}\right)\boldsymbol{G}_{\text{D}}\right]^{-1} \\ &= (\boldsymbol{G}_{\text{D}}^T\boldsymbol{P}\boldsymbol{G}_{\text{D}})^{-1} \\ &= \boldsymbol{\Sigma}_{\hat{\boldsymbol{\theta}}_{\text{D}},\text{TOA/WLS}} && (29)\end{aligned}$$

where $\tilde{\boldsymbol{G}}_{\text{D}} = \boldsymbol{\Sigma_v}^{-1/2}\boldsymbol{G}_{\text{D}}$ and $\tilde{\boldsymbol{D}} = \boldsymbol{D}\boldsymbol{\Sigma_v}^{1/2}$ and $\tilde{\boldsymbol{1}} = \boldsymbol{\Sigma_v}^{-1/2}\boldsymbol{1}$. $\tilde{\boldsymbol{D}}^T(\tilde{\boldsymbol{D}}\tilde{\boldsymbol{D}}^T)^{-1}\tilde{\boldsymbol{D}}$ is the projection matrix to the range of $\tilde{\boldsymbol{D}}^T$. As $\boldsymbol{1}$ is orthogonal to $\boldsymbol{D}$, $\tilde{\boldsymbol{1}}$ is orthogonal to $\tilde{\boldsymbol{D}}$. Therefore $\tilde{\boldsymbol{D}}^T(\tilde{\boldsymbol{D}}\tilde{\boldsymbol{D}}^T)^{-1}\tilde{\boldsymbol{D}} = \boldsymbol{I} - \tilde{\boldsymbol{1}}\tilde{\boldsymbol{1}}^T/(\tilde{\boldsymbol{1}}^T\tilde{\boldsymbol{1}})$ and $\boldsymbol{D}^T(\boldsymbol{D}\boldsymbol{\Sigma_v}\boldsymbol{D}^T)^{-1}\boldsymbol{D} = \boldsymbol{P}$. (29) proves the equivalence of the position covariances of the TOA/WLS and the TDOA/WLS.

The position solutions of the TOA/WLS and the TDOA/WLS are given in (16) and (18).

$$\hat{\boldsymbol{\theta}}_{\text{TOA/WLS}} = (\boldsymbol{G}^T\boldsymbol{\Sigma}_{\boldsymbol{v}}^{-1}\boldsymbol{G})^{-1}\boldsymbol{G}^T\boldsymbol{\Sigma}_{\boldsymbol{v}}^{-1}\delta\boldsymbol{\rho} = \begin{bmatrix}\boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12}\\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22}\end{bmatrix}\begin{bmatrix}\boldsymbol{G}_{\text{D}}^T\\ \boldsymbol{1}^T\end{bmatrix}\boldsymbol{\Sigma}_{\boldsymbol{v}}^{-1}\delta\boldsymbol{\rho} \quad (30)$$

Again, only the position related parts of the TOA/WLS solution need to be obtained. $\boldsymbol{\Sigma}_{11}$ and $\boldsymbol{\Sigma}_{12}$ are given in (26) and (27).

$$\begin{aligned}&\hat{\boldsymbol{\theta}}_{\text{D,TOA/WLS}}\\ &= (\boldsymbol{\Sigma}_{11}\boldsymbol{G}_{\text{D}}^T + \boldsymbol{\Sigma}_{12}\boldsymbol{1}^T)\boldsymbol{\Sigma}_{\boldsymbol{v}}^{-1}\delta\boldsymbol{\rho}\\ &= (\boldsymbol{G}_{\text{D}}^T\boldsymbol{P}\boldsymbol{G}_{\text{D}})^{-1}\boldsymbol{G}_{\text{D}}^T\left(\boldsymbol{\Sigma}_{\boldsymbol{v}}^{-1} - \frac{\boldsymbol{\Sigma}_{\boldsymbol{v}}^{-1}\boldsymbol{1}\boldsymbol{1}^T\boldsymbol{\Sigma}_{\boldsymbol{v}}^{-1}}{\boldsymbol{1}^T\boldsymbol{\Sigma}_{\boldsymbol{v}}^{-1}\boldsymbol{1}}\right)\delta\boldsymbol{\rho}\\ &= (\boldsymbol{G}_{\text{D}}^T\boldsymbol{P}\boldsymbol{G}_{\text{D}})^{-1}\boldsymbol{G}_{\text{D}}^T\boldsymbol{P}\delta\boldsymbol{\rho}\\ &= \hat{\boldsymbol{\theta}}_{\text{D,TDOA/WLS}} \quad (31)\end{aligned}$$

which proves the equivalence of the position solutions of the TOA/WLS and the TDOA/WLS. ∎

Within this paper, the positioning system is supposed to have a single clock bias and thus $\delta\boldsymbol{x} = [\delta\boldsymbol{u}^T, \delta b]^T$ which represents the GPS only case or the integrated system using the GPS signal and land based signals with synchronization between the GPS receiver and the land signal receiver. It would be a desirable platform for an integrated system and is assumed to be throughout this paper.

The optimal weighting matrices for the TOA and the TDOA given in (19) and (20) can be simplified when the measurement noises are uncorrelated. If $\boldsymbol{\Sigma}_{\boldsymbol{v}} = \sigma^2\boldsymbol{I}$, then $\boldsymbol{W}^* = \boldsymbol{I}$ but $\boldsymbol{W}_{\text{D}}^*$ becomes $\boldsymbol{I}+$ extra terms to decorrelate the correlation created by the differencing.

$$\boldsymbol{W}^* = \boldsymbol{I} \quad (32)$$

$$\begin{aligned}\boldsymbol{W}_{\text{D}}^* &= (\boldsymbol{D}\boldsymbol{\Sigma}_{\boldsymbol{v}}\boldsymbol{D}^T)^{-1/2}\\ &= \left(\boldsymbol{I}_{(n-1)\times(n-1)} - \frac{1}{n}\boldsymbol{1}_{(n-1)\times(n-1)}\right)^{1/2}\\ &= \boldsymbol{I}_{(n-1)\times(n-1)} - \frac{1}{n \pm \sqrt{n}}\boldsymbol{1}_{(n-1)\times(n-1)} \quad (33)\end{aligned}$$

$\boldsymbol{W}_{\text{D}}^*$ can be expressed as an upper triangular matrix by the Cholesky factorization [11], [12]. If $\boldsymbol{\Sigma}_{\boldsymbol{v}} = diag(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2)$, then $\boldsymbol{W}^*$ remains diagonal but $\boldsymbol{W}_{\text{D}}^*$ becomes really complex. Instead $(\boldsymbol{W}_{\text{D}}^T\boldsymbol{W}_{\text{D}})^*$ is given which is still within printable complexity.

$$\boldsymbol{W}^* = diag(\sigma_1^{-1}, \sigma_2^{-1}, \ldots, \sigma_n^{-1}) \quad (34)$$

where $\boldsymbol{W}_{i,i}^* = \sigma_i^{-1}$ and $\boldsymbol{W}_{i,j}^* = 0$ if $i \neq j$ for the TOA.

$$(\boldsymbol{W}_{\text{D}}^T\boldsymbol{W}_{\text{D}})^* = \quad (35)$$

$$\begin{bmatrix}\left(\sigma_1^{-2}\sum_{k=1}^n \sigma_k^{-2}\right) - 1 & \cdots & \sigma_1^{-2}\sigma_{n-1}^{-2}\\ \sigma_2^{-2}\sigma_1^{-2} & \cdots & \sigma_2^{-2}\sigma_{n-1}^{-2}\\ \vdots & \ddots & \vdots\\ \sigma_{n-1}^{-2}\sigma_1^{-2} & \cdots & \left(\sigma_{n-1}^{-2}\sum_{k=1}^n \sigma_k^{-2}\right) - 1\end{bmatrix}$$

where $(\boldsymbol{W}_{\text{D}}^T\boldsymbol{W}_{\text{D}})_{i,i}^* = \left(\sigma_1^{-2}\sum_{k=1}^n \sigma_k^{-2}\right) - 1$ and $(\boldsymbol{W}_{\text{D}}^T\boldsymbol{W}_{\text{D}})_{i,j}^* = -\sigma_i^{-2}\sigma_j^{-2}$ if $i \neq j$ for the TDOA. $(\boldsymbol{W}_{\text{D}}^T\boldsymbol{W}_{\text{D}})^*$ is obtained by evaluating $(\boldsymbol{D}\boldsymbol{\Sigma}_{\boldsymbol{v}}\boldsymbol{D}^T)^{-1}$ and removing the common scaler $(\sum_{k=1}^n \sigma_k^{-2})^{-1}$. In the above two examples with diagonal covariance matrices, the weighting matrices for the TDOA are nondiagonal with complex expressions of the individual noise variances while those of the TOA remain in diagonal forms easily obtainable by elementwise inversions. The complexity of the weighting matrix for the TDOA is due to the matrix inversion whose operation count is $\mathcal{O}(n^3)$ and is a disadvantage compared to the TOA when there is no difference in their performance. To avoid the complexity, the approximations of the noise covariance matrix would be preferred in certain low cost receivers with low computational power. The simplest and practical approach would be approximating the covariance matrix by only its diagonal elements. It is to be compared with the LS and the WLS in the next section by simulation.

IV. Simulation Results

There are six different methods to be compared in combination of the TOA and the TDOA with the LS, the WLS and the diagonal WLS (DWLS) which uses the approximated weighting matrix considering only the diagonal elements of the noise covariance matrix. For the TOA/LS and the TDOA/LS, $\boldsymbol{W} = \boldsymbol{I}$ and for the TOA/WLS and the TDOA/WLS, $\boldsymbol{W} = \boldsymbol{W}^*$. For the TOA/DWLS and the TDOA/DWLS, the weightings are respectively

$$\boldsymbol{W} = \widetilde{\boldsymbol{W}} = diag(\sigma_1^{-1}, \sigma_2^{-1}, \ldots, \sigma_n^{-1}) \quad (36)$$

$$\boldsymbol{W}_{\text{D}} = \widetilde{\boldsymbol{W}}_{\text{D}} = diag\left(\frac{1}{\sqrt{\sigma_1^2+\sigma_n^2}}, \frac{1}{\sqrt{\sigma_2^2+\sigma_n^2}}, \ldots, \frac{1}{\sqrt{\sigma_{n-1}^2+\sigma_n^2}}\right) \quad (37)$$

For simulation, the geometry matrix $\boldsymbol{G}$ is generated based on randomly located transmitters on the surface of a half sphere centered by a user. The covariance matrix of the user variables $\boldsymbol{\Sigma}_{\hat{\boldsymbol{\theta}}}$ is evaluated according to (21)–(24) and here $\boldsymbol{\theta}$ is assumed to be the 3 dimensional user position variable $\delta\boldsymbol{u}$. Then the XDOP values are calculated which should be called as the XPDOP. The comparison is based on the XPDOP where a higher XPDOP means higher position variance. The RMS average of the XPDOP in 10^5 random trials is plotted against the number of ranging sources. To provide the relative performance of the approaches, an excess ratio is defined as a percentage of the XPDOP of a given scheme exceeding the XPDOP of the optimal TOA/WLS.

$$\text{Excess Ratio} = \left(\frac{\text{XPDOP}}{\text{XPDOP}_{\text{TOA/WLS}}} - 1\right) \times 100\ [\%] \quad (38)$$

which indicates how much more error is generated by a suboptimal scheme than the optimal ones.

The noise covariance is assumed to be perfectly known to the user. Regarding the nature of ranging sources, their homogeneity and correlation are in interest corresponding

TABLE I

EXCESS RATIO IN HOMOGENEOUS UNCORRELATED RANGING SOURCES [%]

n	TOA/LS	TOA/DWLS	TDOA/LS	TDOA/DWLS
5	0	0	2.45	2.45
10	0	0	15.88	15.88
15	0	0	27.77	27.77
20	0	0	38.86	38.86

to the diagonal elements and the off-diagonal elements of the noise covariance respectively. Homogeneous sources have equal diagonal terms and correlated sources have nonzero off-diagonal terms.

A. Homogeneous Uncorrelated Ranging Sources

In the space based positioning system like the GPS, the satellites are usually supposed to be homogeneous and the noises are uncorrelated thanks to the constant monitoring and calibration by the ground monitor stations. The noise covariance matrix becomes an identity matrix multiplied by a common variance $\boldsymbol{\Sigma_v} = \sigma^2 \boldsymbol{I}$. Then the TOA/LS becomes equivalent to the TOA/WLS and the TDOA/WLS because $\boldsymbol{W}^* = \boldsymbol{I}$ but the TDOA/LS remains inferior to them because $\boldsymbol{W}_{\mathrm{D}}^* \neq \boldsymbol{I}$. The approximated cases, the TOA/DWLS and the TDOA/DWLS, are equivalent to the TOA/LS and the TDOA/LS respectively. In other words, only the TDOA/LS and the TDOA/DWLS are suboptimal to the rest. Their position variance are in the order of

$$\begin{aligned} &\sigma_{\text{TOA/WLS}} = \sigma_{\text{TDOA/WLS}} = \sigma_{\text{TOA/DWLS}} = \sigma_{\text{TOA/LS}} \\ &< \sigma_{\text{TDOA/DWLS}} = \sigma_{\text{TDOA/LS}} \end{aligned}$$

Because of the optimality, the TOA/LS always generates lower estimation error variances than the TDOA/LS. In Fig. 3, the ratio between the XPDOPs of the TDOA/LS and the TOA/LS is shown to be always higher than 1. The gap between them goes to zero as the number of the transmitters n comes close to 4 because $n = 4$ forces the solution to be exact and increases as n increases. Each data point in Fig. 1 represents the RMS average of the XPDOPs for 100,000 random geometries where the XPDOPs are shown to be inversely proportional to the number of ranging sources. They are translated into the excess ratios (38) in Fig. 2. The gap between the optimal sets and the suboptimal sets are almost linearly increasing as more transmitters are used. For $n = 10$, there is approximately 16 % and for $n = 15$, 28 % increase of the positioning error if the TDOA/LS or the TDOA/DWLS is used while all other methods do not experience such loss which is summarized in Table I where the excess ratios of the subopitmal schemes are listed. If Fig. 3 is revisited, we can see that the excess ratio can be as high as 50 % for n=10 which is an substantial degradation.

B. Nonhomogeneous Correlated Ranging Sources

In case of the integrated positioning system combining the GPS signal and other terrestrial signals, there would be significant variations in their behavior and the terretrial signals

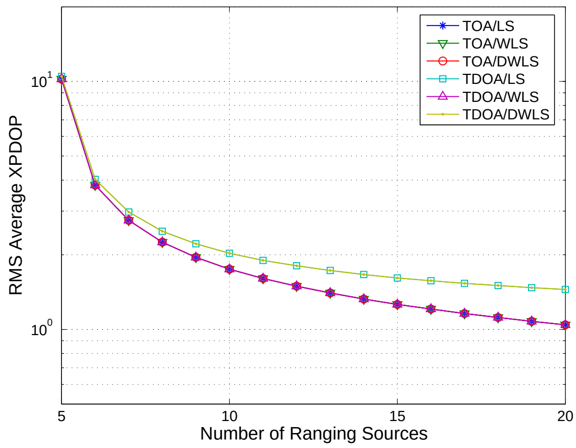

Fig. 1. XPDOP in homogeneous uncorrelated ranging sources (n = 5–20)

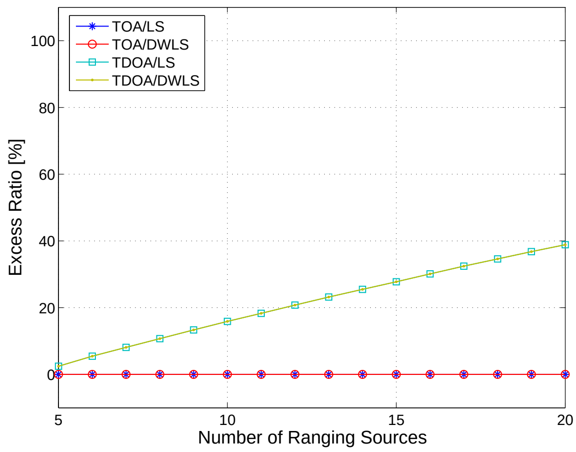

Fig. 2. Excess ratio of XPDOP compared to TOA/WLS in homogeneous uncorrelated ranging sources (n = 5–20)

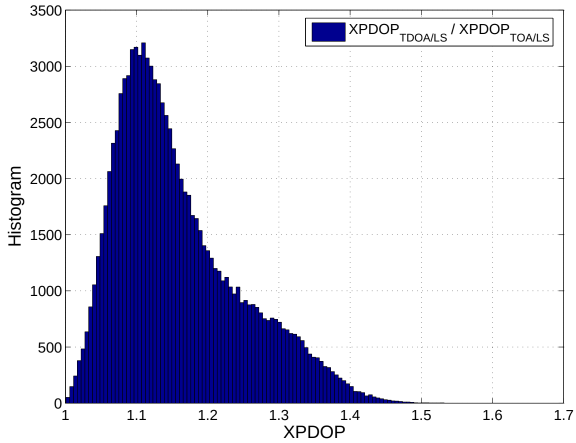

Fig. 3. Ratio between XPDOPs of TOA/LS and TDOA/LS in homogeneous uncorrelated ranging sources (n = 10)

are more likely to be correlated to one another. Thus they can be considered as non-homogeneous correlated ranging sources. For simulation, the maximum deviation between the noise variances which are the diagonal elements of the noise covariance matrix is set to be 20 dB and randomly generated uniformly between 0–20 dB which is a modest assumption considering much wider variations in real systems. The off-diagonal elements are proportional to the corresponding diagonal terms with random attenuations $\mathbf{\Sigma}_{\boldsymbol{v},i,j} = a\sqrt{\mathbf{\Sigma}_{\boldsymbol{v},i,i}\mathbf{\Sigma}_{\boldsymbol{v},j,j}}$ where $a \sim U[-0.2, 0.2]$. In other words, it is assuemd that there is in average 10 % correlation between channels which is again another modest assumption. Because of these mild assumptions, the result is less severe than the reality where more degardation is expected.

The major change from the homogeneous uncorrelated case is that all schemes are benefiting from the non-homogeneity. If Fig. 8 is compared to Fig. 1, all schemes are performing better than previous which is because there are very bad channels but at the same time very good channels which improve the performance especially the WLS and the DWLS schemes. The DWLS forms a second group closely following the optimal WLS while the equal weighting methods form a distant third group.

For non-homogeneous and correlated sources, $\sigma_{\text{TOA/WLS}} = \sigma_{\text{TDOA/WLS}} < \sigma_{\text{TOA/DWLS}} < \sigma_{\text{TDOA/DWLS}} < \sigma_{\text{TDOA/LS}} < \sigma_{\text{TOA/LS}}$

One thing to note is the switch between the TOA/LS and the TDOA/DWLS. The TOA/LS is best for the homogeneous sources but worst for the non-homogeneous ones while the TDOA/DWLS is worst for the homogeneous but very close to the best for the non-homogeneous case. It means that their performance are dependent on the type of the sources and neither of them could be an universal solution. Another change from previous two cases is the widened performance gap between the optimal groups and the LS schemes.

For the non-homogeneous correlated ranging sources, the weighting strategy is a decisive factor while the TOA and the TDOA are not differentiated much. Especially the DWLS schemes generate small degradations represented by the excess ratio less than 7 % and 13 % respectively for the TOA/DWLS and the TDOA/DWLS for upto 20 sources in Fig. 9. Thus they could be an simple and good alternative to the WLS in this case. Contrarily the TOA/LS and the TDOS/LS cause significant degradations and for $n = 10$ there are approximately 47 % and 40 % and for $n = 15$, 71 % and 61 % increase of positioning errors respectively in Table II. More detailed pictures of the LS schemes and the DWLS schemes are given in Fig. 6 and Fig. 7 respectively. In both case, one method is better than the other but not always.

C. Comparison Between Noise Scenarios

To compare the weighting schemes across the various types of the noise distributions, n is fixed to be 10. 4 different scenarios, homogeneous uncorrelated, homogeneous correlated, non-homogeneous uncorrelated, and non-homogeneous correlated

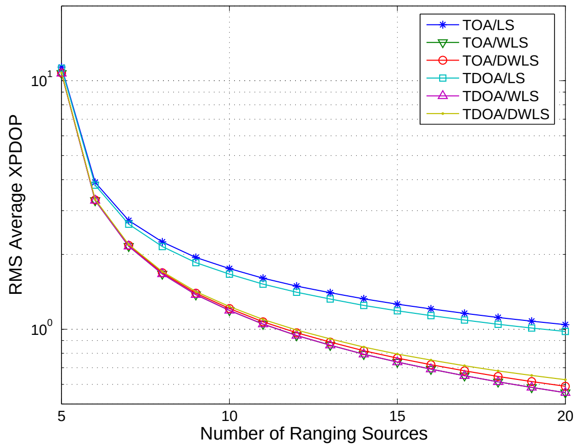

Fig. 4. XPDOP in non-homogeneous correlated ranging sources (n = 5–20)

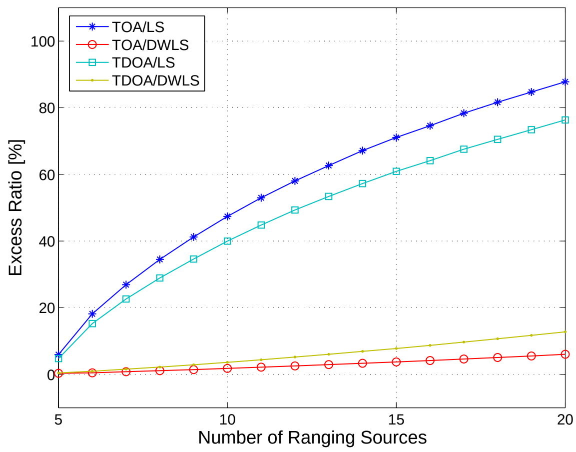

Fig. 5. Excess ratio of XPDOP compared to TOA/WLS in non-homogeneous correlated ranging sources (n = 5–20)

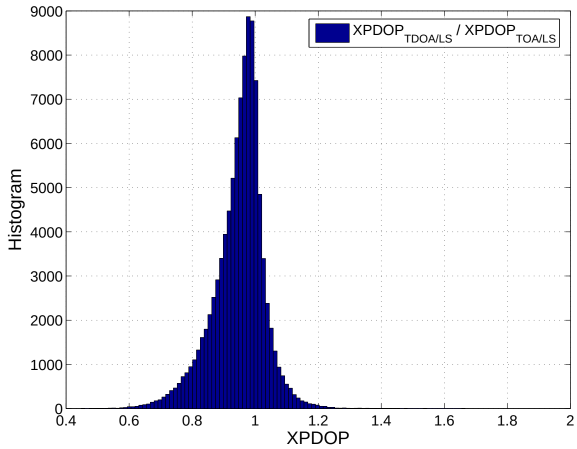

Fig. 6. Ratio between XPDOPs of TOA/LS and TDOA/LS in non-homogeneous correlated ranging sources (n = 10)

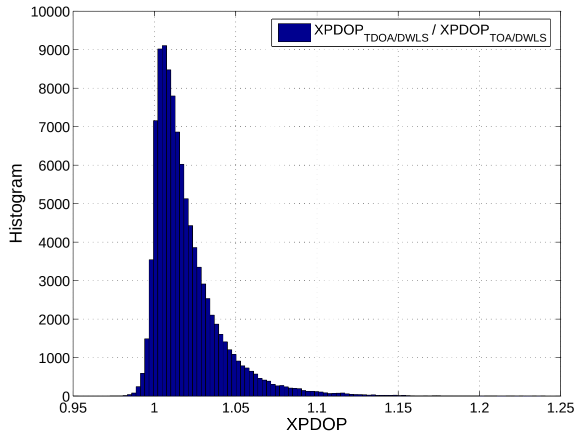

Fig. 7. Ratio between XPDOPs of TOA/DWLS and TDOA/DWLS in non-homogeneous correlated ranging sources (n = 10)

TABLE II
EXCESS RATIO IN NON-HOMOGENEOUS CORRELATED RANGING SOURCES [%]

n	TOA/LS	TOA/DWLS	TDOA/LS	TDOA/DWLS
5	5.93	0.31	4.75	0.44
10	47.38	1.79	39.97	3.63
15	71.07	3.74	60.91	7.78
20	87.78	6.04	76.34	12.76

transmitter networks, are considered. In Fig. 8, the XPDOP is shown to be significantly lower in the non-homogeneous case while correlation does not make noticeable change on it. The robustness of the schemes can be analyzed from Fig. 9. The TOA/LS is very good in the homogeneous cases but worst in the non-homogeneous cases and the TDOA/DWLS is in reverse order. The TDOA/LS is in a poor group regardless of the noise types. Among them, the TOA/DWLS is shown to be best in all cases only affected by the correlation which is still in quite low range less than 2 % for n=10. Considering its simplicity, its robust performance across the noise types is impressive and it could be a good alternative to the optimal weighting schemes for low end receivers. In Table III, the excess ratio is summarized.

V. CONCLUSION

The DOP is the most popular metric of the quality of a particular set of ranging sources measuring its geometry and estimating the resulting variance of user variables. But

TABLE III
EXCESS RATIO IN HOMOGENEOUS UNCORRELATED, HOMOGENEOUS CORRELATED, NON-HOMOGENEOUS UNCORRELATED, AND NON-HOMOGENEOUS CORRELATED RANGING SOURCES [%] (N=10)

	TOA/LS	TOA/DWLS	TDOA/LS	TDOA/DWLS
Homo/Unc	0	0	15.88	15.88
Homo/Cor	1.78	1.78	17.90	17.90
Non/Unc	44.82	0	37.55	1.81
Non/Cor	47.38	1.79	39.97	3.63

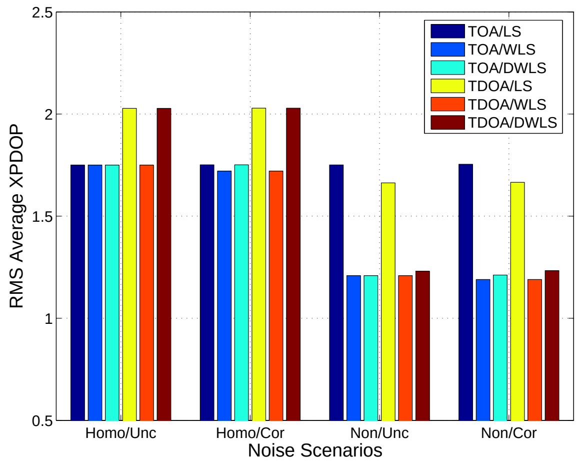

Fig. 8. XPDOP in homogeneous uncorrelated, homogeneous correlated, non-homogeneous uncorrelated, and non-homogeneous correlated ranging sources (n=10)

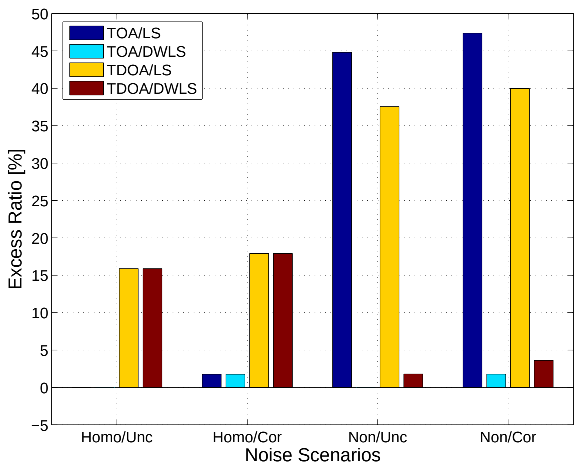

Fig. 9. Excess ratio of XPDOP compared to TOA/WLS in homogeneous uncorrelated, homogeneous correlated, non-homogeneous uncorrelated, and non-homogeneous correlated ranging sources [%] (n=10)

the latter function is only valid in case of the uncorrelated homogeneous ranging sources. Thus, for proper representations in general cases where noises can be non-homogeneous or correlated, the extended DOP (XDOP) is proposed to be a position variance normalized by range variances. It is compatible with the conventional DOP and other definitions of the DOP like the KDOP and the WDOP. Using the XDOP, the TOA and the TDOA can be compared in combination of the LS and the optimal WLS in various scenarios.

The TOA and the TDOA have been known to be equivalent when the WLS is adopted in case of homogeneous uncorrelated ranging sources. In this paper their equivalence is proved and shown to hold by simulation in a general case including non-homogeneous correlated cases. However, in terms of implementation, the TOA is much less complex and straightforward and thus recommended over the TDOA. Besides these optimal methods, there are supoptimal weighting schemes such as the LS with equal weighting and the diagonal WLS (DWLS) with diagonal approximation of the optimal weighting matrix. Among the suboptimal schemes, the TOA/DWLS is the best choice which are universally stable and simple to implement. The TDOA/DWLS and the TOA/LS perform well only for the certain types of transmiter networks and the TDOA/LS is the worst of all. Thus, both for the optimal weighting and the suboptimal weighting methods, the TOA is recommended over the TDOA.

ACKNOWLEDGMENT

The authors would like to thank the Federal Aviation Administration (FAA) and the Rosum Corporation for their generous support for this research.

REFERENCES

[1] D. H. Shin, S. B. Son, and T. K. Sung, "DOP relationship between the TOA and the TDOA positioning," *Proc. ION Annual Meeting* 2000, pp. 436-442, June 2000

[2] C. Park, I. Kim, G. I. Jee, and J. G. Lee, "Relationships between positioning error measures in global positioning system," *AIAA Journal of Guidance, Control, and Dynamics*, vol. 20, no. 5, pp. 1045-1047, 1997

[3] B. Parkinson and J. Spilker, *Global Positioning System: Theory and Applications.* AIAA, 1996

[4] P. Misra and P. Enge, *Global Positioning System: Signals, Measurements, and Performance.* Massachusetts: Ganga-Jamuna Press, 2001

[5] S. M. Kay, *Fundamentals of Statistical Signal Processing: Estimation Theory.* New Jersey: Prentice Hall, 1993

[6] H. Sairo, D. Akopian, and J. Takala, "Weighted dilution of precision as quality measure in satellite positioning," *IEE Proc. Radar Sonar Navig.*, vol. 150, no. 6, pp. 430-436, Dec. 2003

[7] K. Kovach, H. Maquet, and D. Davis, "PPS RAIM algorithms and their performance," *Navigation*, vol. 42, no. 3, pp. 515-529, 1995

[8] C. Park, I. Kim, J. G. Lee, and G. I. Jee, "A satellite selection criterion incorporating the effect of elevation angles in GPS positioning," *Control Eng. Pract.*, vol. 4, no. 12, pp. 1741-1746, 1996

[9] J. D. Bard and F. M. Ham, "Time difference of arrival dilution of precision and applications," *IEEE Trans. Signal Processing*, vol. 47, no. 2, pp. 521-523, Feb. 1999

[10] United States Coast Guard Navigation Center, *Navstar GPS User Equipment Introduction, Public Release Version.* United States Coast Guard Navigation Center, Sept. 1996.

[11] G. Strang and K. Borre, *Linear Algebra, Geodesy, and GPS.* Massachusetts: Wellesley-Cambridge Press, 1997

[12] R. A. Horn and C. R. Johnson, *Matrix Analysis.* Cambridge: Cambridge University Press, 1987