Computational Mechanics (1990) 6, 77–91

Computational
Mechanics

Numerical solution of nonlinear hyperbolic conservation laws using exponential splines

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Abstract. Previous theoretical (McCartin 1989a) and computational (McCartin 1989b) results on exponential splines are herein applied to provide approximate solutions of high order accuracy to nonlinear hyperbolic conservation laws. The automatic selection of certain "tension" parameters associated with the exponential spline allows the sharp resolution of shocks and the suppression of any attendant oscillations. Specifically, spatial derivatives are replaced by nodal derivatives of interpolatory splines and temporal discretization is achieved via a Runge–Kutta time stepping procedure. The fourth order accuracy of this scheme in both space and time (for uniform mesh and tension) is established and a linearized stability analysis is provided. The Lax–Wendroff theorem on convergence to weak solutions (Lax and Wendroff 1960) is then extended to spline approximations in conservation form. An implicit artificial viscosity term (Anderson et al. 1984) is included via upwinding in conservation form in order to assure convergence to the physically relevant weak solution. The efficacy of this procedure is illustrated on the inviscid Burgers' equation where the accurate capture of a travelling shockwave is demonstrated.

1 Introduction

In this paper, we take up the application of prior work on exponential splines (McCartin 1981, 1983, 1989a, b) to the numerical solution of nonlinear hyperbolic conservation laws. Many physical phenomena, particularly in fluid dynamics, are governed by such laws. For reasons to be discussed subsequently, spline approximations have hitherto not been successfully applied to such problems.

McCartin 1989a presented the exponential spline as a generalization of the semi-classical cubic spline in which the presence of certain tension parameters provides for the adjustment of the tautness of individual spline segments. The interpolant so constructed allows the replication of convexity and monotonicity properties of the function being approximated, i.e. the interpolant is shape preserving. A wealth of theoretical results concerning exponential splines can be found in McCartin 1989a.

The existence of tension parameters fulfilling the shape preservation capabilities of exponential splines was originally established in a non-constructive fashion. The lack of a suitable tension parameter selection scheme thus hindered the widespread use of exponential splines. Consequently, with few exceptions (Flaherty and Mathon 1980; Rubin and Graves 1975; Smith and Wiegel 1980), the subject of the application of exponential splines has been largely neglected in the literature. However, McCartin 1989b presented such a reliable algorithm. The removal of this stumbling block has led to the application of exponential splines to a wide variety of geometric and data fitting problems in computational fluid dynamics (McCartin 1983). A

* This work was supported in part under Contract DE-AC02-76ER07077 with the U.S. Department of Energy, Divisıon of Basic Energy Sciences, Applied Mathematical Sciences Program; and NASA Grants Nos. NGT-33-016-800 and NGT-33-016-201

thorough treatment of computational issues concerning exponential splines is available in McCartin 1989b.

Herein, we embark on an investigation of the possibility of using the shape preservation properties of exponential splines to inhibit the emergence of wiggles and overshoots/undershoots in the numerical simulation of flows with shock waves, while retaining a high order of accuracy. The proposed scheme replaces spatial derivatives by derivatives at the nodes of interpolatory exponential splines and temporal discretization is provided by a Runge–Kutta time stepping procedure.

We commence with a linear model problem. Here, we demonstrate the fourth order accuracy of the scheme for uniform mesh and tension. A von Neumann stability analysis provides the C-F-L condition. The dissipative and dispersive properties of the scheme are then established.

Nonlinearity is introduced via the inviscid Burgers' equation. At this juncture, we generalize the Lax–Wendroff theorem (Lax and Wendroff 1960) concerning convergence of approximations to weak solutions to the case of our spline scheme. As Lax 1973 has pointed out, since there are in general an infinite number of weak solutions, we need to impose an entropy inequality to extract the physically correct solution. This is accomplished by the inclusion of an implicit artificial viscosity term via upwinding in conservation form (Jameson 1978) that is compatible with the spatial discretization error. The accurate simulation of shock fronts is illustrated numerically.

The significant achievement of this technique is that the proposed scheme yields a numerical solution that is third order accurate (fourth order accurate for uniform mesh and tension) in smooth regions of the flow, while accurately capturing any discontinuities that arise without introducing significant wiggles or overshoots/undershoots.

2 Review of theory

In this section, we review the theoretical results on exponential splines (McCartin 1989a) that will be of service to us in the remaining pages. We begin with some notation. Below, first/second derivative formulation refers to which derivative of the spline is the unknown.

$a = x_1 < \cdots < x_{N+1} = b$ = spline nodes
$h_i = x_{i+1} - x_i \ (i = 1, \ldots, N)$ = spline interval lengths
$p_i \ (i = 1, \ldots, N)$ = spline interval tensions
f = data
s = cubic spline interpolant
τ = exponential spline interpolant

$$
\begin{aligned}
b_1 &= \frac{f_2 - f_1}{h_1} - f'(a) \\
b_i &= \frac{f_{i+1} - f_i}{h_i} - \frac{f_i - f_{i-1}}{h_{i-1}} \quad (i = 2, \ldots, N), \\
b_{N+1} &= f'(b) - \frac{f_{N+1} - f_N}{h_N}
\end{aligned}
\tag{2.1}
$$

$$s_i \ (i = 1, \ldots, N) = \sinh(p_i h_i)$$
$$c_i \ (i = 1, \ldots, N) = \cosh(p_i h_i)$$
$$d_i \ (i = 1, \ldots, N) = \left[p_i \frac{c_i}{s_i} - \frac{1}{h_i} \right] \Big/ p_i^2$$
$$e_i \ (i = 1, \ldots, N) = \left[\frac{1}{h_i} - \frac{p_i}{s_i} \right] \Big/ p_i^2$$

We then have the second derivative formulation on $[x_i, x_{i+1}]$:

$$\tau(x) = \frac{1}{p_i^2 s_i}\{\tau_i'' \sinh p_i(x_{i+1} - x) + \tau_{i+1}'' \sinh p_i(x - x_i)\}$$
$$+ \left[f_i - \frac{\tau_i''}{p_i^2}\right]\cdot\frac{x_{i+1} - x}{h_i} + \left[f_{i+1} - \frac{\tau_{i+1}''}{p_i^2}\right]\cdot\frac{x - x_i}{h_i} \tag{2.2}$$

where $\tau_i''(i = 1, \ldots, N+1)$ are the solution of the tridiagonal system

$$\begin{aligned} d_1\tau_1'' + e_1\tau_2'' &= b_1 \\ e_{i-1}\tau_{i-1}'' + (d_{i-1} + d_i)\tau_i'' + e_i\tau_{i+1}'' &= b_i \quad (i = 2, \ldots, N) \\ e_N\tau_N'' + d_N\tau_{N+1}'' &= b_{N+1}. \end{aligned} \tag{2.3}$$

In the above, we have used specified first derivative end conditions.

We also have the first derivative formulation on $[x_i, x_{i+1}]$:

$$\tau(x) = f_i\cdot\frac{x_{i+1} - x}{h_i} + f_{i+1}\cdot\frac{x - x_i}{h_i} + \frac{1}{e_i - d_i}\cdot\frac{f_{i+1} - f_i}{h_i}$$
$$\cdot\left[\frac{\sinh p_i(x - x_i) - \sinh p_i(x_{i+1} - x)}{p_i^2 s_i} + \frac{x_{i+1} - 2x + x_i}{p_i^2 h_i}\right]$$
$$+ \tau_i'\left\{\frac{d_i}{e_i^2 - d_i^2}\cdot\left[\frac{\sinh p_i(x_{i+1} - x)}{p_i^2 s_i} - \frac{x_{i+1} - x}{p_i^2 h_i}\right] - \frac{e_i}{e_i^2 - d_i^2}\cdot\left[\frac{\sinh p_i(x - x_i)}{p_i^2 s_i} - \frac{x - x_i}{p_i^2 h_i}\right]\right\}$$
$$+ \tau_{i+1}'\left\{\frac{e_i}{e_i^2 - d_i^2}\cdot\left[\frac{\sinh p_i(x_{i+1} - x)}{p_i^2 s_i} - \frac{x_{i+1} - x}{p_i^2 h_i}\right] - \frac{d_i}{e_i^2 - d_i^2}\cdot\left[\frac{\sinh p_i(x - x_i)}{p_i^2 s_i} - \frac{x - x_i}{p_i^2 h_i}\right]\right\} \tag{2.4}$$

where $\tau_i'(i = 1, \ldots, N+1)$ are the solution to the tridiagonal system

$$\left[\frac{d_1}{d_1^2 - e_1^2}\right]\tau_1' + \left[\frac{e_1}{d_1^2 - e_1^2}\right]\tau_2' = \left[\frac{1}{d_1 - e_1}\right]\left[\frac{f_2 - f_1}{h_1}\right] - \tau_1''$$

$$\left[\frac{e_{i-1}}{d_{i-1}^2 - e_{i-1}^2}\right]\tau_{i-1}' + \left[\frac{d_{i-1}}{d_{i-1}^2 - e_{i-1}^2} + \frac{d_i}{d_i^2 - e_i^2}\right]\tau_i' + \left[\frac{e_i}{d_i^2 - e_i^2}\right]\tau_{i+1}' = \left[\frac{1}{d_{i-1} - e_{i-1}}\right]\left[\frac{f_i - f_{i-1}}{h_{i-1}}\right]$$
$$+ \left[\frac{1}{d_i - e_i}\right]\left[\frac{f_{i+1} - f_i}{h_i}\right]\left[\frac{e_N}{d_N^2 - e_N^2}\right]\tau_N' + \left[\frac{d_N}{d_N^2 - e_N^2}\right]\tau_{N+1}' = \tau_{N+1}'' + \left[\frac{1}{d_N - e_N}\right]\left[\frac{f_{N+1} - f_N}{h_N}\right]. \tag{2.5}$$

In the above, we have used specified second derivative end conditions.

We note that, although the second derivative formulation is more compact, the first derivative formulation is sometimes preferred. For example, this occurs when using exponential splines to approximate the solution of a first order differential equation, as is the case in this paper. Algorithms for the automatic selection of the exponential spline tension parameters are presented in Appendix A.

3 Stability, dissipativity, and dispersivity

In the following section, we will apply spline approximations to the numerical solution of nonlinear hyperbolic conservation laws. The stability, dissipativity, and dispersivity properties of such a scheme can most easily be studied by considering model equations with constant coefficients.

Therefore, in this section we investigate the prototypical equation

$$u_t + cu_x = 0, \qquad (x,t)\varepsilon(-\infty,\infty)\times[0,\infty) \tag{3.1}$$

with $u(x,0)=\phi(x)$. All calculations are to be performed on a mesh with uniform Δx and Δt.

Consider the Runge–Kutta scheme of fourth order

$$\begin{aligned} u^{(0)} &= u^n & u^{(3)} &= u^{(0)} - c\Delta t D_x u^{(2)} \\ u^{(1)} &= u^{(0)} - \frac{c\Delta t}{2} D_x u^{(0)} & u^{(4)} &= u^{(0)} - \frac{c\Delta t}{6} D_x (u^{(0)} + 2u^{(1)} + 2u^{(2)} + u^{(3)}) \\ u^{(2)} &= u^{(0)} - \frac{c\Delta t}{2} D_x u^{(1)} & u^{n+1} &= u^{(4)}. \end{aligned}$$

If we use the spline approximation to D_x we obtain the expression for the amplification factor (Vishnevetsky and Bowles 1982) as follows.

Let

$$u_j^n = e^{iqx_j}, \quad u_j^{n+1} = g e^{iqx_j}, \quad \xi = q\Delta x, \quad \lambda = c\frac{\Delta t}{\Delta x}. \tag{3.3}$$

Then

$$g = 1 - 2\rho i - 2\rho^2 + \tfrac{4}{3}\rho^3 i + \tfrac{2}{3}\rho^4, \tag{3.4}$$

where

$$\rho = \frac{\lambda(d+e)\sin\xi}{2(e\cos\xi + d)}. \tag{3.5}$$

Thus

$$|g|^2 = (1 - 2\rho^2 + \tfrac{2}{3}\rho^4)^2 + 4\rho^2(\tfrac{2}{3}\rho^2 - 1)^2. \tag{3.6}$$

The C-F-L condition for stability requires that the magnitude of the amplification factor, g, be less than unity. Accordingly, we restrict

$$|\rho| < \sqrt{2}, \tag{3.7}$$

which requires the following time–step restriction

$$|\lambda| < 2\sqrt{2}\cdot\sqrt{\frac{2d}{d+e} - 1}. \tag{3.8}$$

Hence, for the cubic spline ($p = 0$, $e/d = 1/2$) we have

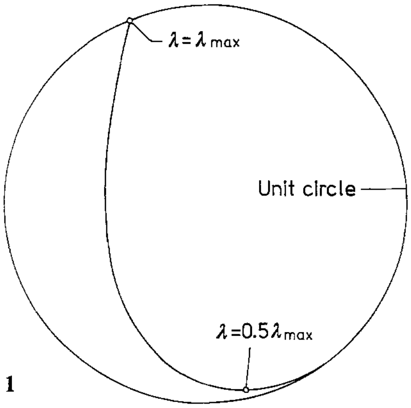
$$|\lambda| < 2\sqrt{\tfrac{2}{3}} \approx 1.633, \tag{3.9}$$

while for the linear spline ($p = \infty$, $e/d = 0$) we have

$$|\lambda| < 2\sqrt{2} \approx 2.828. \tag{3.10}$$

Our explicit expression for g can also be used to study the dissipative and dispersive properties of the spline-based Runge–Kutta scheme. This is accomplished by studying polar plots of its amplitude and phase versus the frequency, ξ. Complete details of this brand of analysis are available in Anderson et al. (1984).

Figure 1 displays the locus of the amplification factor, g, in the complex plane as the frequency, ξ, varies. Curves are shown for both maximum and mid-range Courant numbers (the curves overlap). We observe that the mid-range choice clearly results in less numerical distortion of the initial wavefront.

1


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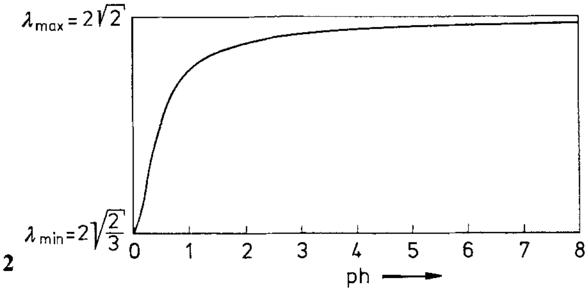
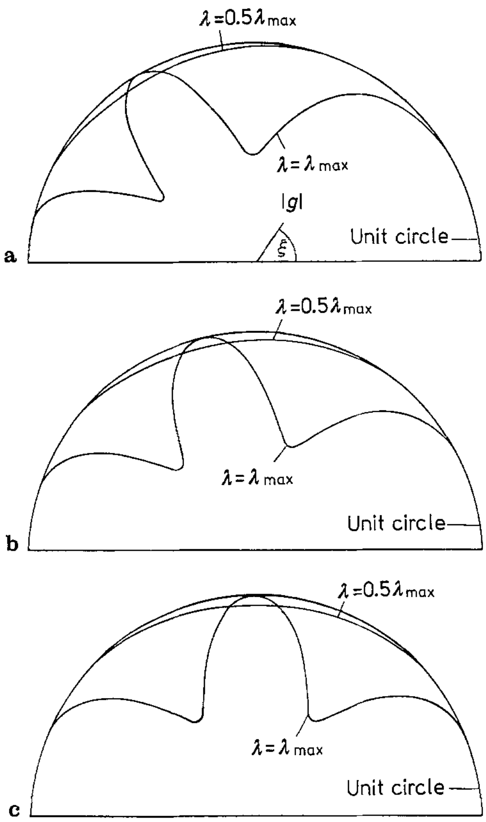


Figs. 1 and 2. 1 Amplification factor. **2** C-F-L number

3


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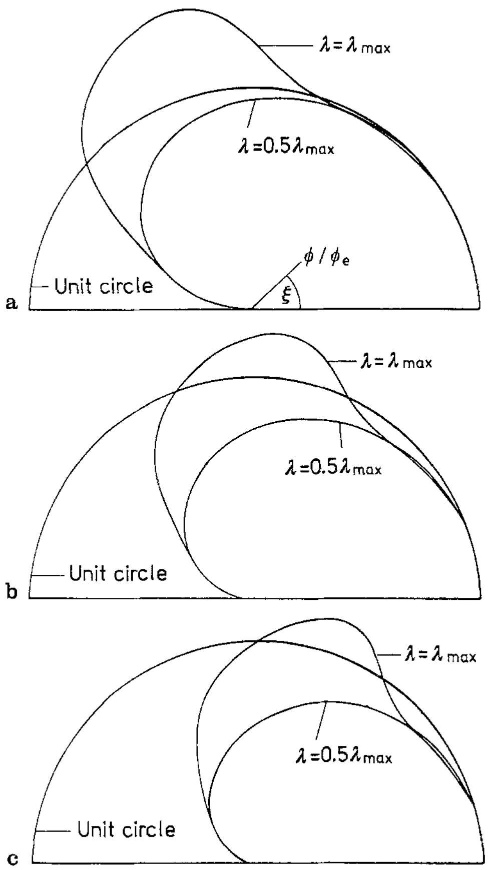

Figs. 3 and 4. 3a–c Modulus of amplification factor; **4a–c** phase error. **a** $ph = 0$; **b** $ph = 5$; **c** $ph = \infty$

Figure 2 presents the maximum Courant number as a function of normalized tension. Note that the addition of a small amount of tension substantially improves the maximum permissible time step and that the combination ph is the relevant parameter. However, for $ph > 5$ we have clearly reached the point of diminishing returns.

Figure 3 is a polar plot of the modulus of the amplification factor as a function of frequency for various values of applied tension. The dissipative properties of our scheme are readily apparent. Firstly, we again see that the mid-range Courant number prevents heavy damping of the Fourier modes. Secondly, we observe a shift of the damping to lower frequencies as the tension is increased.

Figure 4 is a corresponding plot of the phase error. The dispersive properties of our scheme are thus revealed. Firstly, the upper-range Courant number results in leading phase errors while the mid-range produces strictly lagging phase errors. Secondly, we see that increased tension results in increased phase error in the low frequency modes.

The above results are not too surprising since at the nodes (Appendix B)

$$D_x\tau = u_x + \frac{h^4}{180}(p^2 u_{xxx} - u_{xxxxx}) + O(h^6), \tag{3.11}$$

so that approximation by spline derivatives is seen to be primarily dispersive as is evidenced by the leading order odd derivatives in the truncation error (Anderson et al. 1984).

Because of the accuracy (fourth order temporal and spatial on a uniform mesh with uniform tension) and stability of this exponential spline/Runge–Kutta scheme, it is used exclusively in our subsequent calculations. In addition, no modifications are needed at the boundaries since the spline is fourth order accurate over the full range of interpolation and no loss of accuracy results.

4 Convergence

Consider the scalar conservation law (Lax 1973)

$$u_t = f_x(u) \tag{4.1}$$

subject to the initial condition

$$u(x,0) = \phi(x). \tag{4.2}$$

This can be rewritten as

$$u_t = a u_x \tag{4.3}$$

with

$$a = \frac{df}{du}. \tag{4.4}$$

As such, we recognize it as a nonlinear analogue of the linear model problem of the previous section.

Since $f(u)$ is in general nonlinear, we cannot guarantee the existence of a smooth solution for all time. Instead, we seek weak solutions satisfying

$$\int_0^\infty \int_{-\infty}^\infty (w_t u - w_x f)\,dx\,dt + \int_{-\infty}^\infty w(x,0)\phi(x)\,dx = 0, \tag{4.5}$$

for all smooth test functions, w, which vanish for $|x| + t$ large enough.

One must consider whether a numerical solution technique has the capability of capturing such a weak solution. We next present the following convergence result which is a generalization of a theorem of Lax and Wendroff 1960. First, recall that the exponential spline, $\tau(x)$, can be defined in terms of its first derivatives at the knots. These values are determined by the tridiagonal system of linear algebraic equations

$$A_i\tau'_{i-1} + B_i\tau'_i + C_i\tau'_{i+1} = \alpha_i\cdot\frac{f_{i+1} - f_i}{h_i} + \beta_i\cdot\frac{f_i - f_{i-1}}{h_{i-1}} \tag{4.6}$$

where $A_i, B_i, C_i, \alpha_i, \beta_i$ were provided in our review of theory. What is of key importance to us here is that

$$A_i + B_i + C_i = \alpha_i + \beta_i = 1. \tag{4.7}$$

In what follows, we use the notation $\Delta g^n = g^{n+1} - g^n$ and avail ourselves of the summation by parts formula

$$\sum_{n=0}^{\infty} u^n \Delta v^n = -u^0 v^0 - \sum_{n=0}^{\infty} \Delta u^n v^{n+1}. \tag{4.8}$$

Theorem 4.1. *If we approximate* $u_t = f_x$ *by*

$$\frac{\Delta v^n}{k_n} = \tau_x \tag{4.9}$$

and if v *converges boundedly almost everywhere to some function* $u(x,t)$ *as* $k = \max_i k_i$ *and* $h = \max_i h_i$ *tend to zero with uniformly bounded tension parameters, then* $u(x,t)$ *is a weak solution.*

Proof. We have by assumption

$$\frac{v_i^{n+1} - v_i^n}{k_n} = (\tau_x)_i^n, \quad v_i^0 = \phi_i \tag{4.10}$$

and hence

$$A_i^n\left[\frac{v_{i-1}^{n+1} - v_{i-1}^n}{k_n}\right] + B_i^n\left[\frac{v_i^{n+1} - v_i^n}{k_n}\right] + C_i^n\left[\frac{v_{i+1}^{n+1} - v_{i+1}^n}{k_n}\right] = \alpha_i^n \cdot \frac{f_{i+1}^n - f_i^n}{h_i} + \beta_i^n \cdot \frac{f_i^n - f_{i-1}^n}{h_{i-1}}. \tag{4.11}$$

Multiplying both sides by $w_i^n k_n((h_{i-1} + h_i)/2)$ and summing over all grid points yields

$$\sum_{n=0}^{\infty} \sum_{i=-\infty}^{\infty} \frac{w_i^n}{2}[A_i^n(v_{i-1}^{n+1} - v_{i-1}^n) + B_i^n(v_i^{n+1} - v_i^n) + C_i^n(v_{i+1}^{n+1} - v_{i+1}^n)](h_{i-1} + h_i)$$
$$= \sum_{n=0}^{\infty} \sum_{i=-\infty}^{\infty} \frac{w_i^n k_n}{2}\left[\alpha_i^n \cdot \frac{f_{i+1}^n - f_i^n}{h_i} + \beta_i^n \cdot \frac{f_i^n - f_{i-1}^n}{h_{i-1}}\right](h_{i-1} + h_i) \tag{4.12}$$

or $LHS = RHS$ for convenience.

$$LHS = \sum_{i=-\infty}^{\infty} \frac{h_{i-1} + h_i}{2} \sum_{n=0}^{\infty} [(w_i^n A_i^n)\Delta v_{i-1}^n + (w_i^n B_i^n)\Delta v_i^n + (w_i^n C_i^n)\Delta v_{i+1}^n]$$
$$= -\sum_{i=-\infty}^{\infty} \frac{h_{i-1} + h_i}{2} \sum_{n=0}^{\infty} [\Delta(w_i^n A_i^n) v_{i-1}^{n+1} + \Delta(w_i^n B_i^n) v_i^{n+1} + \Delta(w_i^n C_i^n) v_{i+1}^{n+1}]$$
$$- \sum_{i=-\infty}^{\infty} \frac{h_{i-1} + h_i}{2} (w_i^0 A_i^0 v_{i-1}^0 + w_i^0 B_i^0 v_i^0 + w_i^0 C_i^0 v_{i+1}^0). \tag{4.13}$$

Thus,

$$LHS \to -\int_0^{\infty} \int_{-\infty}^{\infty} w_t u \, dx \, dt - \int_{-\infty}^{\infty} w(x,0)\phi(x) \, dx. \tag{4.14}$$

$$RHS = \sum_{n=0}^{\infty} \frac{k_n}{2} \sum_{i=-\infty}^{\infty} \left[w_i^n \alpha_i^n \frac{h_{i-1} + h_i}{h_i} f_{i+1}^n + w_i^n \left(\frac{\beta_i^n}{h_{i-1}} - \frac{\alpha_i^n}{h_i}\right)(h_{i-1} + h_i) f_i^n - w_i^n \beta_i^n \frac{h_{i-1} + h_i}{h_{i-1}} f_{i-1}^n \right]$$
$$= \sum_{n=0}^{\infty} \frac{k_n}{2} \sum_{i=-\infty}^{\infty} f_i^n \left[w_{i-1}^n \alpha_{i-1}^n \frac{h_{i-2} + h_{i-1}}{h_{i-1}} + w_i^n \left(\frac{\beta_i^n}{h_{i-1}} - \frac{\alpha_i^n}{h_i}\right)(h_{i-1} + h_i) - w_{i+1}^n \beta_{i+1}^n \frac{h_i + h_{i+1}}{h_i} \right]. \tag{4.15}$$

As shown in Appendix B, as $h \to 0$ with uniformly bounded tension

$$\alpha_i^n \to \frac{h_i}{h_{i-1} + h_i}, \quad \beta_i^n \to \frac{h_{i-1}}{h_{i-1} + h_i}. \tag{4.16}$$

Thus,

$$RHS \to -\int_0^{\infty}\int_{-\infty}^{\infty} f w_x \, dx \, dt. \tag{4.17}$$

That is, $u(x,t)$ is a weak solution. □

5 Inviscid Burgers' equation

In this section, we provide our treatment of nonlinear hyperbolic conservation laws using splines. The subject of discussion is a nonlinear analogue of our model problem, namely the inviscid Burgers' equation (Sod 1985)

$$u_t + uu_x = 0, \quad (x,t)\varepsilon(-\infty,\infty)\times[0,\infty). \tag{5.1}$$

This equation, which is in quasilinear form, can be put into the conservation form

$$u_t + \left(\frac{u^2}{2}\right)_x = 0. \tag{5.2}$$

This equation admits travelling shock wave solutions (Lax 1973). We consider the following initial data

$$\phi(x) = \begin{cases} 1 \quad , & x \leqq 0 \\ 1-x, & 0 < x < 1 \\ 0 \quad , & 1 \leqq x \end{cases} \tag{5.3}$$

This wave front steepens until at $t = 1$ a discontinuity develops. Specifically, for $t < 1$

$$u(x,t) = \begin{cases} 1 \quad , & x \leqq t \\ \dfrac{x-1}{t-1}, & t < x < 1 \\ 0 \quad , & 1 \leqq x \end{cases} \tag{5.4}$$

while for $t \geqq 1$

$$u(x,t) = \begin{cases} 1, & x < \dfrac{1+t}{2} \\ 0, & \dfrac{1+t}{2} < x \end{cases} \tag{5.5}$$

The physically relevant solution is the one satisfying the so-called entropy inequality. This condition stipulates that the characteristics issuing from either from either side of the discontinuity curve in the direction of increasing t should intersect the line of discontinuity.

The task before us is now clear. We must devise a numerical scheme that will capture discontinuities as well as select the physically relevant solution from the, generally infinite, collection of weak solutions to an initial value problem.

This is accomplished by adding a suitable implicit artificial viscosity term (Jameson 1978) to the exponential spline/Runge–Kutta scheme as previously applied to the model problem. This is achieved by shifting the spatial derivative evaluation upstream thus modelling the correct domain of dependence while prohibiting expansion shocks.

Specifically, we solve

$$u_t = f_x(u(x + sgn(f'(u))\varepsilon h, t)) \tag{5.6}$$

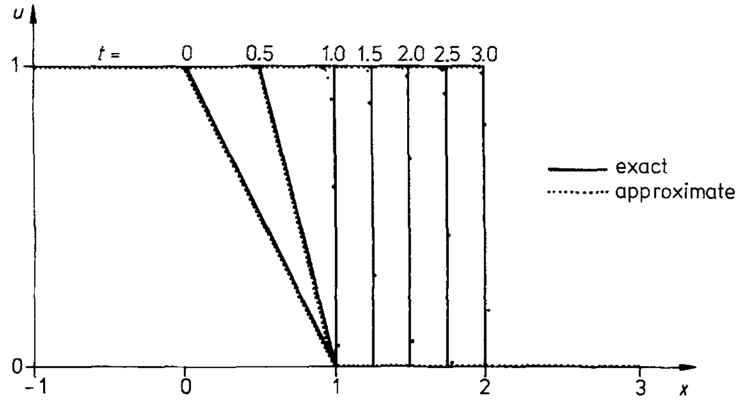

Fig. 5. Shock wave computation

where

$$\varepsilon = \min(\alpha h^3 |u_x|^4 + \beta h^3 |u_{xx}|^2, 1/2). \tag{5.7}$$

Expansion in Taylor series reveals that

$$u_t = f_x(u) + sgn(f'(u))\varepsilon h f_{xx}(u) + O(\varepsilon^2). \tag{5.8}$$

Several observations should be made. Firstly, the artificial viscosity so introduced is $O(h^4)$ so that fourth order accuracy is retained. Secondly, the modified equation is still in conservation form so that weak solutions are still captured. Thirdly, the specific form for ε was selected to switch on in regions of high gradient or curvature and the diffusion parameters α and β were included to allow control of the width of the shock transition region. Finally, the above scheme should reduce the dispersion errors previously observed since the artificially induced diffusivity should serve to attenuate the higher order modes.

Figure 5 shows our computed solution together with the exact solution. In this calculation, values of $\Delta t = \Delta x = 0.02$ were used. These parameters were selected to be compatible with those of Sod 1985 where several other schemes are presented for comparative purposes. Also, $\alpha = \beta = 0.5$ were used. Evident from these numerical results are the correct shock location, the sharp shock resolution, and the conspicuous lack of overshoots and undershoots at the shock.

6 Conclusions

Building upon previous theoretical and computational work on exponential splines, we have introduced a numerical scheme for highly accurate approximation to nonlinear hyperbolic conservation laws. In addition to its high order of accuracy, this scheme provides numerical simulation of discontinuities without any attendant oscillations. Numerical results have been presented illustrating the efficacy of this technique.

The analysis presented in this paper can be extended to systems of conservation laws. McCartin 1981 presents such an extension and applies the technique to the shock tube problem. Furthermore these results can be extended to multiple spatial dimensions. McCartin 1981 presents this extension and applies the method to transonic channel flow. Both of these topics will be the subject of separate publications.

A) Tension parameter selection algorithms. The following algorithms provide for the automatic selection of the exponential spline tension parameters. Algorithm COCONVEX incorporates convexity constraints while Algorithm COMONOTONE incorporates monotonicity constraints. Convexity and monotonicity constraints can be enforced simultaneously by taking the pointwise maximum of the tensions supplied by Algorithms COCONVEX and COMONOTONE individually. Complete details are available in McCartin 1989b.

A.1) Algorithm COCONVEX

do $i \leftarrow 1(1)N$

$\dot{p}_i \leftarrow p_i$

end do

if $\tau_1'' b_1 < 0$ then

$$\bar{\lambda} \leftarrow \frac{\max(|b_1|, d_1|\tau_1''|)}{|\tau_2''|}; \quad \bar{p} \leftarrow \max((\bar{\lambda}h_1)^{-1/2}, p_1); \quad \dot{p}_1 \leftarrow \max(\bar{p}, \dot{p}_1)$$

end if

do $i \leftarrow 2(1)N$

if $\tau_i'' = 0$ then

$\dot{p}_{i-1} \leftarrow \max(p_{i-1} + \epsilon, \dot{p}_{i-1})$

end if

if $\tau_i'' b_i < 0$

$$\bar{\lambda} \leftarrow \frac{\max(|b_i|, (d_{i-1} + d_i)|\tau_i''|)}{2\max(|\tau_{i-1}''|, |\tau_{i+1}''|)}$$

$\bar{p} \leftarrow \max((\bar{\lambda}h_{i-1})^{-1/2}, p_{i-1});$ $\quad \dot{p}_{i-1} \leftarrow \max(\bar{p}, \dot{p}_{i-1})$

$\bar{p} \leftarrow \max((\bar{\lambda}h_i)^{-1/2}, p_i);$ $\quad \dot{p}_i \leftarrow \max(\bar{p}, \dot{p}_i)$

end if

end do

if $\tau_{N+1}'' b_{N+1} < 0$ then

$$\bar{\lambda} \leftarrow \frac{\max(|b_{N+1}|, d_N|\tau_{N+1}''|)}{|\tau_N''|}; \quad \bar{p} \leftarrow \max((\bar{\lambda}h_N)^{-1/2}, p_N); \quad \dot{p}_N \leftarrow \max(\bar{p}, \dot{p}_N)$$

end if

do $i \leftarrow 1(1)N$

$p_i \leftarrow p_i + \omega(\dot{p}_i - p_i)$

end do

A.2) Algorithm COMONOTONE

do $i \leftarrow 1(1)N$

$\dot{p}_i \leftarrow p_i$

end do

do $i \leftarrow 2(1)N - 1$

if $m_{i-1}m_i \geq 0$ and $m_i \cdot m_{i+1} \geq 0$ then

if $\tau_i' \cdot m_i < 0$ then

if $p_{i-1} = 0$ then

$\dot{p}_{i-1} \leftarrow \delta$

else

$$\bar{p} \leftarrow \frac{1 + p_{i-1}h_{i-1}}{p_{i-1}h_{i-1}} \cdot \frac{2\max(|\tau''_{i-1}|, |\tau''_i|, |\tau''_{i+1}|)}{|m_{i-1} + m_i|} + \delta$$

$\dot{p}_{i-1} \leftarrow \max(\bar{p}, \dot{p}_{i-1})$

end if

if $p_i = 0$ then

$p_i \leftarrow \delta$

else

$$\bar{p} \leftarrow \frac{1 + p_i h_i}{p_i h_i} \cdot \frac{2\max(|\tau''_{i-1}|, |\tau''_i|, |\tau''_{i+1}|)}{|m_{i-1} + m_i|} + \delta$$

$\dot{p}_i \leftarrow \max(\bar{p}, \dot{p}_i)$

end if

end if

if $\tau'_{i+1} \cdot m_{i+1} < 0$ then

if $p_i = 0$ then

$\dot{p}_i \leftarrow \delta$

else

$$\bar{p} \leftarrow \frac{1 + p_i h_i}{p_i h_i} \cdot \frac{2\max(|\tau''_i|, |\tau''_{i+1}|, |\tau''_{i+2}|)}{|m_i + m_{i+1}|} + \delta$$

$\dot{p}_i \leftarrow \max(\bar{p}, \dot{p}_i)$

end if

if $p_{i+1} = 0$ then

$\dot{p}_{i+1} \leftarrow \delta$

else

$$\bar{p} \leftarrow \frac{1 + p_{i+1}h_{i+1}}{p_{i+1}h_{i+1}} \cdot \frac{2\max(|\tau''_i|, |\tau''_{i+1}|, |\tau''_{i+2}|)}{|m_i + m_{i+1}|}$$

$\dot{p}_{i+1} \leftarrow \max(\bar{p}, \dot{p}_{i+1})$

end if

end if

if $\tau'_i \cdot m_i \geq 0$ and $\tau'_{i+1} m_{i+1} \geq 0$ then

if $C_i > 0$ and $D_i < 0$ then

if $m_i \leq 0$ then

$\dot{p}_i \leftarrow \max(p_i, \dot{p}_i)$

else

$\bar{p} \leftarrow -B_i/(2\sqrt{-C_i D_i}) + \delta$

$\dot{p}_i \leftarrow \max(\bar{p}, p_i)$

end if

end if

if $C_i < 0$ and $D_i > 0$ then

if $m_i \geq 0$ then

$\dot{p}_i \leftarrow \max(p_i, \dot{p}_i)$

```
                          else
                                    p̄ ← B_i/(2 √(−C_iD_i)) + δ

                                    p̂_i ← max(p̄, p̂_i)
                          end if
                     end if
                end if
           end if
end do
do    i ← 1(1)N
     p_i ← p_i + ω(p̂_i − p_i)
end do
```

B) Expansion of exponential spline derivatives. Before tabulating power series expansions for exponential spline derivatives, we assemble the following expansions for the spline coefficients themselves. The complete derivation of all ensuing expressions is contained in McCartin 1981.

$$\begin{aligned}
e &= \frac{h}{6}[1 - \tfrac{7}{60}p^2h^2 + \tfrac{31}{2520}p^4h^4 + O(p^6h^6)] \\
d &= \frac{h}{3}[1 - \tfrac{1}{15}p^2h^2 + \tfrac{2}{315}p^4h^4 + O(p^6h^6)] \\
\frac{1}{d-e} &= \frac{1}{h}[6 + \tfrac{1}{10}p^2h^2 - \tfrac{1}{1400}p^4h^4 + O(p^6h^6)] \\
\frac{1}{d+e} &= \frac{1}{h}[2 + \tfrac{1}{6}p^2h^2 - \tfrac{1}{360}p^4h^4 + O(p^6h^6)] \\
\frac{1}{h(d-e)} &= \frac{1}{h^2}[6 + \tfrac{1}{10}p^2h^2 - \tfrac{1}{1400}p^4h^4 + O(p^6h^6)] \\
\frac{1}{d^2-e^2} &= \frac{1}{h^2}[12 + \tfrac{6}{5}p^2h^2 - \tfrac{1}{700}p^4h^4 + O(p^6h^6)] \\
\frac{e}{d^2-e^2} &= \frac{1}{h}[2 - \tfrac{1}{30}p^2h^2 + \tfrac{13}{12600}p^4h^4 + O(p^6h^6)] \\
\frac{d}{d^2-e^2} &= \frac{1}{h}[4 + \tfrac{2}{15}p^2h^2 - \tfrac{11}{6300}p^4h^4 + O(p^6h^6)]
\end{aligned} \tag{B.1}$$

B.1) First derivative. We require the following parameter definitions in order to describe the accuracy of spline first derivatives.

$$\begin{aligned}
a_0 &= 6\left(\frac{1}{h_{i-1}} + \frac{1}{h_i}\right) + \tfrac{1}{10}(p_{i-1}^2h_{i-1} + p_i^2h_i) - \tfrac{1}{1400}(p_{i-1}^4h_{i-1}^3 + p_i^4h_i^3) + \cdots \infty \\
a_1 &= \tfrac{1}{30}(p_{i-1}^2h_{i-1}^2 - p_i^2h_i^2) - \tfrac{13}{12600}(p_{i-1}^4h_{i-1}^4 - p_i^4h_i^4) + \cdots \infty \\
a_2 &= (h_{i-1} + h_i) - \tfrac{1}{60}(p_{i-1}^2h_{i-1}^3 + p_i^2h_i^3) + \tfrac{13}{25200}(p_{i-1}^4h_{i-1}^5 + p_i^4h_i^5) + \cdots \infty \\
a_3 &= -\tfrac{1}{3}(h_{i-1}^2 - h_i^2) + \tfrac{1}{180}(p_{i-1}^2h_{i-1}^4 - p_i^2h_i^4) - \tfrac{13}{75600}(p_{i-1}^4h_{i-1}^6 - p_i^4h_i^6) + \cdots \infty \\
a_4 &= \tfrac{1}{12}(h_{i-1}^3 + h_i^3) - \tfrac{1}{720}(p_{i-1}^2h_{i-1}^5 + p_i^2h_i^5) + \tfrac{13}{302400}(p_{i-1}^4h_{i-1}^7 + p_i^4h_i^7) + \cdots \infty \\
a_5 &= -\tfrac{1}{60}(h_{i-1}^4 - h_i^4) + \tfrac{1}{3600}(p_{i-1}^2h_{i-1}^6 - p_i^2h_i^6) - \tfrac{13}{1512000}(p_{i-1}^4h_{i-1}^8 - p_i^4h_i^8) + \cdots \infty
\end{aligned} \tag{B.2}$$

$$b_0 = \frac{1}{a_0}$$

$$b_1 = -\frac{a_1}{a_0^2}$$

$$b_2 = \frac{a_1^2}{a_0^3} - \frac{a_2}{a_0^2}$$

$$b_3 = -\frac{a_1^3}{a_0^4} + \frac{2a_1a_2}{a_0^3} - \frac{a_3}{a_0^2} \tag{B.3}$$

$$b_4 = \frac{a_1^4}{a_0^5} - \frac{3a_1^2a_2}{a_0^4} + \frac{a_2^2 + 2a_1a_3}{a_0^3} - \frac{a_4}{a_0^2}$$

$$b_5 = -b_4\frac{a_1}{a_0} - b_3\frac{a_2}{a_0} - b_2\frac{a_3}{a_0} - b_1\frac{a_4}{a_0} - b_0\frac{a_5}{a_0}$$

$$c_1 = a_0$$

$$c_2 = -\tfrac{1}{20}(p_{i-1}^2h_{i-1}^2 - p_i^2h_i^2) + \tfrac{1}{2800}(p_{i-1}^4h_{i-1}^4 - p_i^4h_i^4) + \cdots \infty$$

$$c_3 = (h_{i-1} + h_i) + \tfrac{1}{60}(p_{i-1}^2h_{i-1}^3 + p_i^2h_i^3) - \tfrac{1}{8400}(p_{i-1}^4h_{i-1}^5 + p_i^4h_i^5) + \cdots \infty \tag{B.4}$$

$$c_4 = -\tfrac{1}{4}(h_{i-1}^2 - h_i^2) - \tfrac{1}{240}(p_{i-1}^2h_{i-1}^4 - p_i^2h_i^4) + \tfrac{1}{33600}(p_{i-1}^4h_{i-1}^6 - p_i^4h_i^6) + \cdots \infty$$

$$c_5 = \tfrac{1}{20}(h_{i-1}^3 + h_i^3) + \tfrac{1}{1200}(p_{i-1}^2h_{i-1}^5 + p_i^2h_i^5) - \tfrac{1}{168000}(p_{i-1}^4h_{i-1}^7 + p_i^4h_i^7) + \cdots \infty.$$

In terms of these parameters, we may express

$$\tau_i' = [(b_0c_1)D + (b_1c_1 + b_0c_2)D^2 + (b_2c_1 + b_1c_2 + b_0c_3)D^3 + (b_3c_1 + b_2c_2 + b_1c_3 + b_0c_4)D^4 + (b_4c_1 + b_3c_2 + b_2c_3 + b_1c_4 + b_0c_5)D^5 + \cdots]f_i, \tag{B.5}$$

where D is the differentiation operator, $Df_i = f'(x_i)$. Note that $b_0c_1 = 1$.

We $p_ih_i = ph$ for all i, we have

$$b_1c_1 + b_0c_2 = 0, \quad b_2c_1 + b_1c_2 + b_0c_3 = O(h^4), \quad b_3c_1 + b_2c_2 + b_1c_3 + b_0c_4 = 0 \tag{B.6}$$

while all succeeding terms are $O(h^4)$. That is, for uniform tension and mesh width, the spline first derivative is a fourth order accurate approximation to the first derivative of the approximated function (at the nodes).

B.2) Second derivative. We next proceed to study the accuracy of spline second derivatives for which we need the following parameters.

$$a_0 = \tfrac{1}{2}(h_{i-1} + h_i) - \tfrac{1}{24}(p_{i-1}^2h_{i-1}^3 + p_i^2h_i^3) + \tfrac{1}{240}(p_{i-1}^4h_{i-1}^5 + p_i^4h_i^5) + \cdots \infty$$

$$a_1 = -\tfrac{1}{6}(h_{i-1}^2 - h_i^2) + \tfrac{7}{360}(p_{i-1}^2h_{i-1}^4 - p_i^2h_i^4) - \tfrac{31}{15120}(p_{i-1}^4h_{i-1}^6 - p_i^4h_i^6) + \cdots \infty$$

$$a_2 = \tfrac{1}{12}(h_{i-1}^3 + h_i^3) - \tfrac{7}{720}(p_{i-1}^2h_{i-1}^5 + p_i^2h_i^5) + \tfrac{31}{30240}(p_{i-1}^4h_{i-1}^7 + p_i^4h_i^7) + \cdots \infty$$

$$a_3 = -\tfrac{1}{36}(h_{i-1}^4 - h_i^4) + \tfrac{7}{2160}(p_{i-1}^2h_{i-1}^6 - p_i^2h_i^6) - \tfrac{31}{90720}(p_{i-1}^4h_{i-1}^8 - p_i^4h_i^8) + \cdots \infty$$

$$a_4 = \tfrac{1}{144}(h_{i-1}^5 + h_i^5) - \tfrac{7}{8640}(p_{i-1}^2h_{i-1}^7 + p_i^2h_i^7) + \tfrac{31}{362880}(p_{i-1}^4h_{i-1}^9 + p_i^4h_i^9) + \cdots \infty$$

$$a_5 = -\tfrac{1}{720}(h_{i-1}^6 - h_i^6) + \tfrac{7}{43200}(p_{i-1}^2h_{i-1}^8 - p_i^2h_i^8) - \tfrac{31}{1814400}(p_{i-1}^4h_{i-1}^{10} - p_i^4h_i^{10}) + \cdots \infty \tag{B.7}$$

$$c_2 = \tfrac{1}{2}(h_{i-1} + h_i)$$

$$c_3 = -\tfrac{1}{6}(h_{i-1}^2 - h_i^2)$$

$$c_4 = \tfrac{1}{24}(h_{i-1}^3 + h_i^3) \tag{B.8}$$
$$c_5 = -\tfrac{1}{120}(h_{i-1}^4 - h_i^4)$$

With the b's as previously defined, we have

$$\tau_i'' = [(b_0c_2)D^2 + (b_1c_2 + b_0c_3)D^3 + (b_2c_2 + b_1c_3 + b_0c_4)D^4 + \cdots]f_i \tag{B.9}$$

When $p_ih_i = ph$ for all i, we have

$$\tau_i'' = D^2f_i + \left[\frac{p^2}{12}D^2f_i - \frac{1}{12}D^4f_i\right]h^2 + O(h^4). \tag{B.10}$$

We see that the uniformity conditions do not yield the increase in accuracy encountered in our analysis of first derivative approximation.

In spite of this, the above expansion can be exploited, producing

$$\frac{12}{24+(ph)^2}\left[\tau_i'' + \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2}\right] = D^2f_i + O(h^4) \tag{B.11}$$

which is a fourth order accurate approximation. Note that for the cubic spline ($p = 0$) this reduces to the arithmetic mean.

B.3 Fourth derivative. The above expansions also yield the following second order accurate approximation to the fourth derivative.

$$D^4f_i = \frac{12}{h^2}\left\{\frac{12+(ph)^2}{24+(ph)^2}\left[\frac{e}{h}\tau_{i-1}'' + \left(1 + \frac{2d}{h} - \frac{24+(ph)^2}{12+(ph)^2}\right)\tau_i'' + \frac{e}{h}\tau_{i+1}''\right]\right\} + O(h^2) \tag{B.12}$$

Note that in the case of the cubic spline ($p = 0$, $e = h/6$, $d = h/3$) this reduces to

$$D^4f_i = \frac{\tau_{i-1}'' - 2\tau_i'' + \tau_{i+1}''}{h^2} + O(h^2), \tag{B.13}$$

that is, a central differencing of $\tau''(x)$ at $x = x_i$.

Also,

$$D^4f_i = \frac{p}{s}[\tau_{i-1}'' + (ps - 2c)\tau_i'' + \tau_{i+1}''] + O(h^2). \tag{B.14}$$

Hence, we have yet another second order accurate approximation to the fourth derivative.

Acknowledgements

This research has greatly benefited from the encouragement and advice of Charlotte Ansted, Paul Garabedian, Peter Lax, and Olof Widlund.

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Communicated by S. N. Atluri, March 1, 1989