A Tractable Method for Robust Downlink Beamforming in Wireless Communications

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***Abstract*—In downlink beamforming in a multiple-input multiple-output (MIMO) wireless communication system, we design beamformers that minimize the power subject to guaranteeing given signal-to-interference noise ratio (SINR) threshold levels for the users, assuming that the channel responses between the base station and the users are known exactly. In robust downlink beamforming, we take into account uncertainties in the channel vectors, by designing beamformers that minimize the power subject to guaranteeing given SINR threshold levels over the given set of possible channel vectors. When the uncertainties in channel vectors are described by complex uncertainty ellipsoids, we show that the associated worst-case robust beamforming problem can be solved efficiently using an iterative method. The method uses an alternating sequence of optimization and worst-case analysis steps, where at each step we solve a convex optimization problem using efficient interior-point methods. Typically, the method provides a fairly robust beamformer design within 5–10 iterations. The robust downlink beamforming method is demonstrated with a numerical example.**

I. Introduction

We consider the downlink channel of a MIMO wireless communication system, where a base station equipped with many antennas serves remote users, each equipped with a single antenna. The data is transmitted from the base station to the users using modern MIMO coding techniques, such as spatial beamforming; the reader is referred to recent textbooks, *e.g.*, [1], [2], for more on the techniques.

Several researchers have studied "nominal" downlink beamforming in which we design beamformers that minimize the power subject to guaranteeing given SINR threshold levels for the users, assuming that the channel responses between the base station and the users are known exactly; see, *e.g.*, [3], [4]. In practice, the channel vectors are estimated with error from training sequences, and moreover, vary over time. The imperfect estimation and variations in the channels can greatly affect performance of the overall system, resulting in degradation in users' QoS, and possibly service outage.

There are several general approaches for accounting for uncertain parameters in an optimization problem. In worst-case robust optimization (or minimax optimization), we model the parameters as lying in some given set of possible values, but without any known distribution, and we choose a design that minimizes an objective value while guaranteeing the feasibility of constraints over the given set of possible parameters [5]–[7]. In this model, we do not rely on any knowledge of the distribution of uncertain parameters (which, indeed, need not be stochastic). The worst-case robust optimization approach has been applied to a variety of signal processing problems including robust beamforming [8]–[17], robust power control [18], [19], and downlink beamforming with uncertain channel covariance matrices [3].

In this paper, we are interested in designing robust downlink beamformers that minimize the power subject to guaranteeing given SINR threshold levels for the users over the given set of possible uncertainties. When the uncertainties in channel vectors are described by complex uncertainty ellipsoids, we can solve the problem using an iterative procedure which consists of alternating 'optimization' and 'pessimization' steps, which is described in more detail in [20]. Each of these steps requires solving a convex optimization problem, which can be readily done using interior-point algorithms [21]. The iterative procedure can find good conservative solutions for the robust downlink beamforming problem within 5–10 iterations. The computational effort of the iterative robust beamforming method is practically the same order as solving the nominal downlink beamforming problem (but with a substantially larger constant).

We briefly outline the rest of the paper. In Section II we describe the downlink beamforming problem with perfect channel information and give a short review of the SOCP formulation of the (nominal) downlink beamforming problem derived in [22]. In Section III we describe the worst-case robust beamforming problem and give a tractable iterative solution method. In Section IV we present a numerical example. In Section V we give our conclusions.

II. Downlink beamforming

We consider a base station equipped with n antennas, which serves m remote users, each equipped with a single antenna. The base station (transmitter) uses spatial beamforming to convey information to the remote users (receivers). In the beamforming setting, the transmit signal is $x = \sum_i u_i w_i$, where u_i is a complex scalar denoting the information signal and $w_i \in \mathbf{C}^n$ is the vector of beamforming weights for user $i = 1, \ldots, m$. The signal received by user i is

$$y_i = h_i^* \sum_{j=1}^{m} u_j w_j + z_i, \quad i = 1, \ldots, m,$$

where $h_i \in \mathbf{C}^n$ is the channel vector for the channel between user i and the transmitter, and z_i are independent and

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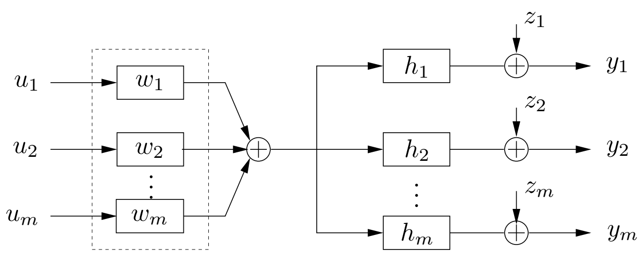

Fig. 1. Downlink wireless channel, with a single base station and m remote users.

identically distributed (i.i.d.) additive complex Gaussian noises with zero mean and variance $\sigma^2 > 0$. Figure 1 illustrates the downlink channel model described above.

The desired signal power at the ith user is given by

$$S_i(w) = |h_i^* w_i|^2,$$

and the interference and noise power at the ith user is

$$I_i(w) = \sum_{j \neq i} |h_i^* w_j|^2 + \sigma^2.$$

Here, without loss of generality, we take $|u_i|^2 = 1$.

An important measure of the system performance is the *signal-to-interference-plus-noise ratio*. The SINR of user i is given by

$$\mathrm{SINR}_i(w) = \frac{S_i(w)}{I_i(w)} = \frac{|h_i^* w_i|^2}{\sum_{j \neq i} |h_i^* w_j|^2 + \sigma^2}. \quad (1)$$

We can also express it as

$$\mathrm{SINR}_i(w) = \frac{h_i^*(w_i w_i^*) h_i}{h_i^*(\sum_{j \neq i} w_j w_j^*) h_i + \sigma^2}. \quad (2)$$

Given beamformers $w_1, \ldots, w_m \in \mathbf{C}^n$, the SINR value provides a *quality-of-service* guarantees for the users. If the SINR goes below a threshold $\gamma > 0$, *i.e.*, $\mathrm{SINR}(w) < \gamma$, then the user experiences a *service outage*.

A. Problem statement

The goal of the *downlink beamforming* or the *downlink power control* problem is to find optimal beamforming weights $w_1, \ldots, w_m \in \mathbf{C}^n$, which achieve the required QoS guarantees for all users, while minimizing the power consumed by the overall system. The power is given by

$$P(w) = \sum_{i=1}^{n} \|w_i\|_2^2. \quad (3)$$

In the case of *perfect* channel side information (CSI), we exactly know the channel vectors $\bar{h}_i$ in both the transmitter and the receivers. (We often call the channel vectors $\bar{h}_i$ the nominal or reference channels.) The downlink beamforming problem with perfect CSI or the *nominal problem* is given by

$$\begin{array}{ll} \text{minimize} & P(w) \\ \text{subject to} & \mathrm{SINR}_i(w) \geq \gamma_i, \quad i = 1 \ldots, m. \end{array} \quad (4)$$

We refer to any solution as the nominal optimal weights.

B. SOCP formulation

As shown by several researchers, the nominal downlink beamforming problem can be solved using convex optimization, and in particular, using second-order cone programming (SOCP), *e.g.*, see a recent survey [23]. The SOCP formulation is given by

$$\begin{array}{ll} \text{minimize} & P(w) \\ \text{subject to} & \beta_i \, \mathbf{Re}(\bar{h}_i^* w_i) \geq \left(\sum_{j=1}^{m} |\bar{h}_i^* w_j| + \sigma^2 \right)^{1/2}, \\ & i = 1, \ldots, m, \end{array} \quad (5)$$

where

$$\beta_i = \left(1 + \frac{1}{\gamma_i} \right)^{1/2}. \quad (6)$$

A drawback of nominal downlink beamforming is that it can be very sensitive to a variation in the channel, meaning that the QoS constraints are often violated even with a small variation in the system.

III. Robust downlink beamforming

In most cases the channels h_i are unknown, and we are given *partial* CSI described in stochastic or deterministic (set-based) terms. In a stochastic setting, h_i are independent random vectors with mean vector $\bar{h}_i \in \mathbf{C}^n$ and covariance matrix $\Sigma_i \in \mathbf{C}^{n \times n}$. (It is often assumed that h_i are complex Gaussian.) In a set-based setting, the channel vectors h_i belong to known and bounded sets $\mathcal{H}_i$ (which include the nominal or reference channel vector $\bar{h}_i$). In practice, we usually estimate $\bar{h}_i$, Σ_i, and $\mathcal{H}_i$, from training sequences to learn the channel, and through feedback between the transmitter and the receivers.

In this paper we consider a set-based uncertainty description, in which we assume that the channels are uncertain, but belong to a known compact sets of possible channels. In particular, we assume that channel vectors h_i belong to known *ellipsoidal* uncertainty sets

$$\mathcal{H}_i = \{ h_i \mid \|F_i(h_i - \bar{h}_i)\|_2 \leq 1 \}, \quad (7)$$

where $\bar{h}_i \in \mathbf{C}^n$ are the nominal channel vectors and $F_i \in \mathbf{C}^{n \times n}$ describe the shapes of the ellipsoids. This ellipsoidal model serves as a conservative approximation of the stochastic model, where we take the ellipsoids to be the confidence ellipsoids for some high confidence, *e.g.*, 95%.

The goal of (worst-case) robust downlink beamforming is to find robust beamforming weights that minimize the power consumption in the system, while guaranteeing the QoS specifications in spite of channel variations. The robust downlink beamforming problem, or simply the robust problem, can be formulated as

$$\begin{array}{ll} \text{minimize} & P(w) \\ \text{subject to} & \inf_{h_i \in \mathcal{H}_i} \mathrm{SINR}_i(w) \geq \gamma_i, \quad i = 1, \ldots, m. \end{array} \quad (8)$$

In the robust problem we require the SINR constraints to exceed the threshold value for all possible channel response vectors h_i in the uncertainty ellipsoids $\mathcal{H}_i$.

We will present a tractable method for solving the robust downlink beamforming problem. The key idea is based on the fact that using the S-procedure, we can evaluate the worst-case channel from the ellipsoidal set of possible channels, for given beamformer weights. (The reader is referred to [21, App. B] or the recent survey [24] for more on the S-procedure.) Therefore, we can approximately solve robust beamforming problem using cutting set methods [20].

A. Worst-case channel analysis

In the worst-case channel analysis problem, we want to find a channel $h_i \in \mathcal{H}_i$ that violates the SINR constraint (the most) or we want to claim that all channels from the uncertainty set meet the constraint, *i.e.*, we want to evaluate

$$\mathrm{SINR}_i^{\mathrm{wc}}(w) = \inf_{h_i \in \mathcal{H}_i} \frac{|h_i^* w_i|^2}{\sum_{j \neq i} |h_i^* w_j|^2 + \sigma^2} \geq \gamma_i,$$

for some fixed beamformer w. An equivalent problem is to find the optimal value of the following constrained problem

$$\begin{array}{ll} \text{minimize} & h_i^* \left(\frac{1}{\gamma_i} w_i w_i^* - \sum_{j \neq i} w_j w_j^* \right) h_i \\ \text{subject to} & h_i \in \mathcal{H}_i, \end{array} \tag{9}$$

and verify that the optimal value is greater than σ^2, since then $\mathrm{SINR}_i^{\mathrm{wc}}(w) \geq \gamma_i$ will hold.

When $\mathcal{H}_i$ are ellipsoids, the worst-case analysis problem (9), under the minor technical condition of strict feasibility (so-called Slater's condition), can be globally solved using the well-known "S-procedure"; see, *e.g.*, [21, App. B], [24]. For each channel (and the corresponding user), the worst-case analysis problem (9) with ellipsoidal uncertainty set (7) is

$$\begin{array}{ll} \text{minimize} & h_i^* \left(\frac{1}{\gamma_i} w_i w_i^* - \sum_{j \neq i} w_j w_j^* \right) h_i \\ \text{subject to} & h_i^* F_i^* F_i h_i - 2\,\mathbf{Re}(\bar{h}_i^* F_i^* F_i h_i) + \bar{h}_i^* F_i^* F_i \bar{h}_i \leq 1. \end{array} \tag{10}$$

Its dual problem is given by the SDP

$$\begin{array}{ll} \text{maximize} & \mu_i \\ \text{subject to} & \lambda_i \geq 0 \\ & A_i \succeq 0, \end{array} \tag{11}$$

with two variables $\mu_i, \lambda_i \in \mathbf{R}$ and

$$A_i = \left[\begin{array}{cc} \frac{1}{\gamma_i} w_i w_i^* - \sum_{j \neq i} w_j w_j^* + \lambda_i F_i^* F_i & -\lambda_i F_i \bar{h}_i \\ -\bar{h}_i^* F_i^* \lambda_i & \lambda_i \bar{h}_i^* F_i^* F_i \bar{h}_i - \lambda_i - \mu_i \end{array} \right],$$

where $i = 1, \ldots, m$. Since strong duality holds in this case, the primal and dual problems have the same optimal value $\mu_i^\star$ (see [21, App. B]), and a worst-case channel is given by

$$h_{\mathrm{wc},i}^\star = \left(\frac{1}{\gamma_i} w_i w_i^* - \sum_{j \neq i} w_j w_j^* + \lambda_i^\star F_i^* F_i \right)^{-1} \lambda_i^\star F_i \bar{h}_i,$$

where $\lambda_i^\star$ is an optimal solution of the SDP (11). Note that if $\mu_i^\star < \sigma^2$, then the ith SINR constraint is violated given the worst-case channel above.

B. Tractable method for robust design

In this section, we present an algorithm for solving the robust problem (8), using the worst-case analysis given in the previous section. The algorithm is motivated by the cutting set methods given in [20]. It is based on solving an alternating sequence of optimization and worst-case analysis problems (pessimizations), with an expanding set of worst-case channels added to the optimization problem at each step.

Let $(h_1^{(k)}, \ldots, h_m^{(k)})$ denote the worst-case channels found by performing worst-case analysis (10) for each user at the kth iteration of the algorithm, where $h_i^{(1)} = \bar{h}_i$. Let $\widehat{\mathcal{H}}_k$ denote the subset of these channels for which the SINR constraint was violated, and let $\widehat{\mathcal{H}}$ denote the discrete set

$$\widehat{\mathcal{H}} = \{\, h_1^{(1)}, \ldots, h_m^{(1)}, \widehat{\mathcal{H}}_2, \ldots, \widehat{\mathcal{H}}_K \,\}$$

found after K optimization-pessimization iterations. At each optimization step we solve the following *multi-scenario* robust problem given the set of worst-case channels collected so far

$$\begin{array}{ll} \text{minimize} & P(w) \\ \text{subject to} & \beta_i\, \mathbf{Re}(h_i^{(k)*} w_i) \geq \left(\sum_{j=1}^m |h_i^{(k)*} w_j| + \sigma^2 \right)^{1/2}, \end{array} \tag{12}$$

where $i = 1 \ldots, m$, $k = 1, \ldots, K$ (from the set $\widehat{\mathcal{H}}$), and β_i is a positive constant defined in (6). The optimal solution of this problem will satisfy the SINR constraints for all the given channels in the (discrete) uncertainty set $\widehat{\mathcal{H}}$. This method can be viewed as an iterative sampling of the uncertainty sets $\mathcal{H}_i$.

The algorithm is given as follows.

given initial nominal channels $\widehat{\mathcal{H}} = \{\bar{h}_1, \ldots, \bar{h}_m\}$.

repeat

1. *Optimization.*
   Solve the robust problem (12) with $\widehat{\mathcal{H}}$ and return w.
2. *Pessimization.*
   2a. Find worst-case violating channels at the current w.
   2b. Append them to the set $\widehat{\mathcal{H}}$.
3. *Sign-off criterion.*
   quit if robust analysis is satisfactory.

The algorithm alternates between optimization and pessimization steps, until a sign-off criterion is satisfied. In the pessimization step, we perform worst-case analysis for violating constraints. If the analysis returns a violating constraint, then we append the obtained worst-case channels to the set $\widehat{\mathcal{H}}$. We repeat this process until worst-case analysis does not produce violating channels. In our numerical simulations, we have found that this process usually converges within 5–10 iterations to fairly robust beamformer weights.

We close by addressing the convergence of the method described above. It is straightforward to prove the convergence of the basic cutting-set algorithm; the reader is referred to [20] for more on convergence issues.

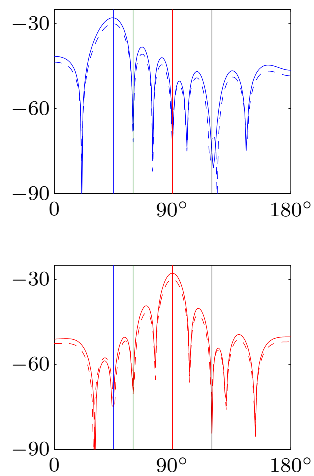

Fig. 2. Gain pattern comparison. Dashed curves: Gain patterns for the four users with the nominal beamformer. Solid curves: Gain patterns for the four users with the robust beamformer. Vertical lines represent the nominal directional locations of the users.

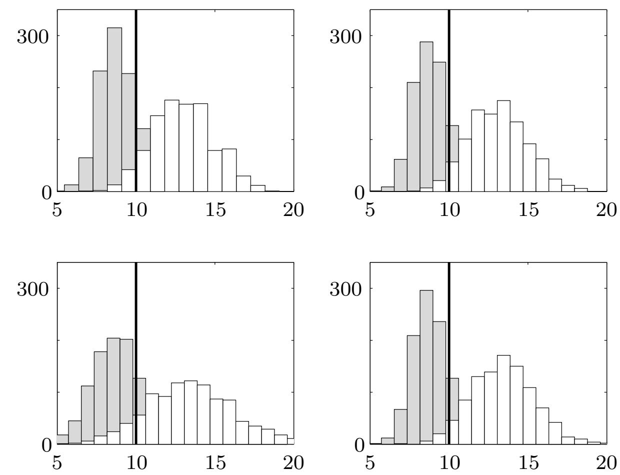

Fig. 3. MC simulation results. Dark shaded histograms: SINRs achieved by the nominal downlink beamformer. Light shaded histograms: SINRs achieved by the robust downlink beamformer. Dark vertical lines correspond to the SINR threshold level $\gamma_i = 10$.

IV. Numerical Example

We illustrate the proposed methods by a numerical example with $m = 4$ users served by a base station with $n = 10$ antennas arranged in a linear array and spaced half a wavelength apart. The users are located at directions $\theta_1 = 45°$, $\theta_2 = 60°$, $\theta_3 = 90°$, and $\theta_4 = 120°$. We use a simple model for the channels given by

$$h(\theta)_i = \exp(2\pi j/\lambda(x_i \cos\theta + y_i \sin\theta)),$$

where (x_i, y_i) is the location of the ith antenna element, and $j = \sqrt{-1}$.

We take the nominal channel to be given with the perfect angle information $\bar{h}_i$, while the sets are described by unit disks around the nominal channels, *i.e.*, $F_i = 1/\rho_i I$, where $\rho_i > 0$ gives the radius of the uncertainty disk for the ith channel. We take the noise power $\sigma = 0.01$, SINR thresholds $\gamma_i = 10$ (20dB), and $\rho_i = 0.05$ for all $i = 1, \dots, m$.

We solve the nominal beamformer problem (5) and implement our proposed methods for robust beamforming using the CVX software package [25], a Matlab-based modeling system for convex optimization. (The CVX package internally uses SDPT3 [26] as the solver.)

Figure 2 shows the gain patterns for the nominal beamformer designed with the nominal channels and the robust beamformer designed with the uncertainty model described above. Using the worst-case analysis we have found that the nominal beamformer violated all of the SINR constraints, while the robust beamformer does not violate any of them.

As another robustness analysis, we carry out Monte-Carlo (MC) analysis of the nominal and robust optimal beamforming method. Here we estimate the probability density function (PDF) of SINRs using MC simulations with 1000 realizations of Gaussian perturbations around the nominal channels $\bar{h}_i$, such that the 95% confidence ellipsoids coincides with our disk uncertainty.

Figure 3 shows the histogram of the SINR values for each of the four users. We observe that the nominal beamformer is very sensitive to the variations in the channel, while the robust beamformer performs very well and satisfies the SINR constraints with about 95% probability, as expected. However, the robustness has its price, *i.e.*, the total power of the robust beamformer will increase. For the given simulation setup, the total power of the nominal beamformer is $P(w_{\text{nom}}) = 0.020$, while the total power of the robust beamformer is $P(w_{\text{rob}}) = 0.026$. Here the total power of a beamformer is defined in (3).

V. Conclusions

In this paper, we have presented a worst-case robust optimization method for beamforming in the downlink channel of a wireless system. We have shown that robust downlink power control with ellipsoidal uncertainty in the channel response can be solved efficiently. Our computational experience with the method so far suggests that the method is far superior to the nominal optimal design (*i.e.*, the design obtained by ignoring statistical variation).

With a proper choice of the uncertainty model, the robust beamforming method can handle probabilistic QoS specifications that for every user the outage probability, *i.e.*, the probability of the signal-to-interference-plus-noise ratio (SINR) being below some threshold, is kept below a specified level. An important question is how suboptimal the robust allocation is compared with the stochastically optimal one which minimizes the power subject to guaranteeing the probabilistic QoS specification.

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REFERENCES

[1] E. Biglieri, A. Constantinides, R. Calderbank, A. Goldsmith, A. Paulraj, and V. Poor, *Introduction to MIMO Wireless*. Cambridge University Press, Nov. 2006.

[2] A. Goldsmith, *Wireless Communcations*. Cambridge Univ. Press, 2005.

[3] M. Bengtsson, "Robust and constrained downlink beamforming," in *Proc. of European Signal Processing Conference, EURASIP'00*, pp. 1433–1436, Sept. 2000.

[4] W. Yu and T. Lan, "Transmitter optimization for the multi-antenna downlink with per-antenna power constraints," *IEEE Transactions on Signal Processing*, vol. 55, no. 6, pp. 2646–2660, 2007.

[5] A. Ben-Tal and A. Nemirovski, "Robust convex optimization," *Mathematics of Operations Research*, vol. 23, no. 4, pp. 769–805, 1998.

[6] L. E. Ghaoui and H. Lebret, "Robust solutions to least-squares problems with uncertain data," *SIAM Journal on Matrix Analysis and Applications*, vol. 18, no. 4, pp. 1035–1064, 1997.

[7] A. Ben-Tal and A. Nemirovski, "Robust optimization: methodology and applications," *Mathematical Programming Series B*, vol. 92, pp. 453–480, May 2002.

[8] A. Mutapcic, S.-J. Kim, and S. Boyd, "Beamforming with uncertain weights," *IEEE Signal Processing Letters*, vol. 14, no. 5, pp. 352–354, 2007.

[9] S. Vorobyov, A. Gershman, and Z.-Q. Luo, "Robust adaptive beamforming using worst-case performance optimization: A solution to the signal mismatch problem," *IEEE Transactions on Signal Processing*, vol. 51, pp. 313–324, Apr. 2003.

[10] S. Vorobyov, A. Gershman, Z.-Q. Luo, and N. Ma, "Adaptive beamforming with joint robustness against mismatched signal steering vector and interference nonstationarity," *IEEE Signal Processing Letters*, vol. 11, pp. 108–111, Feb. 2004.

[11] S. Shahbazpanahi, A. Gershman, Z.-Q. Luo, and K. Wong, "Robust adaptive beamforming for general-rank signal models," *IEEE Transactions on Signal Processing*, vol. 51, pp. 2257–2269, Sept. 2003.

[12] P. Stoica, Z. Wang, and J. Li, "Robust Capon beamforming," *IEEE Transactions on Signal Processing*, vol. 10, pp. 172–175, June 2003.

[13] J. Li, P. Stoica, and Z. Wang, "On robust Capon beamforming and diagonal loading," *IEEE Transactions on Signal Processing*, vol. 51, pp. 1702–1715, July 2003.

[14] J. Li, P. Stoica, and Z. Wang, "Doubly constrained robust Capon beamformer," *IEEE Transactions on Signal Processing*, vol. 52, no. 9, pp. 2407–2423, 2004.

[15] R. Lorenz and S. Boyd, "Robust minimum variance beamforming," *IEEE Transactions on Signal Processing*, vol. 53, no. 5, pp. 1684–1696, 2005.

[16] N. Wong, V. Balakrishnan, and T.-S. Ng, "A second-order cone bounding algorithm for robust minimum variance beamforming," in *Switching and Learning in Feedback Systems* (R. Muarry-Smith and R. Shorten, eds.), pp. 223–247, Springer, 2005.

[17] S.-J. Kim, A. Magnani, A. Mutapcic, S. Boyd, and Z.-Q. Luo, "Robust beamforming via worst-case SINR maximization," 2007. To appear in *IEEE Transactions on Signal Processing*. Available from http://stanford.edu/~boyd/papers/robust_bf.html.

[18] T.-L. Tung and K. Yao, "Robust downlink power-control for DS-CDMA system with multimedia services," in *Proc. of 4th IEEE Workshop on Signal Processing Advances in Wireless Communications SPAWC '03.*, pp. 532–536, June 2003.

[19] A. Fridman, R. Grote, S. Weber, K. Dandekar, and M. Kam, "Robust optimal power control for ad hoc networks," in *Proc. of 2006 Conference on Information Sciences and Systems, Princeton University*, march 2006.

[20] A. Mutapcic and S. Boyd, "Cutting-set methods for robust convex optimization with worst-case oracles," November 2007. In preparation.

[21] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, 2004.

[22] A. Wiesel, Y. Eldar, and S. Shamai (Shitz), "Linear precoding via conic programming for fixed MIMO receivers," *IEEE Transactions on Signal Processing*, vol. 54, no. 1, pp. 161–176, 2006.

[23] Z.-Q. Luo and W. Yu, "An introduction to convex optimization for communications and signal processing," *IEEE J. on Sel. Areas in Comm.*, vol. 24, no. 8, pp. 1426–1438, 2006.

[24] I. Pólik and T. Terlaky, "A survey of the S-Lemma," *SIAM Review*, vol. 49, no. 3, pp. 371–418, 2007.

[25] M. Grant, S. Boyd, and Y. Ye, "CVX: Matlab software for disciplined convex programming, ver. 1.1." Available at www.stanford.edu/~boyd/cvx/, Nov. 2007.

[26] K.-C. Toh, M. J. Todd, and R. H. Tutuncu, "SDPT3: Matlab software for semidefinite-quadratic-linear programming, ver. 4.0 (beta)." Available at www.math.nus.edu.sg/~mattohkc/sdpt3.html, July 2006.