Limit of fluctuations of solutions of Wigner Equation

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Abstract

We consider fluctuations of the solution $W_\varepsilon(t,x,k)$ of the Wigner equation, which describes energy evolution of a solution of the Schrödinger equation with a random white noise in time potential. The expectation of $W_\varepsilon(t,x,k)$ converges as $\varepsilon \to 0$ to $\bar{W}(t,x,k)$ which satisfies the radiative transport equation. We prove that when the initial data is singular in the x variable, that is, $W_\varepsilon(0,x,k) = \delta(x)f(k)$ and $f \in \mathcal{S}(\mathbb{R}^d)$, then the laws of the rescaled fluctuation $Z_\varepsilon(t) := \varepsilon^{-1/2}[W_\varepsilon(t,x,k) - \bar{W}(t,x,k)]$ converge, as $\varepsilon \to 0+$, to the solution of the same radiative transport equation but with a random initial data. This complements the result of [6], where the limit of the covariance function has been considered.

1 Introduction

A weak random potential in the Schrödinger equation

$$\mathrm{i}\frac{\partial \phi}{\partial t} + \frac{1}{2}\Delta_x \phi - \delta V(t,x)\phi = 0 \tag{1.1}$$

with the parameter $\delta \ll 1$, strongly effects the behavior of solutions on times scales of the order $t \sim O(\delta^{-2})$ and larger. The corresponding rescaled problem for $\phi_\varepsilon(t,x) := \varepsilon^{-d/2}\phi(t/\varepsilon, x/\varepsilon)$ reads

$$\mathrm{i}\varepsilon\frac{\partial \phi_\varepsilon}{\partial t} + \frac{\varepsilon^2}{2}\Delta_x \phi_\varepsilon - \sqrt{\varepsilon} V\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)\phi_\varepsilon = 0, \tag{1.2}$$

with $\varepsilon = \delta^2$. Physically, on this time scale waves undergo multiple scattering and propagate in all directions. In such regimes behavior of the spatial energy density $E_\varepsilon(t,x) = |\phi_\varepsilon(t,x)|^2$ is best described in terms of the Wigner transform [14, 18, 24] defined as

$$W_\varepsilon(t,x,k) = \int \mathrm{e}^{\mathrm{i}k\cdot y}\phi_\varepsilon\left(t, x - \frac{\varepsilon y}{2}\right)\phi_\varepsilon^*\left(t, x + \frac{\varepsilon y}{2}\right)\frac{dy}{(2\pi)^d}.$$

Here and everywhere below a^* denotes the complex conjugate of $a \in \mathbb{C}$. Note that $W_\varepsilon^*(t,x,k) = W_\varepsilon(t,x,k)$. Since the wave energy density can be decomposed as

$$E_\varepsilon(t,x) = \int W_\varepsilon(t,x,k)dk,$$

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it is customary to interpret the Wigner transform as a "phase space resolved" wave energy density. It is well known that, provided that $\|\phi_\varepsilon\|_{L^2} \le C$, in the limit $\varepsilon \to 0$ the Wigner transform converges (possibly after passing to a subsequence) to a limit $W(t,x,k)$ which is a non-negative measure. When $V(t,x)$ is a spatially and temporally statistically homogeneous random process with sufficiently fast decaying correlations, the expected value of the limit, $\bar{W}(t,x,k) = \mathbb{E}W(t,x,k)$ satisfies the radiative transport equation [2, 8, 9, 13, 19, 24, 25]

$$\bar{W}_t + k \cdot \nabla \bar{W} = \int \hat{R}\left(\frac{k^2}{2} - \frac{p^2}{2}, k-p\right) [\bar{W}(t,x,p) - \bar{W}(t,x,k)] \frac{dp}{(2\pi)^d}, \tag{1.3}$$

where $R(t,x) = \mathbb{E}[V(s,y)V(t+s,x+y)]$ is the two-point correlation function of the process $V(t,x)$ and $\hat{R}(\omega,k)$ is its power spectrum, that is, the Fourier transform of $R(t,x)$. Moreover, it has been shown in [3, 9, 20] that in many situations the limit is actually self-averaging, or, more precisely, that for any test function $\theta \in \mathcal{S}(\mathbb{R}^{2d})$ the process $\langle W_\varepsilon, \theta\rangle$ converges, as $\varepsilon \to 0+$, to the deterministic limit $\langle \bar{W}, \theta\rangle$ in probability. However, this result was generally established only when the initial data for the Wigner transform is sufficiently regular, which may be achieved, for instance, by consideration of mixtures of states. The restriction on the regularity of the initial data is not simply technical: sufficiently singular initial data $W_0(x,k)$ does lead to the absence of self-averaging. In particular, it has been shown in [1] that when the initial data is localized both in space and wave vectors: $W_0(x,k) = \delta(x)\delta(k-k_0)$ then the Wigner transform is not self-averaging.

The problem of the self-avaraging of the energy density has a practical aspect as recent inversion techniques based on the kinetic limits for wave propagation in random media have been developed and even tested in physical experiments [4, 5, 6]. This makes important understanding of the statistics of the fluctuation $W_\varepsilon - \bar{W}$, as well as its dependence on the regularity of the initial data. The first step in this direction was made recently in [7], where the limit of the scintillation of $\langle W_\varepsilon, \theta\rangle$ was analyzed in the simplest case when the random process $V(t,x)$ is a white noise in time. The purpose of the present paper is obtain the weak limit of the full fluctuation process

$$Z_\varepsilon(t,x,k) = \frac{W_\varepsilon(t,x,k) - \bar{W}(t,x,k)}{\sqrt{\varepsilon}} \tag{1.4}$$

in the case of the initial data of the form $W(0,x,k) = \delta(x)f(k)$. This Cauchy data corresponds to the physically perhaps most interesting case of a point source in a random medium with a smooth distribution of energy over wave vectors. One consequence of [7] is that the size of $W_\varepsilon - \bar{W}$ should depend on the regularity of the initial data: if $W_0(x,k)$ is smooth then $W_\varepsilon - \bar{W}$ is of the size $O(\varepsilon^{d/2})$ while for a spatially localized initial data as above the fluctuation is $O(\sqrt{\varepsilon})$ independent of the dimension. We choose the random potential to be still of the white noise type to simplify the analysis but we believe that the main asymptotic results hold for a much larger class of the random potentials.

Let us explain our main result informally. We consider the solution of the random Wigner equation

$$\mathrm{d}W_\varepsilon(t,x,k) = -k \cdot \nabla_x W_\varepsilon(t,x,k)\mathrm{d}t + \mathrm{i}\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}p\cdot x/\varepsilon}\Big[W_\varepsilon(t,x,k-\frac{p}{2}) - W_\varepsilon(t,x,k+\frac{p}{2})\Big] \frac{\widehat{B}(\mathrm{d}_S t, \mathrm{d}p)}{(2\pi)^d}, \tag{1.5}$$

with the initial data $W_\varepsilon(0,x,k) = \delta(x)f(k)$. The stochastic integral in the right side is understood in the Stratonovich sense and $\{B(t),\, t \ge 0\}$ is a spatially homogeneous Wiener process with the spectral measure μ. That is, it can be written as

$$B(t) = \sum_n B_n(t)\mathcal{F}(e_n\mu), \qquad t \ge 0,$$

where e_n is a basis of L^2 space over the spectral measure μ and $\mathcal{F}$, or $\hat{}$ denote the Fourier transform. Then the expectation $\bar{W}(t) = \mathbb{E}W_\varepsilon(t)$ does not depend on ε and satisfies the kinetic equation, also known as the radiative transport equation

$$\begin{aligned} &\frac{\partial \bar{W}}{\partial t} + k \cdot \nabla_x \bar{W} = \int \left[\bar{W}(t,x,k-p) - \bar{W}(t,x,k)\right] \mu(dp) \\ &\bar{W}(0) = \delta(x) f(k). \end{aligned} \tag{1.6}$$

Note that in case when $V(t,x)$ in (1.1) is a white noise in t then $\hat{R}(\omega,p) = \hat{R}(p)$ and equation (1.6) is a special case of (1.3) with $\mu(\mathrm{d}p) = (2\pi)^{-d}\hat{R}(p)\mathrm{d}p$.

We show that the correction Z_ε defined by (1.4) converges weakly to a random limit $Z(t,x,k)$ which satisfies the same kinetic equation (1.6) but with a random initial data:

$$\begin{aligned} &\frac{\partial Z}{\partial t} + k \cdot \nabla_x Z = \int \left[Z(t,x,k-p) - Z(t,x,k)\right] \mu(\mathrm{d}p) \\ &Z(0) = \delta(x) X(k). \end{aligned} \tag{1.7}$$

Here $X(k)$ is a distribution valued Gaussian random variable given by

$$X(k) := -\mathrm{i}\sum_n \sum_{\sigma=\pm 1} \sigma \int_0^{+\infty} \mathrm{d}B_n(s) \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}p\cdot(k+\sigma p/2)s} f(k+\sigma p/2) e_n(p) \mu(\mathrm{d}p). \tag{1.8}$$

From a physical standpoint the fact that Z satisfies a deterministic problem with a random initial data is quite natural, the reason being once again the fact that the size of the fluctuation depends on the regularity of the initial data. Solution of the radiative transport equation $\bar{W}(t,x,k)$ is less singular for $t > 0$ then the initial data $W_0(x,k) = \delta(x)f(k)$ – hence, the fluctuation that is produced at positive times is smaller than $O(\sqrt{\varepsilon})$, and the main random contribution to Z_ε comes from an initial time layer when $\bar{W}(t,x,k)$ still has a spatially localized singularity. Hence, the stochastic nature of $Z\varepsilon(t,x,k)$ manifests itself, in the leading order, only as the initial data for the limiting kinetic equation.

The exact form of the angular distribution $X(k)$ may be deduced formally from the initial layer problem for the fluctuation. Let us write (1.5) in a formal differential form

$$\frac{\partial W_\varepsilon}{\partial t} + k \cdot \nabla_x W_\varepsilon = \frac{\mathrm{i}}{\sqrt{\varepsilon}} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}p\cdot x/\varepsilon} \hat{V}(\frac{t}{\varepsilon}, p) \left[W_\varepsilon(t,x,k-\frac{p}{2}) - W_\varepsilon(t,x,k+\frac{p}{2})\right] \frac{dp}{(2\pi)^d},$$

where we replaced the white noise by a potential of the form $\sqrt{\varepsilon}V(t/\varepsilon, x/\varepsilon)$ to make the scaling in the subsequent computation more transparent. In the fast variables $s = t/\varepsilon$, $y = x/\varepsilon$ this problem may be re-written as

$$\frac{\partial W'_\varepsilon}{\partial s} + k \cdot \nabla_y W'_\varepsilon = -\mathrm{i}\sqrt{\varepsilon} \sum_{\sigma=\pm 1} \sigma \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}p\cdot y} \hat{V}(s,p) W'_\varepsilon(s,y,k+\frac{\sigma p}{2}) \frac{dp}{(2\pi)^d},$$

where $W'_\varepsilon(s,y,k) := W_\varepsilon(\varepsilon s, \varepsilon y, k)$. We introduce a formal asymptotic expansion

$$W_\varepsilon(t,x,k) = \bar{W}(t,x,k) + \sqrt{\varepsilon} Z(t,x,k) + \ldots$$

then,

$$W'_\varepsilon(s,y,k) = \bar{W}'(s,y,k) + \sqrt{\varepsilon} Z'(s,y,k) + \ldots$$

The leading order term satisfies the homogeneous transport equation

$$\bar{W}'_s + k \cdot \nabla_y \bar{W}' = 0, \quad \bar{W}'(0,y,k) = \varepsilon^{-d} \delta(y) f(k),$$

and is, therefore, given by $\bar{W}'(s,y,k)=\varepsilon^{-d}\delta(y-ks)f(k)$. The equation for $Z'(s,y,k)$ is

$$\partial_s Z'(s,y,k)+k\cdot\nabla_y Z'(s,y,k)=-\mathrm{i}\sum_{\sigma=\pm1}\sigma\int_{\mathbb{R}^d}\mathrm{e}^{\mathrm{i}p\cdot y}\hat{V}(s,p)\bar{W}'\left(s,y,k+\frac{\sigma p}{2}\right)\mathrm{d}p$$

with the initial data $Z'(0,y,k)=0$. For a random potential of the form (1) this gives an explicit formula for $Z(s,y,k)$:

$$\begin{aligned}Z'(s,y,k)&=-\mathrm{i}\sum_n\sum_{\sigma=\pm1}\sigma\int_0^s\int_{\mathbb{R}^d}\mathrm{e}^{\mathrm{i}p\cdot(y-k(s-\tau))}\bar{W}'(\tau,y-k(s-\tau),k+\frac{\sigma p}{2})e_n(p)\mu(\mathrm{d}p)dB_n(\tau)\\&=-\mathrm{i}\varepsilon^{-d}\sum_n\sum_{\sigma=\pm1}\sigma\int_0^s\int_{\mathbb{R}^d}\mathrm{e}^{\mathrm{i}p\cdot(k+\sigma p/2)\tau}f\left(k+\frac{\sigma p}{2}\right)\delta(y-ks-\frac{\sigma p}{2}\tau)e_n(p)\mu(\mathrm{d}p)\mathrm{d}B_n(\tau).\end{aligned}$$

We obtain therefore:

$$\begin{aligned}&Z(t,x,k)=Z'(t/\varepsilon,x/\varepsilon,k)\\&=-\mathrm{i}\varepsilon^{-d}\sum_n\sum_{\sigma=\pm1}\sigma\int_0^{t/\varepsilon}\int_{\mathbb{R}^d}\mathrm{e}^{\mathrm{i}p\cdot(k+\sigma p/2)\tau}f\left(k+\frac{\sigma p}{2}\right)\\&\times\delta(\varepsilon^{-1}(x-kt+\varepsilon\sigma p\tau/2))e_n(p)\mu(\mathrm{d}p)\mathrm{d}B_n(\tau)\end{aligned}$$

and since $\varepsilon^{-d}\delta(z/\varepsilon)=\delta(z)$ we obtain that for small $t\ll1$ the quantity $\varepsilon\sigma p\tau\le pt\ll1$ can be neglected, thus

$$Z(0,x,k)\approx-\mathrm{i}\sum_n\sum_{\sigma=\pm1}\sigma\int_0^\infty\int_{\mathbb{R}^d}\mathrm{e}^{\mathrm{i}p\cdot(k+\sigma p/2)\tau}f\left(k+\frac{\sigma p}{2}\right)\delta(x)e_n(p)\mu(\mathrm{d}p)\mathrm{d}B_n(\tau).$$

The paper is organized as follows. We first recall some basic facts about homogeneous Wiener processes in Section 2. The basic existence theory for the Wigner equation with a white-noise potential is described in Section 3. We recall that when the initial data for the Wigner equation (1.5) is the Wigner transform of the initial data for the Schrödinger equation (1.2) then existence of the solution of the Wigner equation can be deduced from the respective property of the Schrödinger equation with a white-noise potential [10]. However, to the best of our knowledge, such a theory for the Wigner equation with an arbitrary initial data is not available in the literature. The main result of this section is Theorem 1. Section 4 contains the main result of this paper, Theorem 2, which describes the asymptotics of the fluctuation process.

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2 Preliminaries

Basic notation

We denote by $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d;\mathbb{C})$ the spaces of rapidly decreasing functions of the Schwartz class and by $\mathcal{S}'(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d;\mathbb{C})$ the corresponding spaces of tempered distributions. The value of a distribution ξ on a test function ψ will be denoted by $\langle\xi,\psi\rangle$. Let $\tau_x\psi(\cdot):=\psi(x+\cdot)$, $x\in\mathbb{R}^d$ be the group of translations on $\mathcal{S}(\mathbb{R}^d)$. It can be extended to $\mathcal{S}'(\mathbb{R}^d)$ by setting $\langle\tau_x\xi,\psi\rangle:=\langle\xi,\tau_{-x}\psi\rangle$.

We denote by

$$\mathcal{F}\psi(p) = \widehat{\psi}(p) := \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}p\cdot x}\psi(x)\mathrm{d}x$$

the Fourier transform of a function $\psi(x)$. Also, we use the notation

$$\mathcal{F}_1(f)(q,k) := \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}q\cdot x} f(x,k)\mathrm{d}x, \quad \mathcal{F}_2(f)(x,y) := \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}y\cdot k} f(x,k)\mathrm{d}k$$

for the partial Fourier transform in just one of the variables. Given $s,u \in \mathbb{R}$ we denote by $H^{s,u}$ the mixed Sobolev space with the norm

$$\|f\|^2_{H^{s,u}} := \int_{\mathbb{R}^{2d}} (1+|q|^2)^{s/2}(1+|y|^2)^{u/2}|\hat{f}(q,y)|^2\mathrm{d}q\mathrm{d}y, \quad f \in \mathcal{S}(\mathbb{R}^{2d}).$$

In the ensuing notation we shall also write $H^s_1 := H^{s,0}$ and $H^u_2 := H^{0,u}$.

Given $p_1, p_2 \in [1,+\infty)$ we denote by $\mathcal{A}_{p_1,p_2}$, $\mathcal{B}_{p_1,p_2}$ the Banach spaces that are the completions of $\mathcal{S}(\mathbb{R}^{2d})$ under the norms

$$\|\phi\|^{p_1}_{p_1,p_2} := \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} |\mathcal{F}_1(\phi)(q,k)|^{p_2}\mathrm{d}k\right]^{p_1/p_2}\mathrm{d}q,$$

and

$$(\|\phi\|^{(\mathcal{B})}_{p_1,p_2})^{p_1} := \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} |\hat{\phi}(q,y)|^{p_2}\mathrm{d}y\right]^{p_1/p_2}\mathrm{d}q,$$

respectively. The definition can be easily extended to cover the case when one, or both of the indices equal $+\infty$.

Some functional spaces formed over the spectral measure

Given a function $\psi(p)$, $p \in \mathbb{R}^d$, we set $\psi_{(s)}(p) := \psi^*(-p)$ and say that ψ is even if $\psi = \psi_{(s)}$. Assume that μ is a finite Borel measure on $\mathbb{R}^d$ that is symmetric, that is, $\mu(\Gamma) = \mu(-\Gamma)$ for all sets $\Gamma \in \mathcal{B}(\mathbb{R}^d)$. The real Hilbert space $L^2_{(s)}(\mu)$ consists of all functions $\psi \in L^2_{\mathbb{C}}(\mu)$ that are even. Note that

$$\langle \psi_1, \psi_2\rangle_\mu := \int_{\mathbb{R}^d} \psi_1(p)\psi_2^*(p)\mu(\mathrm{d}p), \quad \forall\, \psi_1, \psi_2 \in L^2_{\mathbb{C}}(\mu)$$

is a real valued scalar product on $L^2_{(s)}(\mu)$, provided μ is symmetric.

We will need the following proposition.

Proposition 1 *Let $\{e_n\}$ be an orthonormal basis of $L^2_{(s)}(\mu)$. Then for any $\psi_1, \psi_2 \in L^2_{\mathbb{C}}(\mu)$ we have*

$$\sum_n \langle \psi_1, e_n\rangle_\mu \langle \psi_2, e_n\rangle_\mu = \int_{\mathbb{R}^d} \psi_1(p)\psi_2(-p)\mu(\mathrm{d}p). \tag{2.1}$$

Proof. Given $\psi \in L^2_{\mathbb{C}}(\mu)$, consider its symmetrization $S[\psi] \in L^2_{(s)}(\mu)$,

$$S[\psi](p) := \frac{1}{2}\left[\psi(p) + \psi^*(-p)\right].$$

For any $\phi \in L^2_{(s)}(\mu)$ and $\psi_1 \in L^2_{\mathbb{C}}(\mu)$ we have, using the symmetry of μ:

$$\langle S[\psi_1], \phi\rangle_\mu = \frac{1}{2}\int_{\mathbb{R}^d} \left(\psi_1(p)\phi^*(p) + \psi_1^*(-p)\phi^*(p)\right)\mu(\mathrm{d}p) = \frac{1}{2}\left(\langle \psi_1, \phi\rangle_\mu + \langle \phi, \psi_1\rangle_\mu\right) = \mathrm{Re}\,\langle \psi_1, \phi\rangle_\mu,$$

and thus

$$\langle S[\mathrm{i}\psi_1], \phi\rangle_\mu = \mathrm{Re}\,\langle \mathrm{i}\psi_1, \phi\rangle_\mu = -\mathrm{Im}\,\langle \psi_1, \phi\rangle_\mu.$$

Therefore, for all $\psi_1 \in L^2_{\mathbb{C}}(\mu)$ and $\phi \in L^2_{(s)}(\mu)$, we have

$$\langle \psi_1, \phi\rangle_\mu = \langle S[\psi_1], \phi\rangle_\mu - \mathrm{i}\langle S[\mathrm{i}\psi_1], \phi\rangle_\mu,$$

which implies that

$$\begin{aligned} I := \sum_n \langle \psi_1, \mathrm{e}_n\rangle_\mu \langle \psi_2, e_n\rangle_\mu &= \sum_n \left(\langle S[\psi_1], \mathrm{e}_n\rangle_\mu - \mathrm{i}\langle S[\mathrm{i}\psi_1], \mathrm{e}_n\rangle_\mu\right)\left(\langle S[\psi_2], \mathrm{e}_n\rangle_\mu - \mathrm{i}\langle S[\mathrm{i}\psi_2], \mathrm{e}_n\rangle_\mu\right) \\ &= \langle S[\psi_1], S[\psi_2]\rangle_\mu - \mathrm{i}\langle S[\mathrm{i}\psi_1], S[\psi_2]\rangle_\mu - \mathrm{i}\langle S[\psi_1], S[\mathrm{i}\psi_2]\rangle_\mu - \langle S[\mathrm{i}\psi_1], S[\mathrm{i}\psi_2]\rangle_\mu \\ &= \langle S[\psi_1] - \mathrm{i}S[\mathrm{i}\psi_1], S[\psi_2]\rangle_\mu - \langle \mathrm{i}S[\psi_1] + S[\mathrm{i}\psi_1], S[\mathrm{i}\psi_2]\rangle_\mu = \langle \psi_1, S[\psi_2]\rangle_\mu - \mathrm{i}\langle \psi_1, S[\mathrm{i}\psi_2]\rangle_\mu, \end{aligned}$$

since $\psi = S[\psi] - iS[i\psi]$. It follows that

$$I = \langle \psi_1, S[\psi_2]\rangle_\mu - \langle \mathrm{i}\psi_1, S[\mathrm{i}\psi_2]\rangle_\mu = \langle \psi_1, S[\psi_2] + \mathrm{i}S[\mathrm{i}\psi_2]\rangle_\mu = \int_{\mathbb{R}^d} \psi_1(p)\psi_2(-p)\mu(\mathrm{d}p),$$

which is (2.1). $\square$

This result can be further generalized by a standard density argument leading to

Corollary 1 *Suppose that $\Psi \in L^2_{\mathbb{C}}(\mu \otimes \mu)$, then*

$$\sum_n \int_{\mathbb{R}^{2d}} \Psi(p,q) e_n(q) e_n(p)\mu(\mathrm{d}p)\mu(\mathrm{d}q) = \int_{\mathbb{R}^d} \Psi(p,-p)\mu(\mathrm{d}p).$$

Spatially homogeneous Wiener process

Let μ be a non-negative symmetric, Borel measure on $\mathbb{R}^d$. Recall that an $\mathcal{S}'(\mathbb{R}^d)$ -valued, Gaussian process $\{B(t),\, t \geq 0\}$ is called a *spatially homogeneous Wiener process* on $\mathbb{R}^d$ if it has the following properties, see e.g. [21, 22, 23]:

(i) for any $\psi \in \mathcal{S}(\mathbb{R}^d)$, $\{\langle B(t), \psi\rangle,\, t \geq 0\}$ is a real-valued Wiener process,

(ii) for any $t \geq 0$, the law of $B(t)$ is invariant with respect to the group of translations $\left\{\tau_x,\, x \in \mathbb{R}^d\right\}$ acting on $\mathcal{S}'(\mathbb{R}^d)$.

Equivalently, one can prove, see e.g. [21], that $\{B(t),\, t \geq 0\}$ is Gaussian and its covariance is of the form

$$\mathbb{E}\left[\,\langle B(t), \psi_1\rangle\langle B(s), \psi_2\rangle\right] = \langle \hat{\psi}_1, \hat{\psi}_2\rangle_\mu (t \wedge s)\,, \qquad \psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}^d) \tag{2.2}$$

for some Borel measure μ. Since $B(t)$ takes values in the space of tempered distributions, there is an $n \geq 0$ such that

$$\int_{\mathbb{R}^d} \left(1 + |p|^2\right)^{-n} \mu(\mathrm{d}p) < +\infty.$$

The measure μ is called the spectral measure of $\{B(t),\, t \geq 0\}$. It is known that if μ is finite then B is a Gaussian random field on $[0,\infty) \times \mathbb{R}^d$ satisfying

$$\mathbb{E}\left[\,B(t,x)B(s,y)\right] = \widehat{\mu}(x-y)(t \wedge s)\,, \qquad x, y \in \mathbb{R}^d,\ t, s \geq 0.$$

Let

$$\mathcal{H}_\mu := \left[\mathcal{F}\left(\psi\mu\right) : \psi \in L^2_{(s)}(\mu)\right] \subset \mathcal{S}'(\mathbb{R}^d)$$

be the real Hilbert space equipped with the scalar product induced from $L^2_{(s)}(\mu)$ by $\mathcal{F}$, that is, for all $\psi_1, \psi_2 \in L^2_{(s)}(\mu) \cap \mathcal{S}(\mathbb{R}^d; \mathbb{C})$ we have

$$\langle \mathcal{F}(\psi_1 \mu), \mathcal{F}(\psi_2 \mu) \rangle_{\mathcal{H}_\mu} = \langle \psi_1, \psi_2 \rangle_\mu = \langle \mathcal{F}(\psi_1 \mu), \mathcal{F}(\psi_2) \rangle.$$

According to [21], the *reproducing kernel Hilbert space* of $\{B(t),\, t \geq 0\}$ can be identified with $\mathcal{H}_\mu$, that is, $\{B(t),\, t \geq 0\}$ is *the cylindrical Wiener process* on $\mathcal{H}_\mu$. The above property is expressed by the following.

Proposition 2 *For any orthonormal basis $\{e_n\}$ of $L^2_{(s)}(\mu)$ there is a sequence of independent standard real-valued Wiener processes $\{B_n(t),\, t \geq 0\}$ such that*

$$B(t) = \sum_n B_n(t) \mathcal{F}(e_n \mu), \qquad t \geq 0, \tag{2.3}$$

where the series converges in the L^2 sense and $\mathbb{P}$-a.s in any Hilbert space H such that the embedding $\mathcal{H}_\mu \hookrightarrow H$ is Hilbert–Schmidt.

3 The Wigner equation with a spatially homogeneous random potential

The Wigner equation

Let $\{B(t),\, t \geq 0\}$ be a spatially homogeneous Wiener process with spectral measure μ and let $\varepsilon > 0$. We are concerned with the initial problem for the following SPDE, called *the Wigner equation*,

$$\begin{aligned} \mathrm{d}W_\varepsilon(t,x,k) &= -k \cdot \nabla_x W_\varepsilon(t,x,k)\mathrm{d}t \\ &\quad + \mathrm{i}\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}p\cdot x/\varepsilon} \Big[W_\varepsilon(t,x,k-\frac{p}{2}) - W_\varepsilon(t,x,k+\frac{p}{2}) \Big] \frac{\widehat{B}(\mathrm{d}_S t, \mathrm{d}p)}{(2\pi)^d}, \\ W_\varepsilon(0,x,k) &= W_0(x,k). \end{aligned} \tag{3.1}$$

The stochastic integral above is understood in the Stratonovich sense. We give a rigorous definition of the solution to (3.1) in an appropriate functional space that shall be specified later on. For any $\phi \in \mathcal{S}_{(s)}(\mathbb{R}^d; \mathbb{C})$ - the space of Schwartz class functions that are complex even (i.e. $\phi(-p) = \phi^*(p)$) - we have

$$\langle \widehat{B(t)}, \phi \rangle = \sum_n \langle \mathcal{F}(\mathcal{F}(e_n \mu)), \phi \rangle B_n(t), \quad t \geq 0,$$

while

$$\begin{aligned} &\langle \mathcal{F}(\mathcal{F}(e_n \mu)), \phi \rangle = (2\pi)^{2d} \langle e_n, \mathcal{F}^{-1}(\mathcal{F}^{-1}\phi) \rangle_\mu \\ &= (2\pi)^d \int_{\mathbb{R}^d} e_n(p)\phi^*(-p)\mu(\mathrm{d}p) = (2\pi)^d \int_{\mathbb{R}^d} e_n(p)\phi(p)\mathrm{d}\mu. \end{aligned} \tag{3.2}$$

It follows that

$$\langle \widehat{B(t)}, \phi \rangle = (2\pi)^d \sum_n B_n(t) \int_{\mathbb{R}^d} e_n(p)\phi(p)\mathrm{d}\mu, \quad t \geq 0,\, \phi \in \mathcal{S}_{(s)}(\mathbb{R}^d; \mathbb{C}). \tag{3.3}$$

Taking into account (2.3) and (3.2) we can rewrite (3.1) into the following form (recall that $W_\varepsilon(t,x,k)$ is real valued)

$$
\begin{aligned}
&\mathrm{d}W_\varepsilon = AW_\varepsilon \mathrm{d}t + C_\varepsilon[W_\varepsilon]\mathrm{d}_S\widehat{B} = AW_\varepsilon \mathrm{d}t + \sum_n C_\varepsilon[W_\varepsilon]e_n \mathrm{d}_S B_n, \qquad (3.4)\\
&W_\varepsilon(0,x,k) = W_0(x,k),
\end{aligned}
$$

where $A : \mathcal{S}(\mathbb{R}^{2d}) \to \mathcal{S}(\mathbb{R}^{2d})$ is given by

$$
A\psi(x,k) := -k\cdot\nabla_x\psi(x,k), \qquad (3.5)
$$

and the operator

$$
C_\varepsilon : \mathcal{S}(\mathbb{R}^{2d}) \to \mathcal{L}(L^2_{(s)}(\mu), C^\infty(\mathbb{R}^{2d}))
$$

is given by

$$
C_\varepsilon[\psi]\varphi(x,k) := -\mathrm{i}\sum_{\sigma=\pm1}\sigma\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}(p\cdot x)/\varepsilon}\psi\left(x, k+\frac{\sigma p}{2}\right)\varphi(p)\mu(\mathrm{d}p) \qquad (3.6)
$$

for $\psi \in \mathcal{S}(\mathbb{R}^{2d})$ and $\varphi \in L^2_{(s)}(\mu)$. Here $\mathcal{L}(X,Y)$ denotes the space of continuous linear operators between linear topological spaces X and Y. We shall further specify the above operators later on when we define the notion of a solution to (3.4).

The Wigner equation in the Itô form

Equation (3.1) can be rewritten in the Itô form

$$
\begin{aligned}
&\mathrm{d}W_\varepsilon = \left(AW_\varepsilon + \frac{1}{2}L_\varepsilon W_\varepsilon\right)\mathrm{d}t + \sum_n C_\varepsilon[W_\varepsilon]e_n \mathrm{d}B_n, \qquad (3.7)\\
&W_\varepsilon(0) = W_0,
\end{aligned}
$$

where

$$
L_\varepsilon\psi := \sum_n C_\varepsilon[C_\varepsilon[\psi]e_n]e_n.
$$

We have, more explicitly:

$$
\begin{aligned}
L_\varepsilon\psi(x,k) &= -\mathrm{i}\sum_n\sum_{\sigma=\pm1}\sigma\int_{\mathbb{R}^d}\mathrm{e}^{\mathrm{i}(p\cdot x)/\varepsilon}C_\varepsilon[\psi]e_n\left(x,k+\frac{\sigma p}{2}\right)\mu(\mathrm{d}p)\\
&= -\sum_n\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathrm{e}^{\mathrm{i}(p+q)\cdot x/\varepsilon}\Psi(x,k,p,q)e_n(q)e_n(p)\mu(\mathrm{d}p)\mu(\mathrm{d}q),
\end{aligned}
$$

where

$$
\Psi(x,k,p,q) := \sum_{\sigma,\sigma'=\pm1}\sigma\sigma'\psi\left(x, k+\frac{\sigma p}{2}+\frac{\sigma' q}{2}\right).
$$

By Proposition 1, the definition of L_ε does not in fact depend on ε. We shall drop therefore ε, from this point on, from its notation. The explicit expression for this operator is given by

$$
\begin{aligned}
L\psi(x,k) &= -\int_{\mathbb{R}^d}\Psi(x,k,p,-p)\mu(\mathrm{d}p) = \int_{\mathbb{R}^d}\left[\psi(x,k-p) - 2\psi(x,k) + \psi(x,k+p)\right]\mu(\mathrm{d}p)\\
&= 2M\psi(x,k) - 2\Sigma\psi(x,k), \qquad (3.8)
\end{aligned}
$$

where

$$M\psi(x,k) := \int_{\mathbb{R}^d} \psi(x,k+p)\mu(\mathrm{d}p) \tag{3.9}$$

and

$$\Sigma := \mu(\mathbb{R}^d) < +\infty. \tag{3.10}$$

The last equality in (3.8) follows from the fact that $\mu(-\mathrm{d}p) = \mu(\mathrm{d}p)$. A simple consequence of (3.7) is that $\bar{W}(t) := \mathbb{E}W_\varepsilon(t)$ does not depend on ε and it satisfies the linear kinetic equation

$$\begin{aligned} \frac{d\bar{W}(t)}{dt} &= \left(A + \frac{1}{2}L\right)\bar{W}(t), \\ \bar{W}(0) &= W_0. \end{aligned} \tag{3.11}$$

The probabilistic representation of a solution of the kinetic equation

We now recall a probabilistic formula for the solution to (3.11) treating it as the solution of Kolmogorov's equation for a certain Markov jump process. The results of this section are standard and their proofs can be found, for instance, in Apppendix 2 of [17]. Define the probability measure $\nu(A) := \Sigma^{-1}\mu(A)$, where Σ is given by (3.10) and A is a Borel set. Let $\{L_i,\, i \ge 0\}$ be a sequence of i.i.d. random variables (momenta i-th jump) distributed according to ν and set

$$K_0 := 0, \quad K_n := \sum_{i=0}^{n-1} L_i, \quad n \ge 1.$$

Let $\sigma_0, \sigma_1, \ldots$ be i.i.d. random variables (times between the jumps), independent of $L_0, L_1, \ldots$ such that σ_0 is exponentially distributed with the intensity parameter Σ. Consider the "jump times"

$$t_0 := 0, \quad t_n := \sum_{i=0}^{n-1} \sigma_i, \quad n \ge 1.$$

The jump process $K(t)$ is defined as $K(t) := K_n$, for $t \in [t_n, t_{n+1})$. For any function $\phi \in L^\infty(\mathbb{R}^d)$ we have

$$\mathbb{E}\phi(k + K(t)) = e^{-t\Sigma}\phi(k) + \sum_{n=1}^{+\infty} \phi_n(t,k), \tag{3.12}$$

where

$$\phi_n(t,k) := e^{-\Sigma t}\frac{(\Sigma t)^n}{n!}\mathbb{E}\phi\left(K_n + k\right), \qquad n \ge 1.$$

Since the laws of K_1 and $-K_1$ are identical we have

$$\int_{\mathbb{R}^d} \mathbb{E}[\psi_1(k + K(t))]\psi_2(k)dk = \int_{\mathbb{R}^d} \mathbb{E}[\psi_2(k + K(t))]\psi_1(k)dk \tag{3.13}$$

for any pair of functions $\psi_i \in L^\infty(\mathbb{R}^d)$, $i = 1, 2$

Let $X(t) := kt + \int_0^t K(s)ds$. The process $\{(-X(t), K(t)),\, t \ge 0\}$ is Markovian with the generator $A + 1/2L$. Thus, the solution of (3.11) can be written as

$$\bar{W}(t,x,k) = \mathbb{E}\left\{W_0(x - X(t), k + K(t))\right\}. \tag{3.14}$$

Using (3.12) we obtain therefore

$$\bar{W}(t,x,k) = e^{-t\Sigma} W_0(x-kt,k) + \sum_{n=1}^{+\infty} W_n(t,x,k),$$

where

$$\begin{aligned} W_0(t,x,k) &:= e^{-t\Sigma} W_0(x-kt,k), \\ W_n(t,x,k) &:= e^{-t\Sigma}\Sigma^n \int_{\Delta_n(t)} \mathbb{E}\, W_0\left(x-kt-\mathcal{X}_n, k+K_n\right) \mathrm{d}\tau(n) \end{aligned} \tag{3.15}$$

for $n \geq 1$, and

$$\begin{aligned} &\Delta_n(t) := [(\tau_0,\ldots,\tau_{n-1}) : \, t \geq \sum_{i=0}^{n-1}\tau_i,\, \tau_i \geq 0], \qquad \tau_n := t - \sum_{i=0}^{n-1}\tau_i, \\ &\mathrm{d}\tau(n) := \mathrm{d}\tau_0 \ldots \mathrm{d}\tau_{n-1}, \qquad \mathcal{X}_n := \sum_{i=1}^{n} K_i \tau_i. \end{aligned}$$

We shall introduce a semigroup of operators given by $\bar{W}(t) := S(t)W_0$, $t \geq 0$.

Proposition 3 *The family $\{S(t),\, t\geq 0\}$ extends to a C_0-semigroup of contractions on spaces $\mathcal{A}_{p_1,p_2}$, $\mathcal{B}_{p_1,p_2}$ for all $p_1,p_2 \in [1,+\infty)$ and $H^{s,u}$ for all $s,u \in \mathbb{R}$.*

Proof. Note that for any $W_0 \in \mathcal{A}_{p_1,p_2}$ we have

$$\|S(t)W_0\|_{p_1,p_2}^{p_1} = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |\mathcal{F}_1(S(t)W_0)(q,k)|^{p_2} \mathrm{d}k \right)^{p_1/p_2} \mathrm{d}q.$$

Using formula (3.14) we obtain that the right hand side equals

$$\begin{aligned} &\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \left| \mathbb{E}\left\{ \exp\left\{ iq\cdot kt + i\int_0^t q\cdot K(s)\mathrm{d}s \right\} \mathcal{F}_1(W_0)(q,k+K(t)) \right\} \right|^{p_2} \mathrm{d}k \right)^{p_1/p_2} \mathrm{d}q \\ &\leq \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \mathbb{E}\, |\mathcal{F}_1(W_0)(q,k+K(t))|^{p_2}\, \mathrm{d}k \right)^{p_1/p_2} \mathrm{d}q = \|W_0\|_{p_1,p_2}^{p_1}. \end{aligned}$$

The proofs for $\mathcal{B}_{p_1,p_2}$ and $H^{s,u}$ are similar. Continuity easily follows from contractivity of the semigroup and the fact that the property in question holds on $\mathcal{S}(\mathbb{R}^d)$. $\square$

For any θ belonging to $\mathcal{A}_{p_1,p_2}$ (or $\mathcal{B}_{p_1,p_2}$, or $H^{s,u}$) define

$$S^*(t)\theta(x,k) = \mathbb{E}\left\{\theta(x+X(t),k+K(t))\right\}. \tag{3.16}$$

Using integration by parts we easily conclude that for $W_0 \in \mathcal{A}_{p_1,p_2}$ and $\theta \in \mathcal{A}_{p_1',p_2'}$ we have the following duality relation

$$\langle S(t)W_0, \theta\rangle = \langle W_0, S^*(t)\theta\rangle. \tag{3.17}$$

A similar statement holds if $\mathcal{A}_{p_1,p_2}$ is replaced by $\mathcal{B}_{p_1,p_2}$, or by $H^{s,u}$ and their dual counterparts.

Existence and uniqueness result for solutions to the Wigner equation

Note that for $\psi \in \mathcal{S}(\mathbb{R}^{2d})$ the Hilbert-Schmidt norm of the operator $C_\varepsilon[\psi]$ equals

$$\|C_\varepsilon[\psi]\|^2_{L_{(HS)}(L^2_{(s)}(\mu),H_1^{-s})} = \sum_n \|C_\varepsilon[\psi]e_n\|^2_{H_1^{-s}} = \sum_n \int_{\mathbb{R}^{2d}} \left|\mathcal{F}_1\left(C_\varepsilon[\psi]e_n\right)(q,k)\right|^2 \frac{\mathrm{d}k\mathrm{d}q}{(1+|q|^2)^{-s/2}}.$$

Here $\mathcal{F}_1$ denotes the Fourier transform performed with respect to the first variable. The sum inside the integral is

$$\begin{aligned}\sum_n \left|\mathcal{F}_1\left(C_\varepsilon[\psi]e_n\right)(q,k)\right|^2 &= \sum_n \left|\sum_{\sigma=\pm 1} \sigma \int_{\mathbb{R}^{2d}} \mathrm{e}^{\mathrm{i}(q+p/\varepsilon)\cdot x}\psi(x,k+\frac{\sigma p}{2})e_n(p)\mu(\mathrm{d}p)\mathrm{d}x\right|^2 \\ &= \sum_n \left|\int_{\mathbb{R}^d} \Phi(p,q,k)e_n(p)\mu(\mathrm{d}p)\right|^2,\end{aligned}$$

where

$$\Phi(p,q,k) := \sum_{\sigma=\pm 1} \int_{\mathbb{R}^d} \sigma \mathrm{e}^{\mathrm{i}(q+p/\varepsilon)\cdot x}\psi(x,k-\frac{\sigma p}{2})\mathrm{d}x.$$

We have

$$\begin{aligned}\sum_n \left|\int_{\mathbb{R}^d} \Phi(p,q,k)e_n(p)\mu(\mathrm{d}p)\right|^2 &= \sum_n \int_{\mathbb{R}^d} \Phi(p,q,k)e_n(p)\mu(\mathrm{d}p) \int_{\mathbb{R}^d} \Phi^*(p',q,k)e_n(-p')\mu(\mathrm{d}p') \\ &= \sum_n \int_{\mathbb{R}^{2d}} \Phi(p,q,k)\Phi^*(-p',q,k)e_n(p)e_n(p')\mu(\mathrm{d}p)\mu(\mathrm{d}p'). \end{aligned} \tag{3.18}$$

Therefore, by Corollary 1, we obtain

$$\sum_n \left|\int_{\mathbb{R}^d} \Phi(p,q,k)e_n(p)\mu(\mathrm{d}p)\right|^2 = \int_{\mathbb{R}^d} \left|\Phi(p,q,k)\right|^2 \mu(\mathrm{d}p),$$

and, consequently,

$$\|C_\varepsilon[\psi]\|^2_{L_{(HS)}(L^2_{(s)}(\mu),H_1^{-s})} = \int_{\mathbb{R}^{2d}} \left|\Phi(p,q,k)\right|^2 \frac{\mu(\mathrm{d}p)\mathrm{d}q}{(1+|q|^2)^{s/2}}.$$

Now, write

$$\Phi_\pm(p,q,k) := \pm \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}(q+p/\varepsilon)\cdot x}\psi(x,k\pm\frac{p}{2})\mathrm{d}x,$$

so that $\Phi = \Phi_- + \Phi_+$, and, moreover,

$$\int_{\mathbb{R}^d} \left|\Phi_-(p,q,k)\right|^2 \mathrm{d}k = \int_{\mathbb{R}^d} \left|\Phi_+(p,q,k)\right|^2 \mathrm{d}k = \int_{\mathbb{R}^d} \left|\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x\cdot(q+p/\varepsilon)}\psi(x,k)\mathrm{d}x\right|^2 \mathrm{d}k.$$

Hence, the Hilbert-Schmidt norm of the operator $C_\varepsilon[\psi]$ is bounded as

$$\begin{aligned}\|C_\varepsilon[\psi]\|^2_{L_{(HS)}(L^2_{(s)}(\mu),H_1^{-s})} &\le 2\int_{\mathbb{R}^{3d}} \left|\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x\cdot(q+p/\varepsilon)}\psi(x,k)\mathrm{d}x\right|^2 \frac{\mu(\mathrm{d}p)\mathrm{d}k\mathrm{d}q}{(1+|q|^2)^{s/2}} \\ &= 2\int_{\mathbb{R}^{3d}} \left|\mathcal{F}_1(\psi)\left(q+\frac{p}{\varepsilon},k\right)\right|^2 \frac{\mu(\mathrm{d}p)\mathrm{d}k\mathrm{d}q}{(1+|q|^2)^{s/2}} \le a_\varepsilon \|\psi\|^2_{H_1^{-s}},\end{aligned} \tag{3.19}$$

where

$$a_\varepsilon := 2 \sup_{q\in\mathbb{R}^d} \int_{\mathbb{R}^d} \left(1 + \left|q + \frac{p}{\varepsilon}\right|^2\right)^{-s/2} \left(1 + |q|^2\right)^{s/2} \mu(\mathrm{d}p) < +\infty. \tag{3.20}$$

We have shown that for each ε fixed, the operator $C_\varepsilon : \ H_1^{-s} \to L_{(HS)}(L^2_{(s)}(\mu), H_1^{-s}))$ is bounded. In addition, we show in Proposition 3 below that the operator $A - (1/2)L$ generates a C_0-semigroup on H_1^{-s}. We have shown therefore the following existence and uniqueness result, see Theorem 7.13, p. 203 of [11]

Theorem 1 *Under condition* (3.20) *for any* $W_\varepsilon(0) \in H_1^{-s}$ *and* $\varepsilon > 0$, *there exists a unique solution to* (3.4) *starting from* $W_\varepsilon(0)$ *and such that* $W_\varepsilon(\cdot) \in C([0, +\infty), H_1^{-s})$ *a.s. Moreover,* (3.4) *defines a Markov family on* H_1^{-s} *satisfying Feller property.*

Remark. Observe that when $s > 0$ a sufficient condition for (3.20) is

$$\int_{\mathbb{R}^d} \left(1 + |p|^2\right)^{s/2} \mu(\mathrm{d}p) < +\infty. \tag{3.21}$$

Indeed, suppose with no loss of generality that $\varepsilon = 1$ in (3.20). Then for $|q| \geq 2|p|$ we have

$$1 + |q + p|^2 \geq 1 + (|q| - |p|)^2 \geq 1 + (|q|/2)^2.$$

Hence, there exists $C > 0$ such that

$$1 + |q|^2 \leq C(1 + |p + q|^2)(1 + |p|^2).$$

This of course, in light of (3.21), implies (3.20). One can easily show an example of a measure μ such that $\int_{\mathbb{R}^d} \left(1 + |p|^2\right)^{s/2} \mu(\mathrm{d}p) = +\infty.$ for which condition (3.20) fails.

An a priori estimate

Suppose that $\{f_n(t),\, t \geq 0\}$, $n \geq 1$ are $H_1^{-s} \cap \mathcal{A}_{p_1,p_2}$-valued processes that together with $\{W_\varepsilon(t),\, t \geq 0\}$ are adapted with respect to the filtration corresponding to the Brownian motions $\{B_n(t),\, t \geq 0\}$, $n \geq 0$. Suppose also that $U_\varepsilon(t)$ is an H_1^{-s}-valued process that satisfies

$$\langle U_\varepsilon(t), \theta\rangle = \langle S(t)W_0, \theta\rangle + \sum_{n=1}^{\infty} \int_0^t \langle S(t-s)C_\varepsilon[U_\varepsilon(s)]e_n, \theta\rangle \mathrm{d}B_n(s) \tag{3.22}$$
$$+ \sum_{n=1}^{\infty} \int_0^t \langle S(t-s)f_n, \theta\rangle \mathrm{d}B_n(s)$$

for all $\theta \in \mathcal{S}(\mathbb{R}^{2d})$.

We formulate here a certain a priori estimate for $\mathbb{E}\langle U_\varepsilon(t), \theta\rangle^2$, that will be useful in what follows. Recall that $p_i' := p_i/(p_i - 1)$ when $p_i > 1$, or $p_i' := +\infty$ if $p_i = 1$ for $i = 1, 2$.

Proposition 4 *Suppose that* $W_0 \in H_1^{-s} \cap \mathcal{A}_{p_1',p_2'}$, *for some* $p_1, p_2 \geq 1$. *Then,*

$$\sup_{\|\theta\|_{p_1,p_2}\leq 1} \mathbb{E}[\langle U_\varepsilon(t), \theta\rangle^2] \leq 3e^{3\Sigma t} \left\{ \|W_0\|^2_{p_1',p_2'} + \sup_{\|\theta\|_{p_1,p_2}\leq 1} \int_0^t \sum_n \mathbb{E}\langle f_n(s), \theta\rangle^2 \mathrm{d}s \right\} \tag{3.23}$$

for all $t > 0$. *A similar result holds also when the norm* $\|\cdot\|_{p_1,p_2}$ *is replaced by* $\|\cdot|^{(\mathcal{B})}_{p_1,p_2}$.

Proof. Suppose that $\|\theta\|_{p_1,p_2} \le 1$. Observe that from Lemma 1 we have

$$\sum_n \mathbb{E}\left[\langle S(t-s)C_\varepsilon[U_\varepsilon(s)]e_n,\theta\rangle^2\right] = \sum_{\sigma,\sigma'=\pm 1}\mathbb{E}\left\{\int_{\mathbb{R}^{4d}} \exp\left\{i\varepsilon^{-1}p\cdot(x-x')\right\}\theta(x,k)\theta(x',k')\right.$$

$$\left.\times\,[S(t-s)U_\varepsilon](s,x,k+\sigma p/2)[S(t-s)W_\varepsilon](s,x',k'+\sigma' p/2)\mathrm{d}x\mathrm{d}x'\mathrm{d}k\mathrm{d}k'\mu(\mathrm{d}p)\right\}$$

$$= \sum_{\sigma,\sigma'=\pm 1}\int_{\mathbb{R}^d}\mathbb{E}\left\{\langle U_\varepsilon(s),S^*(t-s)\theta_\varepsilon^{\sigma,p}\rangle\langle U_\varepsilon(s),S^*(t-s)(\theta_\varepsilon^{\sigma',p})^*\rangle\right\}\mu(\mathrm{d}p),$$

where

$$\theta_\varepsilon^{\sigma,p}(x,k) := \exp\left\{i\varepsilon^{-1}p\cdot x\right\}\theta(x,k+\sigma p/2).$$

Note that

$$\|\theta_\varepsilon^{\sigma,p}\|_{p_1,p_2}^{p_1} = \int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d}|\mathcal{F}_1(\theta)\left(q\mp\frac{p}{\varepsilon},k+\sigma p/2\right)|^{p_2}\mathrm{d}k\right]^{p_1/p_2}\mathrm{d}q = \|\theta\|_{p_1,p_2}^{p_1}.$$

Hence, using the above and the fact that $S^*(t)$ is a contraction on $\mathcal{A}_{p_1,p_2}$ we obtain from (3.22)

$$\mathbb{E}[\langle U_\varepsilon(t),\theta\rangle^2] \le 3\left[\|W_0\|_{p_1',p_2'}^2 + \Sigma\int_0^t \sup_{\|\theta\|_{p_1,p_2}\le 1}\mathbb{E}[\langle U_\varepsilon(s),\theta\rangle^2]\mathrm{d}s\right.$$

$$\left.+\sup_{\|\theta\|_{p_1,p_2}\le 1}\sum_n\int_0^t\mathbb{E}[\langle f_n(s),\theta\rangle^2]\mathrm{d}s.\right] \tag{3.24}$$

Taking the supremum over $\|\theta\|_{p_1,p_2} \le 1$ on the left hand side and using Gronwall's inequality we conclude the proof of the proposition. □

4 Asymptotics of the fluctuations

Assumptions on the spectral measure

We shall assume that the spectral measure μ satisfies the following condition:

$$\sup_{q\in\mathbb{R}^d}\int\left(1+\frac{1}{|p+q|}\right)\mu(\mathrm{d}p) < +\infty. \tag{4.1}$$

We will actually require a more refined version of (4.1) around $p=0$:

$$\Gamma(f) := \limsup_{\varepsilon\to 0+}\varepsilon^{\gamma-1}\int_{[|p|\le\varepsilon]}\frac{\mu(\mathrm{d}p)}{|p|} < +\infty. \tag{4.2}$$

Next, set

$$|det(x,y)| := \{|x|^2|y|^2-(x\cdot y)^2\}^{1/2}.$$

We shall assume that

$$\sup_q\int\int|det(p+q,p+p_1+q)|^{-1}\mu(\mathrm{d}p)\mu(\mathrm{d}p_1) < +\infty. \tag{4.3}$$

The main result

Define the rescaled fluctuation $Z_\varepsilon(t) := \varepsilon^{-1/2}[W_\varepsilon(t) - \bar{W}(t)]$, where $W_\varepsilon(\cdot)$ satisfies the Wigner equation (3.7) and $\bar{W}(\cdot)$ is the solution of the kinetic equation (3.11). Then $Z_\varepsilon(\cdot)$ satisfies

$$\begin{aligned} &\mathrm{d}Z_\varepsilon = \left(A + \frac{1}{2}L\right) Z_\varepsilon \mathrm{d}t + \sum_n C_\varepsilon[Z_\varepsilon] e_n \mathrm{d}B_n + \varepsilon^{-1/2} \sum_n C_\varepsilon[\bar{W}] e_n \mathrm{d}B_n, \\ &Z_\varepsilon(0) = 0. \end{aligned} \tag{4.4}$$

We will consider the initial data for the Wigner equation of the form $W_0(x,k) = \delta(x) f(k)$. For simplicity we assume that the angular distribution $f(k) \ge 0$ is a Schwartz class function: $f \in \mathcal{S}(\mathbb{R}^d)$. This assumption may be greatly relaxed at the expense of more technicalities which we avoid to keep the presentation as simple as possible.

Suppose that $\{\bar{Z}(t),\, t \ge 0\}$ is a unique, $H^{-s,-u}$-valued, solution of equation

$$\begin{aligned} \frac{d\bar{Z}(t)}{dt} &= \left(A + \tfrac{1}{2}L\right) \bar{Z}(t) \quad , \\ \bar{Z}(0) &= \delta \otimes X. \end{aligned} \tag{4.5}$$

Here X is a Gaussian, random $\mathcal{S}'(\mathbb{R}^d)$-valued element given by

$$X(k) := -\mathrm{i} \sum_n \sum_{\sigma=\pm1} \sigma \int_0^{+\infty} \mathrm{d}B_n(s) \int_{\mathbb{R}^d} e^{\mathrm{i}p\cdot(k+\sigma p/2)s} f(k + \sigma p/2) e_n(p) \mu(\mathrm{d}p).$$

Our principal result can be now stated as follows.

Theorem 2 *Assume that* (4.1)-(4.3) *hold. Then, for any $t_0 > 0$ the laws of $\{Z_\varepsilon(t),\, t \ge 0\}$ over $C([t_0, +\infty); H^{-s,-u})$, where $H^{-s,-u}$ is equipped with the weak topology and $s, u > d$, converge, as $\varepsilon \to 0+$, to the law of the solution of* (4.5).

Note that we need an initial time layer after which the weak convergence could be claimed. The reason is that the non-zero initial data for the limit $\bar{Z}(t)$ cannot be a weak limit of $Z_\varepsilon(0) = 0$, as $\varepsilon \to 0+$. We will actually show that after a short time $t = o(1)$ the process $Z_\varepsilon(t)$ is no longer small and thus the initial angular distribution of the limit $\bar{Z}(t)$ is the limit of the outgoing distributon of $Z_\varepsilon(t)$ after a short initial time layer. This of course precludes the claim of the weak convergence on the entire $[0, +\infty)$.

The initial angular distribution

We may describe $X(k)$ as a real distribution-valued, mean zero, random field and covariance function

$$\mathbb{E}\left[\langle X, \theta\rangle \langle X, \theta'\rangle\right] = \mathcal{C}(\theta, \theta'), \tag{4.6}$$

where

$$\mathcal{C}(\theta, \theta') := \sum_{\sigma,\sigma'=\pm1} \sigma\sigma' \int_{\mathbb{R}^d} \mu(\mathrm{d}p) \int_0^{+\infty} \mathrm{e}^{\mathrm{i}s(\sigma+\sigma')|p|^2/2} g_{\sigma,\sigma'}(ps, p) \mathrm{d}s$$

and

$$g_{\sigma,\sigma'}(q,p) := \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}q\cdot(k-k')} f\left(k + \sigma p/2\right) f\left(k' - \sigma' p/2\right) \theta(k)\theta'(k') \mathrm{d}k \mathrm{d}k',$$

with $\theta, \theta' \in \mathcal{S}(\mathbb{R}^d)$. In fact, the law of X is supported in any Sobolev space H^{-u} for $u > d$ equipped with Borel σ algebra generated by the weak topology. To see that consider the approximants

$$X_{N,T}(k) := -\mathrm{i} \sum_{n=1}^{N} \sum_{\sigma=\pm 1} \sigma \int_0^T \mathrm{d}B_n(s) \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}p\cdot(k+\sigma p/2)s} f(k+\sigma p/2) e_n(p) \mu(\mathrm{d}p).$$

We obtain

$$\begin{aligned}
\mathbb{E}\|X_{N,T}\|^2_{H^{-u}} &= -\sum_{n=1}^{N} \sum_{\sigma,\sigma'=\pm 1} \sigma\sigma' \int_0^T \mathrm{d}s \int_{\mathbb{R}^{5d}} \mathrm{e}^{-iq\cdot k+iq\cdot k'} \mathrm{e}^{\mathrm{i}p\cdot(k+\sigma p/2)s} \mathrm{e}^{\mathrm{i}p'\cdot(k'+\sigma' p'/2)s} \\
&\times f(k+\sigma p/2) f(k'+\sigma' p'/2) e_n(p) e_n(p') \mu(\mathrm{d}p) \mu(\mathrm{d}p') \frac{\mathrm{d}k \mathrm{d}k' \mathrm{d}q}{(1+|q|^2)^{u/2}} \\
&= -\sum_{n=1}^{N} \sum_{\sigma,\sigma'=\pm 1} \sigma\sigma' \int_0^T \mathrm{d}s \int_{\mathbb{R}^{5d}} \mathrm{e}^{-iq\cdot k+iq\cdot k'} \mathrm{e}^{-\mathrm{i}p\cdot(k-\sigma p/2)s} \mathrm{e}^{-\mathrm{i}p'\cdot(k'-\sigma' p'/2)s} f(k-\sigma p/2) \\
&\times f(k'-\sigma' p'/2) e_n^*(p) e_n^*(p') \mu(\mathrm{d}p) \mu(\mathrm{d}p') \frac{\mathrm{d}k \mathrm{d}k' \mathrm{d}q}{(1+|q|^2)^{u/2}} \\
&= \sum_{n=1}^{\infty} \int_0^T \mathrm{d}s \int_{\mathbb{R}^{3d}} \mathcal{X}_N(s,q,p) \mathcal{Y}_N(s,q,p') e_n^*(p) e_n^*(p') \frac{\mu(\mathrm{d}p)\mu(\mathrm{d}p')\mathrm{d}q}{(1+|q|^2)^{u/2}}\,,
\end{aligned}$$

where $\mathcal{X}_N$ and $\mathcal{Y}_N$ are the orthogonal projections in $L^2_{(s)}(\mu)$ (in the p-variable) of

$$\mathcal{X}(s,q,p) = \mathrm{i} \sum_{\sigma=\pm 1} \sigma \int_{\mathbb{R}^d} \mathrm{e}^{-iq\cdot k} \mathrm{e}^{-\mathrm{i}p\cdot(k-\sigma p/2)s} f(k-\sigma p/2) \mathrm{d}k = \mathrm{i} \sum_{\sigma=\pm 1} \sigma \mathrm{e}^{-i\sigma q\cdot p/2} \hat{f}(q+ps)$$

and

$$\mathcal{Y}(s,q,p) = \mathrm{i} \sum_{\sigma=\pm 1} \sigma \int_{\mathbb{R}^d} \mathrm{e}^{iq\cdot k} \mathrm{e}^{-\mathrm{i}p\cdot(k-\sigma p/2)s} f(k-\sigma p/2) \mathrm{d}k.$$

Now, Corollary 1 implies that

$$\mathbb{E}\|X_{N,T}\|^2_{H^{-u}} = \int_0^T \mathrm{d}s \int_{\mathbb{R}^{2d}} \mathcal{X}_N(s,q,p) \mathcal{Y}_N(s,q,-p) \frac{\mu(\mathrm{d}p)\mathrm{d}q}{(1+|q|^2)^{u/2}},$$

while $\mathcal{Y}(s,q,-p) = \mathcal{X}^*(s,q,p)$ and thus

$$\begin{aligned}
\mathcal{X}_N^*(s,q,p) &= \sum_{n=1}^{N} \langle \mathcal{X}, e_n \rangle_\mu^* e_n^*(p) = \sum_{n=1}^{N} \int_{\mathbb{R}^d} e_n(p') \mathcal{X}^*(s,q,p') e_n(-p) \mu(\mathrm{d}p') \\
&= \sum_{n=1}^{N} \int_{\mathbb{R}^d} e_n(-p') \mathcal{Y}(s,q,-p') e_n(-p) \mu(\mathrm{d}p') = \sum_{n=1}^{N} \int_{\mathbb{R}^d} e_n^*(p') \mathcal{Y}(s,q,p') e_n(-p) \mu(\mathrm{d}p') \\
&= \mathcal{Y}_N(s,q,-p).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\mathbb{E}\|X_{N,T}\|^2_{H^{-u}} &= \int_0^T \mathrm{d}s \int_{\mathbb{R}^d} \|\mathcal{X}_N(s,q)\|^2_{L^2_{(s)}(\mu)} \frac{\mathrm{d}q}{(1+|q|^2)^{u/2}} \\
&\le \int_0^{+\infty} \mathrm{d}s \int_{\mathbb{R}^d} \|\mathcal{X}(s,q)\|^2_{L^2_{(s)}(\mu)} \frac{\mathrm{d}q}{(1+|q|^2)^{u/2}} \\
&\le 2 \int_{\mathbb{R}^d} \left(\int_0^{+\infty} \int_{\mathbb{R}^d} |\hat{f}(q+ps)|^2 \mu(\mathrm{d}p) \mathrm{d}s \right) \frac{\mathrm{d}q}{(1+|q|^2)^{u/2}} < +\infty
\end{aligned}$$

for $f \in \mathcal{S}(\mathbb{R}^d)$ and $u > d$, under assumption (4.1).

Taking $T = M$, with $M \in \mathbb{N}$, we conclude that the sequence of laws of $\{X_{N,M};\, N, M \in \mathbb{N}\}$ is tight, thus also weakly pre-compact by the results of [16], in the weak topology of H^{-u}. The existence of the limit can be established by verifying that, for all $\theta, \theta' \in \mathcal{S}(\mathbb{R}^d)$,

$$\lim_{N,M} \mathbb{E}[\langle X_{N,M}, \theta\rangle \langle X_{N,M}, \theta'\rangle] = \mathcal{C}(\theta, \theta').$$

We leave this as an exercise to the reader.

5 The proof of Theorem 2

An auxiliary Gaussian process.

As the first step we will approximate $Z_\varepsilon(t)$ by the solution of (4.4) but without the middle term on the right side. That is, suppose that $\{\bar{Z}_\varepsilon(t),\, t \geq 0\}$ is the solution of equation

$$\begin{aligned} \mathrm{d}\bar{Z}_\varepsilon &= \left(A + \tfrac{1}{2}\, L\right) \bar{Z}_\varepsilon \mathrm{d}t + \varepsilon^{-1/2} \sum_n C_\varepsilon[\bar{W}] e_n \mathrm{d}B_n, \qquad (5.1)\\ \bar{Z}_\varepsilon(0) &= 0. \end{aligned}$$

It is Gaussian given explicitly by a stochastic convolution

$$\bar{Z}_\varepsilon(t) = \varepsilon^{-1/2} \sum_n \int_0^t S(t-s) C_\varepsilon[\bar{W}(s)] e_n \mathrm{d}B_n(s). \qquad (5.2)$$

This is simply the same kinetic equation satisfied by $\bar{W}(t)$ but with an additional random forcing which depends on $\bar{W}(t)$.

The following lemma is crucial in estimating the difference between $Z_\varepsilon(t)$ and $\bar{Z}_\varepsilon(t)$.

Lemma 1 *Suppose that μ satisfies the assumptions of Theorem 2 and $s, u > d$. Then, there exists $C > 0$ such that for any $T > 0$ we have*

$$\mathbb{E}\left[\sum_n \int_0^T \langle C_\varepsilon[\bar{Z}_\varepsilon(t)] e_n, \theta\rangle \mathrm{d}B_n(t)\right]^2 \leq C\varepsilon \|\theta\|^2_{H^{s,u}}, \quad \forall\, \varepsilon \in (0,1],\, \theta \in H^{s,u}. \qquad (5.3)$$

Proof. According to Proposition 1, the expectation appearing on the left side of (5.3) equals

$$\begin{aligned} \mathbb{E}\left[\sum_n \int_0^T \langle C_\varepsilon[\bar{Z}_\varepsilon(t)] e_n, \theta\rangle^2 \mathrm{d}t\right] &= \sum_{\sigma,\sigma'=\pm 1} \sigma\sigma' \int_0^T \int_{\mathbb{R}^{5d}} \mathrm{e}^{\mathrm{i}p\cdot(x-x')/\varepsilon} (\theta \otimes \theta)(x,k,x',k') \qquad (5.4)\\ &\times \mathbb{E}\left[\bar{Z}_\varepsilon\left(t, x, k + \frac{\sigma p}{2}\right) \bar{Z}_\varepsilon\left(t, x', k' + \frac{\sigma' p}{2}\right)\right] \mathrm{d}t\mathrm{d}x\mathrm{d}x'\mathrm{d}k\mathrm{d}k'\mu(\mathrm{d}p). \end{aligned}$$

Using (5.2), the definition of $\bar{Z}_\varepsilon(t)$, we can re-write the expectation of the expression appearing in the right side of (5.4) as

$$\varepsilon^{-1} \sum_n \int_0^t K^{\varepsilon,n}_{t,s} \otimes K^{\varepsilon,n}_{t,s} \left(x, k + \frac{\sigma p}{2}, x', k' + \frac{\sigma' p}{2}\right) \mathrm{d}s,$$

where

$$K^{\varepsilon,n}_{t,s}(x,k) := S(t-s)[C_\varepsilon[\bar{W}(s)] e_n]\,(x,k)\,.$$

Substituting into (5.4) we conclude that the expression on its right hand side equals

$$\begin{aligned}
\mathcal{I}_\varepsilon &= \frac{1}{\varepsilon}\sum_n \sum_{\sigma,\sigma'=\pm 1} \sigma\sigma' \int_0^T \int_0^t \mathrm{d}s \int_{\mathbb{R}^{5d}} \mathrm{e}^{\mathrm{i}p\cdot(x-x')/\varepsilon}\theta\left(x,k-\frac{\sigma p}{2}\right)\theta\left(x',k'-\frac{\sigma' p}{2}\right) \\
&\quad\times \left\{S(t-s)[C_\varepsilon[\bar W(s)]e_n]\right\}(x,k)\left\{S(t-s)[C_\varepsilon[\bar W(s)]e_n]\right\}\left(x',k'\right)\mathrm{d}x\mathrm{d}x'\mathrm{d}k\mathrm{d}k'\mu(\mathrm{d}p) \\
&= -\frac{1}{\varepsilon}\sum_n \sum_{\sigma,\sigma'=\pm 1} \sigma\sigma'\sigma_1\sigma_1' \int_0^T \int_0^t \mathrm{d}s \int_{\mathbb{R}^{7d}} S^*(t-s)\left[\mathrm{e}^{\mathrm{i}p\cdot x/\varepsilon}\theta\left(x,k-\frac{\sigma p}{2}\right)\right] \\
&\times S^*(t-s)\left[\mathrm{e}^{-\mathrm{i}p\cdot x'/\varepsilon}\theta\left(x',k'-\frac{\sigma' p}{2}\right)\right]\mathrm{e}^{iq\cdot x/\varepsilon}\mathrm{e}^{iq_1\cdot x'/\varepsilon}\bar W_s(x,k+\frac{\sigma_1 q}{2})\bar W_s(x',k'+\frac{\sigma_1' q_1}{2}) \\
&\times e_n(q)e_n(q_1)\mathrm{d}x\mathrm{d}x'\mathrm{d}k\mathrm{d}k'\mu(\mathrm{d}p)\mu(\mathrm{d}q)d\mu(\mathrm{d}q_1) \\
&= \frac{1}{\varepsilon}\sum_{\sigma,\sigma'=\pm 1} \sigma\sigma'\sigma_1\sigma_1' \int_0^T \mathrm{d}t \int_0^t \mathrm{d}s \int_{\mathbb{R}^{6d}} \mathrm{e}^{iq\cdot x/\varepsilon}\mathrm{e}^{-iq\cdot x'/\varepsilon}S^*(t-s)\left[\mathrm{e}^{\mathrm{i}p\cdot x/\varepsilon}\theta\left(x,k-\frac{\sigma p}{2}\right)\right] \\
&\times S^*(t-s)\left[\mathrm{e}^{-\mathrm{i}p\cdot x'/\varepsilon}\theta(x',k'-\frac{\sigma' p}{2})\right]\bar W_s(x,k+\frac{\sigma_1 q}{2})\bar W_s(x',k'+\frac{\sigma_1' q}{2})\mathrm{d}x\mathrm{d}x'\mathrm{d}k\mathrm{d}k'\mu(\mathrm{d}p)\mu(\mathrm{d}q).
\end{aligned}$$

This can be written more succinctly as

$$\begin{aligned}
\mathcal{I}_\varepsilon &:= \varepsilon^{-1}\sum_{\sigma,\sigma',\sigma_1,\sigma_1'=\pm 1} \sigma\sigma'\sigma_1\sigma_1' \int_0^T \int_0^t \int_{\mathbb{R}^{6d}} \mathrm{e}^{\mathrm{i}p_1\cdot(x-x')/\varepsilon}\bar W_s\otimes\bar W_s\left(x,k+\frac{\sigma_1 p_1}{2},x',k'+\frac{\sigma_1' p_1}{2}\right) \\
&\times L_{t,s}^{\sigma p,p,\varepsilon}\otimes(L_{t,s}^{\sigma' p,p,\varepsilon})^*(x,k,x',k')\mathrm{d}t\mathrm{d}s\mathrm{d}x\mathrm{d}x'\mathrm{d}k\mathrm{d}k'\mu(\mathrm{d}p)\mu(\mathrm{d}p_1). \qquad (5.5)
\end{aligned}$$

Here $\bar W_s := \bar W(s)$, $T_l f(x,k) := f(x,k+l)$ and

$$L_{t,s}^{q,p,\varepsilon}(x,k) := S^*(t-s)T_{-q/2}\tilde\theta_\varepsilon(x,k;p),$$

with $\tilde\theta_\varepsilon(x,k;p) := \mathrm{e}^{\mathrm{i}p\cdot x/\varepsilon}\theta(x,k)$.

Using (3.15) we can represent $\bar W_s$ as a series $\bar W_s = \sum_{n\ge 0} W_s^{(n)}$, where $W_s^{(n)} = W_n(s)$ is given by (3.15). Likewise, we can write

$$L_{t,s}^{q,p,\varepsilon}(x,k) = \sum_{m\ge 0} L_{t,s}^{q,p,\varepsilon,m}(x,k),$$

where

$$\begin{aligned}
L_{t,s}^{q,p,\varepsilon,0}(x,k) &:= \mathrm{e}^{-\Sigma(t-s)}\tilde\theta_\varepsilon\left(x+\left(k-\frac{q}{2}\right)(t-s),k-\frac{q}{2};p\right) \\
L_{t,s}^{q,p,\varepsilon,n}(x,k) &= \Sigma^n\mathrm{e}^{-\Sigma(t-s)}\int\limits_{\Delta_n(t-s)} \mathbb{E}\left\{\tilde\theta_\varepsilon\left(x+\left(k-\frac{q}{2}\right)(t-s)+\mathcal{X}_n,k-\frac{q}{2}+K_n;p\right)\right\}\mathrm{d}\tau(n).
\end{aligned}$$

Here we have maintained the notation introduced in (3.15). Therefore, the expression in (5.5) can be represented accordingly as

$$\mathcal{I}_\varepsilon = \sum_{n,n',m,m'\ge 0} \mathcal{I}_\varepsilon^{n,n',m,m'},$$

where

$$\mathcal{I}_\varepsilon^{n,n',m,m'} := \frac{1}{\varepsilon}\int_0^T \mathrm{d}t\int_0^t \mathrm{d}s\ \mathrm{e}^{-2\Sigma t}\int_{\mathbb{R}^{2d}} \mathcal{W}_{m,n}\mathcal{W}^*_{m',n'}\mu(\mathrm{d}p)\mu(\mathrm{d}p_1)$$

and

$$\mathcal{W}_{m,n} := \sum_{\sigma,\sigma'=\pm 1} \sigma\sigma' \int_{\mathbb{R}^{2d}} \exp\{\mathrm{i}p_1\cdot x/\varepsilon\} W_s^{(n)}\left(x, k+\frac{\sigma_1 p_1}{2}\right) L_{t,s}^{\sigma p,p,\varepsilon,m}(x,k)\mathrm{d}x\mathrm{d}k \tag{5.6}$$

$$= \Sigma^{m+n} \sum_{\sigma,\sigma'=\pm 1} \sigma\sigma' \int_{\Delta_m(t-s)} \mathrm{d}\tau \int_{\Delta_n(s)} \mathrm{d}\rho \int_{\mathbb{R}^{2d}} \exp\{\mathrm{i}(p+p_1)\cdot x/\varepsilon\} \exp\left\{\mathrm{i}\,(t-s)\,p\cdot\left(k-\frac{\sigma p}{2}\right)/\varepsilon\right\}$$

$$\times \mathbb{E}\left\{\exp\{\mathrm{i}p\cdot\mathcal{X}_m/\varepsilon\}\,\theta\left(x+\left(k-\frac{\sigma p}{2}\right)(t-s)+\mathcal{X}_m, k-\frac{\sigma p}{2}+K_m\right)\right.$$

$$\left.\times\, W_0\left(x-\left(k+\frac{\sigma_1 p_1}{2}\right)s-\mathcal{Y}_n, k+\frac{\sigma_1 p_1}{2}+L_n\right)\right\}\mathrm{d}x\mathrm{d}k.$$

Here $\mathcal{X}_m := \sum_{j=0}^m K_j\tau_j$ and $\mathcal{Y}_n := \sum_{j=0}^n L_j\rho_j$ are the random variables arising from the probabilistic interpretation for the kinetic equation. All variables K_j, L_j are i.i.d., each with the law $\nu(\cdot)$. We can further rewrite the right hand side of (5.6) using the law of the random variables representing momentum and obtain

$$\mathcal{W}_{m,n} = \sum_{\sigma,\sigma_1=\pm 1} \sigma\sigma_1 \int_{\hat{\Delta}_m(t-s)} \mathrm{d}\tau \int_{\hat{\Delta}_n(s)} \mathrm{d}\rho \int_{\mathbb{R}^{(m+n)d+2}} \mathrm{d}x\mathrm{d}k \prod_{j=1}^m \mu(\mathrm{d}k_j) \prod_{j=1}^n \mu(\mathrm{d}l_j)$$

$$\times \exp\left\{\mathrm{i}(p+p_1)\cdot x/\varepsilon + \mathrm{i}(t-s)p\cdot\left(k-\frac{\sigma p}{2}\right)/\varepsilon\right\} \times \exp\left\{\mathrm{i}p\cdot(\sum_{j=1}^m k_j\tau_j)/\varepsilon\right\}$$

$$\times\theta\left(x+\left(k-\frac{\sigma p}{2}\right)(t-s)+\sum_{j=1}^m k_j\tau_j, k-\frac{\sigma p}{2}+\sum_{j=1}^m k_j\right)$$

$$\times W_0\left(x-\left(k+\frac{\sigma_1 p_1}{2}\right)s-\sum_{j=1}^n l_j\rho_j, k+\frac{\sigma_1 p_1}{2}+\sum_{j=1}^n l_j\right),$$

where $\hat{\Delta}_n(s) := [(s_1,\ldots,s_n) : s \geq s_n \geq \ldots s_1 \geq 0]$. Thanks to symmetry we can rewrite the above expression as

$$\mathcal{W}_{m,n} = \frac{1}{n!m!} \sum_{\sigma,\sigma_1=\pm 1} \sigma\sigma_1 \int_{\square_m(t-s)} \mathrm{d}\tau \int_{\square_n(s)} \mathrm{d}\rho \int_{\mathbb{R}^{(m+n)d+2}} \mathrm{d}x\mathrm{d}k \prod_{j=1}^m \mu(\mathrm{d}k_j) \prod_{j=1}^n \mu(\mathrm{d}l_j)$$

$$\times \exp\left\{\mathrm{i}(p+p_1)\cdot x/\varepsilon + \mathrm{i}(t-s)p\cdot\left(k-\frac{\sigma p}{2}\right)/\varepsilon\right\} \exp\left\{\mathrm{i}p\cdot(\sum_{j=1}^m k_j\tau_j)/\varepsilon\right\}$$

$$\times\theta\left(x+\left(k-\frac{\sigma p}{2}\right)(t-s)+\sum_{j=1}^m k_j\tau_j, k-\frac{\sigma p}{2}+\sum_{j=1}^m k_j\right)$$

$$\times W_0\left(x-\left(k+\frac{\sigma_1 p_1}{2}\right)s-\sum_{j=1}^n l_j\rho_j, k+\frac{\sigma_1 p_1}{2}+\sum_{j=1}^n l_j\right)$$

with $\square_n(s) := [(s_1,\ldots,s_n) : s_i \in [0,s],\, i=1,\ldots,n]$. Using the fact that $W_0(x,k) = \delta(x)f(k)$ and performing the Fourier transform of both $f(k)$ and $\theta(x,k)$ we conclude that

$$
\begin{aligned}
\mathcal{W}_{m,n} &= \frac{1}{n!m!}\sum_{\sigma,\sigma_1=\pm 1}\sigma\sigma_1\int_{\square_m(t-s)}\mathrm{d}\tau\int_{\square_n(s)}\mathrm{d}\rho\int_{\mathbb{R}^{(m+n)d+4}}\mathrm{d}k\mathrm{d}y\mathrm{d}q\mathrm{d}z\prod_{j=1}^m\mu(\mathrm{d}k_j)\prod_{j=1}^n\mu(\mathrm{d}l_j)\\
&\times\hat{\theta}(q,y)\,\hat{f}(z)\exp\left\{\mathrm{i}[\varepsilon^{-1}(p+p_1+\varepsilon q)s+z]\cdot\left(k+\frac{\sigma_1 p_1}{2}\right)\right\}\\
&\times\exp\left\{\mathrm{i}[\varepsilon^{-1}(p+\varepsilon q)(t-s)+y]\cdot\left(k-\frac{\sigma p}{2}\right)\right\}\\
&\times\exp\left\{\mathrm{i}\sum_{j=1}^m(\varepsilon^{-1}(p+\varepsilon q)\tau_j+y)\cdot k_j\right\}\exp\left\{\mathrm{i}\sum_{j=1}^n[\varepsilon^{-1}(p+p_1+\varepsilon q)\rho_j+z]\cdot l_j\right\}\\
&=\frac{\varepsilon^{m+n}}{m!n!}\int_{\mathbb{R}^{4d}}\mathrm{d}k\mathrm{d}y\mathrm{d}q\mathrm{d}z\hat{\theta}(q,y)\,\hat{f}(z)\exp\left\{\mathrm{i}[\varepsilon^{-1}(p+p_1+\varepsilon q)s+z]\cdot\left(k+\frac{\sigma_1 p_1}{2}\right)\right\}\\
&\times\Theta^m\left(\frac{t-s}{\varepsilon},p+\varepsilon q,y\right)\Theta^n\left(\frac{s}{\varepsilon},p+p_1+\varepsilon q,z\right)\exp\left\{\mathrm{i}[\varepsilon^{-1}(p+\varepsilon q)(t-s)+y]\cdot\left(k-\frac{\sigma p}{2}\right)\right\}.
\end{aligned}
$$

Here we have set

$$
\Theta(t,k,x)=\int_0^t\mathrm{d}\tau\int_{\mathbb{R}^d}e^{ip\cdot(\tau k+x)}\mu(p)\mathrm{d}p. \tag{5.7}
$$

Finally, we change variables $t:=t/\varepsilon$ and $s:=s/\varepsilon$. We have shown that the expression in (5.5) equals $\mathcal{I}_\varepsilon=\varepsilon\tilde{\mathcal{I}}_\varepsilon$, where

$$
\tilde{\mathcal{I}}_\varepsilon=\int_0^{T/\varepsilon}\mathrm{e}^{-2\varepsilon\Sigma t}\mathrm{d}t\int_0^t\mathrm{d}s\int_{\mathbb{R}^{2d}}|\mathcal{F}_\varepsilon|^2\,\mu(\mathrm{d}p)\mu(\mathrm{d}p_1), \tag{5.8}
$$

and

$$
\begin{aligned}
\mathcal{F}_\varepsilon &:= \sum_{\sigma,\sigma_1=\pm 1}\sigma\sigma_1\int_{\mathbb{R}^{4d}}\hat{\theta}(q,y)\,\hat{f}(z)\exp\{\varepsilon[\Theta(t-s,p+\varepsilon q,y)+\Theta(s,p+p_1+\varepsilon q,z)]\}\\
&\times\exp\left\{\mathrm{i}[(p+p_1+\varepsilon q)s+z]\cdot\left(k+\frac{\sigma_1 p_1}{2}\right)\right\}\exp\left\{\mathrm{i}[(p+\varepsilon q)(t-s)+y]\cdot\left(k-\frac{\sigma p}{2}\right)\right\}\mathrm{d}k\mathrm{d}y\mathrm{d}q\mathrm{d}z\\
&=\sum_{\sigma,\sigma_1=\pm 1}\sigma\sigma_1\int_{\mathbb{R}^{2d}}\hat{\theta}(q,y)\,\hat{f}(-y-t(p+\varepsilon q)-sp_1)\\
&\times\exp\{\varepsilon[\Theta(t-s,p+\varepsilon q,y)+\Theta(s,p+p_1+\varepsilon q,-y-t(p+\varepsilon q)-sp_1)]\}\\
&\times\exp\{(\mathrm{i}/2)\,[(p+\varepsilon q)(t-s)+y]\cdot(\sigma_1 p_1+\sigma p)]\}\,\mathrm{d}q\mathrm{d}y.
\end{aligned}
$$

Changing variables $u:=t-s$, $s:=s$ we obtain that

$$
\tilde{\mathcal{I}}_\varepsilon=\int\limits_{0\le s,u,s+u\le T/\varepsilon}\mathrm{e}^{-\varepsilon(u+s)\Sigma}\mathrm{d}u\mathrm{d}s\int_{\mathbb{R}^{2d}}|\mathcal{F}_\varepsilon|^2\,\mu(\mathrm{d}p)\mu(\mathrm{d}p_1),
$$

and, as f is real so that $\hat{f}(y)$ is complex-even in y, we have

$$
\begin{aligned}
\mathcal{F}_\varepsilon &= \sum_{\sigma,\sigma_1=\pm 1}\sigma\sigma_1\int_{\mathbb{R}^{2d}}\hat{\theta}(q,y)\,\hat{f}^*(y+(s+u)(p+\varepsilon q)+sp_1)\\
&\times\exp\{\varepsilon[\Theta(u,p+\varepsilon q,y)+\Theta(s,p+p_1+\varepsilon q,-y-u(p+\varepsilon q)-s(p+p_1+\varepsilon q))]\}\\
&\times\exp\{(\mathrm{i}/2)\,[(p+\varepsilon q)(t-s)+y]\cdot(\sigma_1 p_1+\sigma p)]\}\,\mathrm{d}q\mathrm{d}y.
\end{aligned} \tag{5.9}
$$

Note that directly from the definition (5.7) we have $\Theta(u,p,y)\geq 0$ and it can be estimated as follows

$$\Theta(u,p,y)=\int_0^u\int_{\mathbb{R}^d}\mathrm{e}^{ik\cdot(\tau p+y)}\mathrm{d}\tau\mu(\mathrm{d}k)=\int_{\mathbb{R}^d}\frac{\mathrm{e}^{ik\cdot(up)}-1}{i(k\cdot p)}\,\mathrm{e}^{ik\cdot y}\mu(\mathrm{d}k) \tag{5.10}$$
$$\leq u\int_{\mathbb{R}^d}\left|\frac{\mathrm{e}^{ik\cdot(up)}-1}{k\cdot(up)}\right|\mu(\mathrm{d}k)\leq u\Sigma.$$

Thus, the expression in the exponent in (5.9) can be bounded as

$$\Theta(u,p+\varepsilon q,y)+\Theta(s,p+p_1+\varepsilon q,y+u(p+\varepsilon q)-s(p+p_1+\varepsilon q))\leq\Sigma(u+s)\leq\Sigma T/\varepsilon.$$

Therefore, expression in (5.8) may be estimated by

$$\begin{aligned}
&|\tilde{\mathcal{I}}_\varepsilon|\leq 4e^{\Sigma T}\int_0^{+\infty}\int_0^{+\infty}\mathrm{d}u\mathrm{d}s\int_{\mathbb{R}^{2d}}\mu(\mathrm{d}p)\mu(\mathrm{d}p_1)\\
&\times\left\{\int_{\mathbb{R}^{2d}}|\hat{\theta}\,(q,y)\,||\hat{f}\,(y+u(p+\varepsilon q)+s(p+p_1+\varepsilon q))\,|\mathrm{d}q\mathrm{d}y\right\}^2\\
&\leq 4e^{\Sigma T}\|\theta\|_{1,1}^{(\mathcal{B})}\int_0^{+\infty}\int_0^{+\infty}\mathrm{d}u\mathrm{d}s\int_{\mathbb{R}^{4d}}\mu(\mathrm{d}p)\mu(\mathrm{d}p_1)\mathrm{d}q\mathrm{d}y|\hat{\theta}\,(q,y)\,||\hat{f}\,(y+u(p+\varepsilon q)+s(p+p_1+\varepsilon q))\,|^2.
\end{aligned}$$

The integral in u and s may be treated as

$$\int_0^{+\infty}\int_0^{+\infty}|\hat{f}\,(y+u(p+\varepsilon q)+s(p+p_1+\varepsilon q))\,|^2\mathrm{d}u\mathrm{d}s\leq C(f)|\det(p+\varepsilon q,p+p_1+\varepsilon q)|^{-1}.$$

Taking this into account we can estimate

$$|\tilde{\mathcal{I}}_\varepsilon|\leq 4e^{\Sigma T}C(f)(\|\theta\|_{1,1}^{(\mathcal{B})})^2\sup_q\iint|\det(p+q,p+p_1+q)|^{-1}\mu(\mathrm{d}p)\mu(\mathrm{d}p_1).$$

Finally, to get (5.3) it suffices only to recall assumption (4.3) and observe that when $s,u>d/2$ there exists $C>0$ such that $\|\theta\|_{1,1}^{(\mathcal{B})}\leq C\|\theta\|_{H^{s,u}}$ for all $\theta\in\mathcal{S}(\mathbb{R}^{2d})$. $\square$

Approximating Z_ε by $\bar{Z}_\varepsilon$

We now use Lemma 1 to estimate the difference between the true corrector Z_ε and $\bar{Z}_\varepsilon$. The error $U_\varepsilon(t):=Z_\varepsilon(t)-\bar{Z}_\varepsilon(t)$ satisfies the equation

$$\begin{aligned}
&\mathrm{d}U_\varepsilon=\left(A+\tfrac{1}{2}\,L\right)U_\varepsilon\mathrm{d}t+\sum_n C_\varepsilon[U_\varepsilon]e_n\mathrm{d}B_n+\sum_n C_\varepsilon[\bar{Z}_\varepsilon]e_n\mathrm{d}B_n,\\
&\bar{U}_\varepsilon(0)=0.
\end{aligned}$$

We have the following estimate.

Lemma 2 *For any $t>0$ there exists $C>0$ such that for all $\theta\in\mathcal{S}(\mathbb{R}^{2d})$ we have*

$$\mathbb{E}\left[\langle U_\varepsilon(t),\theta\rangle^2\right]\leq C\varepsilon(\|\theta\|_{1,1}^{(\mathcal{B})})^2.$$

Proof. Using estimate (3.23), with $W_\varepsilon(0)=0$, we obtain

$$\mathbb{E}[\langle U_\varepsilon(t),\theta\rangle^2]\leq 3(\|\theta\|_{1,1}^{(\mathcal{B})})^2\sup_{\|\theta\|_{1,1}^{(\mathcal{B})}\leq 1}\mathbb{E}\left[\sum_n\int_0^t\langle C_\varepsilon[\bar{Z}_\varepsilon(s)]e_n,\theta\rangle\mathrm{d}B_n(s)\right]^2.$$

The result then follows from Lemma 1. $\square$

Tightness of $\bar Z_\varepsilon(t)$, as $\varepsilon\to 0+$

The asymptotics of $Z_\varepsilon(t)$, as $\varepsilon\to 0+$, is therefore the same as that of $\bar Z_\varepsilon(t)$. Using decomposition of generator L as in the last line of (3.8) we can write that

$$
\begin{aligned}
&{\rm d}\bar Z_\varepsilon=(A-\Sigma+\,M)\,\bar Z_\varepsilon{\rm d}t+\varepsilon^{-1/2}\sum_n C_\varepsilon[\bar W]e_n{\rm d}B_n,\\
&\bar Z_\varepsilon(0)=0,
\end{aligned}
$$

where the operator M is given by (3.9). Therefore by Duhamel's formula we have

$$
\bar Z_\varepsilon(t)=\int_0^t S_0(t-s)M\bar Z_\varepsilon(s){\rm d}s+\varepsilon^{-1/2}\sum_n\int_0^t S_0(t-s)C_\varepsilon[\bar W(s)]e_n{\rm d}B_n(s). \tag{5.11}
$$

Here

$$
S_0f(t):={\rm e}^{-t\Sigma}f(x-kt,k) \tag{5.12}
$$

for an appropriate f and $t\in\mathbb{R}$. Suppose we are given a family of Borel probability measures $\{P_\varepsilon,\,\varepsilon>0\}$ defined over a certain topological space. We say that the family is *weakly pre-compact, as $\varepsilon\to 0+$*, if for any sequence $\varepsilon_n\to 0$, as $n\to+\infty$ one can choose a subsequence from $\{P_{\varepsilon_n},\,n\ge 1\}$ that is weakly convergent.

Proposition 5 *Suppose that $s,u>d$, $t_0>0$ and the space $H^{-s,-u}$ is equipped with the weak topology. Then, the family of laws of the processes $\{\bar Z_\varepsilon(t),\,t\ge 0\}$ considered in $C([t_0,+\infty),H^{-s,-u})$ is weakly pre-compact when $\varepsilon\to 0+$.*

Proof. According to [16], Theorem 3.1, p. 276, to show weak pre-compactness of the laws in $D([t_0,+\infty),H^{-s,-u})$ it suffices only to show that for each $\delta>0$, $T_1\ge t_0$ there exists $K>0$ such that

$$
\mathbb{P}\left[\sup_{t\in[t_0,T_1]}\|\bar Z_\varepsilon(t)\|_{-s,-u}\le K\right]\ge 1-\delta \tag{5.13}
$$

and that for any test function $\theta\in H^{s,u}$

$$
\text{the laws of }\{\langle\bar Z_\varepsilon(t),\theta\rangle,\,t\in[0,T]\},\ \varepsilon\in(0,1]\text{ are tight in }C[0,T]. \tag{5.14}
$$

Since $C([t_0,+\infty),H^{-s,-u})$ is a closed subset of $D([t_0,+\infty),H^{-s,-u})$, see Proposition 1.6, p. 267 of [16], this implies weak pre-compactness of the laws in $C([t_0,+\infty),H^{-s,-u})$.

In order to conclude (5.13) it is a actually enough to prove that

$$
\sup_{\varepsilon\in(0,1],T\in[t_0,T_1]}\varepsilon^{-1}\int_0^T(T-t)^{-2\alpha}\|C_\varepsilon[\bar W(t)]\|^2_{L_{(HS)}(L^2_{(s)}(\mu),H^{-s,-u})}{\rm d}t<+\infty \tag{5.15}
$$

for $\alpha\in(0,1/2)$. Using Lemmma 7.2 p. 182 of [11] and estimates (7.11) and (7.12) p. 184 of ibid. we would be able then to conclude that $\mathbb{E}\left[\sup_{t\in[t_0,T_1]}\|\bar Z_\varepsilon(t)\|^2_{-s,-u}\right]<+\infty$, which in particular implies (5.13). Hence, we will now show that (5.15) holds. Note that, by (3.19), the expression under the supremum in (5.15) can be bounded from above by

$$
\begin{aligned}
\mathcal{J}_\varepsilon:=&\frac{2}{\varepsilon}\int_0^T(T-t)^{-2\alpha}{\rm d}t\int_{\mathbb{R}^{3d}}\frac{\mu({\rm d}p){\rm d}y{\rm d}q}{(1+|q|^2)^{s/2}(1+|y|^2)^{u/2}}\\
&\times\left|\sum_\sigma\sigma\int_{\mathbb{R}^{2d}}{\rm e}^{-{\rm i}x\cdot(q+p/\varepsilon)}{\rm e}^{-{\rm i}y\cdot k}\bar W(t,x,k+\sigma p/2){\rm d}x{\rm d}k\right|^2.
\end{aligned} \tag{5.16}
$$

Using probabilistic representation of $\bar{W}(t,x,k)$ in a similar way as it has been done in the proof of Lemma 1 we obtain that

$$\int_{\mathbb{R}^{2d}} \mathrm{e}^{-\mathrm{i}x\cdot(q+p/\varepsilon)}\mathrm{e}^{-\mathrm{i}y\cdot k}\bar{W}(t,x,k+\sigma p/2)\mathrm{d}x\mathrm{d}k = \sum_n \frac{e^{-\Sigma t}}{n!}\int_{\mathbb{R}^{2d}}\mathrm{d}x\mathrm{d}k\int_{\square_n(t)}\mathrm{d}\tau\int_{(\mathbb{R}^d)^n}\prod_{j=1}^n \mu(\mathrm{d}k_j)$$

$$\times \mathrm{e}^{-\mathrm{i}x\cdot(q+p/\varepsilon)}\mathrm{e}^{-\mathrm{i}y\cdot k}W_0\left(x-\left(k+\frac{\sigma p}{2}\right)t-\sum_{j=1}^n k_j\tau_j, k+\frac{\sigma p}{2}+\sum_{j=1}^n k_j\right).$$

Taking into account the fact that $W_0(x,k)=\delta(x)f(k)$ and substituting into (5.16) we obtain

$$\mathcal{J}_\varepsilon = \frac{2}{\varepsilon}\int_0^T e^{-2\Sigma t}\,(T-t)^{-2\alpha}\,\mathrm{d}t\int_{\mathbb{R}^{3d}}\frac{\mu(\mathrm{d}p)\mathrm{d}y\mathrm{d}q}{(1+|q|^2)^{s/2}(1+|y|^2)^{u/2}}$$

$$\times\left|\sum_\sigma \sigma\int_{\mathbb{R}^{2d}}\exp\left\{-\mathrm{i}(t/\varepsilon)(p+\varepsilon q)\cdot\left(k+\frac{\sigma p}{2}\right)-\mathrm{i}y\cdot k+\mathrm{i}z\cdot\left(k+\frac{\sigma p}{2}\right)+\varepsilon\Theta(t/\varepsilon,q,z)\right\}\hat{f}(z)\mathrm{d}k\mathrm{d}z\right|^2.$$

Recall, see (5.10), that $\varepsilon\Theta(t/\varepsilon,q,z)\le T\Sigma$ for $t\in[0,T]$. Let $\omega(p,q):=(p+\varepsilon q)|p+\varepsilon q|^{-1}$ and $g_u(y):=(1+|y|^2)^{-u/2}$. Integrating out the k and z variables and replacing $t:=t/\varepsilon$ we obtain

$$\begin{aligned}
\mathcal{J}_\varepsilon &\le \frac{C_T}{\varepsilon^{2\alpha}}\int_0^{T/\varepsilon}\left(\frac{T}{\varepsilon}-t\right)^{-2\alpha}\mathrm{d}t\int_{\mathbb{R}^{3d}}g_u(y)g_s(q)\left|\hat{f}(y+t(p+\varepsilon q))\right|^2\mu(\mathrm{d}p)\mathrm{d}y\mathrm{d}q\\
&\le C_T\int_{\mathbb{R}^{3d}}\frac{g_u(y)g_s(q)}{|p+\varepsilon q|}\mu(\mathrm{d}p)\mathrm{d}y\mathrm{d}q\\
&\quad\times\left(\frac{T|p+\varepsilon q|}{\varepsilon}\right)^{2\alpha}\int_0^{T|p+\varepsilon q|/\varepsilon}\left(\frac{T|p+\varepsilon q|}{\varepsilon}-t\right)^{-2\alpha}\left|\hat{f}(y+t\omega(p,q))\right|^2\mathrm{d}t\\
&= C_T\int_{\mathbb{R}^{2d}}g_s(q)|p+\varepsilon q|^{-1}\mu(\mathrm{d}p)\mathrm{d}q\times\sup_{S>0,\omega\in\mathbb{S}^{d-1}}S^{2\alpha}\int_0^S(S-t)^{-2\alpha}g_u*\left|\hat{f}\right|^2(t\omega)\mathrm{d}t\\
&\le C(f,T)<+\infty
\end{aligned}$$

for a function $f\in\mathcal{S}(\mathbb{R}^d)$, with a constant $C(f,T)$ that does not depend on $\varepsilon\in(0,1)$. Hence, (5.15) holds.

Next, we establish (5.14). Suppose first that $\theta\in\mathcal{S}(\mathbb{R}^{2d})$. For each $\varepsilon\in(0,1]$ the real valued process $\{\langle\bar{Z}_\varepsilon(t),\theta\rangle,\ t\ge 0\}$ is Gaussian. In order to prove its tightness we will show that its covariance $R_\varepsilon(t,s)$ satisfies

$$|R_\varepsilon(t,s)-R(s,s)|+|R_\varepsilon(t,t)-R_\varepsilon(t,s)|\le C(t_0,T;\theta)(t-s) \tag{5.17}$$

for all $t>s$, $\varepsilon\in(0,1)$, and $t_0\le t,s\le T$. For $t>s$ the covariance $R_\varepsilon(t,s)$ of the process $\langle\bar{Z}_\varepsilon(t),\theta\rangle$ equals

$$R_\varepsilon(t,s)=\frac{1}{\varepsilon}\sum_{\sigma,\sigma'}\sigma\sigma'\int_0^s\int\ e^{i(x-x')\cdot p/\varepsilon}\bar{W}(u,x,k+\sigma p/2)\bar{W}(u,x',k'+\sigma' p/2)$$

$$\times S^*(t-u)\theta(x,k)S^*(s-u)\theta(x',k')\mathrm{d}u\mu(\mathrm{d}p)\mathrm{d}x\mathrm{d}x'\mathrm{d}k\mathrm{d}k'.$$

Hence, we have

$$R_\varepsilon(t,s)-R_\varepsilon(s,s)=\frac{1}{\varepsilon}\sum_\sigma\sigma\sigma'\int_0^s\mathrm{d}u\int_s^t\mathrm{d}u'\int \mathrm{e}^{i(x-x')\cdot p/\varepsilon}\bar{W}(u,x,k+\sigma p/2)\bar{W}(u,x',k'+\sigma' p/2)$$

$$\times S^*(u'-u)\theta_{A,L}(x,k)S^*(s-u)\theta(x',k')\mu(\mathrm{d}p)\mathrm{d}x\mathrm{d}x'\mathrm{d}k\mathrm{d}k',$$

where $\theta_{A,L}(x,k) := (-A+\frac{1}{2}L)\theta(x,k)$. Using the same argument as in the proof of Lemma 1 we obtain

$$R_\varepsilon(t,s) - R_\varepsilon(s,s) = \frac{1}{\varepsilon}\sum_{m,m',n,n'}\sum_{\sigma,\sigma'}\sigma\sigma'\int_0^s \mathrm{d}u\int_s^t \mathrm{d}u' e^{-\Sigma(u'+s)}\int_{\mathbb{R}^d}\tilde{\mathcal{W}}^{A,L}_{m,n,\sigma}\tilde{\mathcal{W}}_{m',n',\sigma'}\mu(\mathrm{d}p), \qquad (5.18)$$

where

$$\tilde{\mathcal{W}}^{A,L}_{m,n,\sigma} = \frac{1}{m!n!}\int_{\square_m(u'-u)}\mathrm{d}\tau\int_{\square_n(u)}\mathrm{d}\rho\int \mathrm{d}x\mathrm{d}k\prod_{j=1}^m\mu(\mathrm{d}k_j)\prod_{j=1}^n\mu(\mathrm{d}l_j)\mathrm{e}^{ix\cdot p/\varepsilon}$$
$$\times\theta_{A,L}\left(x+k(u'-u)+\sum_{j=1}^m k_j\tau_j, k+\sum_{j=1}^m k_j\right)W_0\left(x-\left(k+\frac{\sigma p}{2}\right)u-\sum_{j=1}^n l_j\rho_j, k+\frac{\sigma p}{2}+\sum_{j=1}^n l_j\right).$$

and

$$\tilde{\mathcal{W}}_{m',n',\sigma'} = (m'!n'!)^{-1}\int_{\square_{m'}(s-u)}\mathrm{d}\tau\int_{\square_{n'}(u)}\mathrm{d}\rho\iint \mathrm{d}x'\mathrm{d}k'\prod_{j=1}^{m'}\mu(\mathrm{d}k_j)\prod_{j=1}^{n'}\mu(\mathrm{d}l_j)\mathrm{e}^{-ix'\cdot p/\varepsilon}$$
$$\times\theta\left(x'+k'(s-u)+\sum_{j=1}^m k_j\tau_j, k'+\sum_{j=1}^m k_j\right)W_0\left(x'-\left(k'+\frac{\sigma' p}{2}\right)u-\sum_{j=1}^n l_j\rho_j, k'+\frac{\sigma' p}{2}+\sum_{j=1}^n l_j\right).$$

As before, using the specific form of the initial data $W_0(x,k)=\delta(x)f(k)$ and performing the Fourier transform of $\theta(x,k)$ and $f(k)$ we obtain

$$\begin{aligned}\tilde{\mathcal{W}}^{A,L}_{m,n,\sigma} &= \frac{1}{m!n!}\int \varepsilon^n\Theta^n\left(\frac{u}{\varepsilon}, p+\varepsilon q, z\right)\Theta^m\left(u'-u,q,y\right)\exp\left\{i(\frac{pu}{\varepsilon}+qu'+y+z)\cdot k\right\}\\ &\times\exp\left\{i(\sigma p/2)\cdot\left[\frac{u}{\varepsilon}(p+\varepsilon q)+z\right]\right\}\hat{\theta}_{A,L}\left(q,y\right)\hat{f}(z)\mathrm{d}k\mathrm{d}q\mathrm{d}y\mathrm{d}z\\ &= \frac{\varepsilon^n}{m!n!}\int\Theta^n\left(\frac{u}{\varepsilon}, p+\varepsilon q, -\frac{pu}{\varepsilon}-qu'-y\right)\Theta^m(u'-u,q,y)\\ &\times\exp\left\{i(\sigma p/2)\cdot\left[(u-u')q-y\right]\right\}\hat{\theta}_{A,L}\left(q,y\right)\hat{f}(-\frac{pu}{\varepsilon}-qu'-y)\mathrm{d}q\mathrm{d}y.\end{aligned}$$

Likewise, we have

$$\begin{aligned}\tilde{\mathcal{W}}_{m',n',\sigma'} &= \frac{\varepsilon^{n'}}{m'!n'!}\int\Theta^{n'}\left(\frac{u}{\varepsilon}, -p+\varepsilon q', \frac{pu}{\varepsilon}-q's-y'\right)\Theta^{m'}(u,q',y')\\ &\times\exp\left\{-i(\sigma p/2)\cdot y'\right\}\hat{\theta}\left(q',y'\right)\hat{f}(\frac{pu}{\varepsilon}-q's-y')\mathrm{d}q'\mathrm{d}y'.\end{aligned}$$

In consequence, (5.18) becomes

$$\begin{aligned}R_\varepsilon(t,s)-R_\varepsilon(s,s) &= \frac{1}{\varepsilon}\sum_{\sigma,\sigma'}\sigma\sigma'\int_0^s\mathrm{d}u\int_s^t\mathrm{d}u'e^{-\Sigma(u'+s)}\int\mu(\mathrm{d}p)\mathrm{d}q\mathrm{d}q'\mathrm{d}y\mathrm{d}y'\\ &\times\exp\left\{\varepsilon\Theta\left(\frac{u}{\varepsilon}, p+\varepsilon q, -\frac{pu}{\varepsilon}-qu'-y\right)+\Theta(u'-u,q,y)\right\}\\ &\times\exp\left\{\varepsilon\Theta\left(\frac{u}{\varepsilon}, -p+\varepsilon q', \varepsilon^{-1}pu-q's-y'\right)+\Theta(u,q',y')\right\}\hat{\theta}_{A,L}\left(q,y\right)\hat{\theta}\left(q',y'\right)\\ &\times\exp\left\{i(\sigma p/2)\cdot\left[(u-u')q-y-y'\right]\right\}\hat{f}(-\frac{pu}{\varepsilon}-qu'-y)\hat{f}(\frac{pu}{\varepsilon}-q's-y').\end{aligned}$$

Changing variable $u_{new} := u/\varepsilon$ we obtain that

$$|R_\varepsilon(t,s) - R_\varepsilon(s,s)| \le C(T,f)\|\theta_{A,L}\|_{1,1}^{(\mathcal{B})}\|\theta\|_{1,1}^{(\mathcal{B})}(t-s) \tag{5.19}$$

for all $\varepsilon \in (0,1]$, $t,s \in [0,T]$. This estimates the first term in (5.17).

On the other hand, for $t > s$ we also have

$$\begin{aligned}
&R_\varepsilon(t,t) - R_\varepsilon(t,s) = \frac{1}{\varepsilon}\sum_\sigma \sigma\sigma' \int_0^s \mathrm{d}u \int_s^t \mathrm{d}u' \int \mathrm{e}^{i(x-x')\cdot p/\varepsilon}\bar{W}(u,x,k+\sigma p/2)\bar{W}(u,x',k'+\sigma' p/2) \\
&\times S^*(u'-u)\theta_{A,L}(x,k)S^*(t-u)\theta(x',k')\mu(\mathrm{d}p)\mathrm{d}x\mathrm{d}x'\mathrm{d}k\mathrm{d}k' \\
&+\frac{1}{\varepsilon}\sum_\sigma \sigma\sigma' \int_s^t \mathrm{d}u \int \mathrm{e}^{i(x-x')\cdot p/\varepsilon}\bar{W}(u,x,k+\sigma p/2)\bar{W}(u,x',k'+\sigma' p/2) \\
&\times S^*(t-u)\theta(x,k)S^*(t-u)\theta(x',k')\mu(\mathrm{d}p)\mathrm{d}x\mathrm{d}x'\mathrm{d}k\mathrm{d}k'.
\end{aligned} \tag{5.20}$$

Denote the first and the second terms on the right hand side of (5.20) by $\mathcal{R}_1$ and $\mathcal{R}_2$ respectively. The first term $\mathcal{R}_1$ can be estimated exactly in the same way as $|R_\varepsilon(t,s) - R_\varepsilon(s,s)|$ and we obtain

$$|\mathcal{R}_1| \le C\|\theta_{A,L}\|_{1,1}^{(\mathcal{B})}\|\theta\|_{1,1}^{(\mathcal{B})}(t-s)$$

for a certain constant independent of $\varepsilon > 0$ and θ. On the other hand, the term $\mathcal{R}_2$ equals

$$\begin{aligned}
\mathcal{R}_2 = &\frac{1}{\varepsilon}\sum_{\sigma,\sigma'} \sigma\sigma' e^{-2\Sigma t}\int_s^t \mathrm{d}u \int \mu(\mathrm{d}p)\mathrm{d}q\mathrm{d}q'\mathrm{d}y\mathrm{d}y' \exp\left\{-i(\sigma p/2)\cdot\left(y+y'\right)\right\}\hat{\theta}\left(q,y\right)\hat{\theta}\left(q',y'\right) \\
&\times \exp\left\{\varepsilon\Theta\left(\frac{u}{\varepsilon}, p+\varepsilon q, -\frac{pu}{\varepsilon} - q(t-u) - y\right) + \Theta(t-u,q,y)\right\} \\
&\times \exp\left\{\varepsilon\Theta\left(\frac{u}{\varepsilon}, p+\varepsilon q', -\frac{pu}{\varepsilon} - q(t-u) - y'\right) + \Theta(t-u,q',y')\right\} \\
&\times \hat{f}(-\frac{pu}{\varepsilon} - q(t-u) - y)\hat{f}(\frac{pu}{\varepsilon} - q'(t-u) - y').
\end{aligned}$$

We can further decompose $\mathcal{R}_2$ as $\mathcal{R}_2 = \mathcal{R}_{21} + \mathcal{R}_{22}$, where the terms $\mathcal{R}_{21}$, $\mathcal{R}_{22}$ correspond to integration with respect to the p variable over the regions $[|p| \le \varepsilon^\gamma]$ and $[|p| > \varepsilon^\gamma]$ with some $\gamma \in (0,1)$:

$$|\mathcal{R}_{21}| \le C_T(t-s)\|\hat{f}\|_\infty^2(\|\theta\|_{1,1}^{(\mathcal{B})})^2\varepsilon^{\gamma-1}\int_{[|p|\le\varepsilon^\gamma]}\frac{\mu(\mathrm{d}p)}{|p|}.$$

On the other hand, for $\mathcal{R}_{22}$ we note that

$$\begin{aligned}
|\mathcal{R}_{22}| \le &\frac{C_T}{\varepsilon}\int_s^t \mathrm{d}u \int_{[|p|>\varepsilon^\gamma]} \mu(\mathrm{d}p)\int \mathrm{d}q\mathrm{d}q'\mathrm{d}y\mathrm{d}y'|\hat{\theta}\left(q,y\right)||\hat{\theta}\left(q',y'\right)| \\
&\times|\hat{f}(-\frac{pu}{\varepsilon} - q(t-u) - y)||\hat{f}(\frac{pu}{\varepsilon} - q'(t-u) - y')|.
\end{aligned}$$

Let $c > 0$ be a fixed constant. We can split the region of integration over q and y variables over the region A consisting of those (q,y,q',y'), for which at least one of these variable is greater than $c\varepsilon^{\gamma-1}$, and its complement A^c. Denote the respective terms by $\mathcal{R}'_{22}$ and $\mathcal{R}''_{22}$. Note that since $s \ge t_0 > 0$ in the latter case we can find an appropriate $c > 0$ such that

$$|\mathcal{R}''_{22}| \le \frac{C_T}{\varepsilon}(t-s)(\|\theta\|_{1,1}^{(\mathcal{B})})^2 \sup_{[|z|\ge c\varepsilon^{\gamma-1}]}|\hat{f}(z)|^2 \le C(t-s)$$

since $f \in \mathcal{S}(\mathbb{R}^d)$. Finally if one of the variables (q, y, q', y') is greater than $c\varepsilon^{\gamma-1}$ we can use the fact that $\theta \in \mathcal{S}(\mathbb{R}^{2d})$ to obtain that $|\mathcal{R}'_{22}| \le C(t-s)$ for some constant $C > 0$ independent of $\varepsilon > 0$. We conclude that for any $T_1 > t_0 > 0$ there exists a constant, independent of $\varepsilon > 0$, such that

$$|R_\varepsilon(t,t) - R_\varepsilon(t,s)| \le C(t-s) \tag{5.21}$$

for any $t > s$ belonging to $[t_0, T_1]$. Combining (5.19) with (5.21) and using Gaussianity of $\{\bar{Z}_\varepsilon(t),\, t \ge 0\}$ we deduce (5.17) and hence tightness of the laws of $\{\langle \bar{Z}_\varepsilon(t), \theta\rangle,\, t \ge 0\}$, $\varepsilon \in (0,1]$ in $C[t_0, +\infty)$ when $\theta \in \mathcal{S}(\mathbb{R}^{2d})$ and $t_0 > 0$ is fixed. To show tightness for an arbitrary $\theta \in H^{s,u}$ it suffices only to use density of $\mathcal{S}(\mathbb{R}^{2d})$ in $H^{s,u}$ and boundedness estimate (5.13). $\square$

Convergence of the initial data

Finally, we prove that for any $t_0 > 0$ the laws of the processes

$$\mathcal{G}_\varepsilon(t) := \varepsilon^{-1/2} \sum_n \int_0^t S_0(-s) C_\varepsilon[\bar{W}(s)] e_n \mathrm{d}B_n(s),$$

which appear in the right side of (5.11), converge weakly, over $C([t_0, +\infty); H^{-s,-u})$, to the law of a constant process $\mathcal{G}(t) \equiv X$. As we have pointed out the law of the latter is supported in this space, provided that $s, u > d$. The proof of tightness essentially follows the same argument as the one for tightness of $\bar{Z}_\varepsilon(t) = \varepsilon^{-1/2} \sum_n \int_0^t S(t-s) C_\varepsilon[\bar{W}(s)] e_n \mathrm{d}B_n(s)$. We focus therefore on the limit identification. Thanks to Gaussianity of $\{\mathcal{G}_\varepsilon(t),\, t \ge 0\}$ it suffices only to calculate the limit of covariance

$$\mathcal{C}_\varepsilon(t,s;\theta,\theta') := \mathbb{E}\left[\langle \mathcal{G}_\varepsilon(t), \theta\rangle \langle \mathcal{G}_\varepsilon(s), \theta'\rangle\right]$$

as $\varepsilon \to 0+$ for $t > s$ and $\theta, \theta' \in \mathcal{S}(\mathbb{R}^{2d})$. A simple calculation shows that

$$\begin{aligned}
\mathcal{C}_\varepsilon(t,s;\theta,\theta') &= \frac{1}{\varepsilon} \sum_{\sigma,\sigma'} \sigma\sigma' \int_0^s \int_{\mathbb{R}^{5d}} \mathrm{e}^{i(x-x')\cdot p/\varepsilon} \bar{W}(u, x, k + \sigma p/2) \bar{W}(u, x', k' + \sigma' p/2) \\
&\times S_0^*(-u)\theta(x,k) S_0^*(-u)\theta'(x',k') \mathrm{d}u \mu(\mathrm{d}p) \mathrm{d}x \mathrm{d}x' \mathrm{d}k \mathrm{d}k' \\
&= \frac{1}{\varepsilon} \sum_{n,n'} \sum_{\sigma,\sigma'=\pm 1} \sigma\sigma' \int_0^s \mathrm{d}u \int \tilde{\mathcal{W}}_n \tilde{\mathcal{W}}'_{n'} \mu(\mathrm{d}p),
\end{aligned}$$

where $S_0^*(-u)$ is the adjoint of $S_0(-u)$ defined in (5.12) and

$$\begin{aligned}
\tilde{\mathcal{W}}_n := &\frac{1}{n!} \int_{\square_n(u)} \mathrm{d}\rho \int_{\mathbb{R}^{(n+2)d}} \mathrm{d}x \mathrm{d}k \prod_{j=1}^n \mu(\mathrm{d}l_j) \\
&\mathrm{e}^{ix\cdot p/\varepsilon} \theta\left(x - ku, k\right) W_0 \left(x - \left(k + \frac{\sigma p}{2}\right) u - \sum_{j=1}^n l_j \rho_j, k + \frac{\sigma p}{2} + \sum_{j=1}^n l_j \right).
\end{aligned}$$

The formula for $\tilde{\mathcal{W}}'_{n'}$ is similar, except θ, n, k, x are replaced by θ', n', k', x' and $\mathrm{e}^{ix\cdot p/\varepsilon}$ by $\mathrm{e}^{-ix'\cdot p/\varepsilon}$. Using the same approach as in the proof of Lemma 1 we obtain

$$\tilde{\mathcal{W}}_n = \frac{\varepsilon^n}{n!} \int_{\mathbb{R}^{2d}} \mathcal{F}_1(\theta)\,(q,k)\, \hat{f}(z) \Theta^n\left(u/\varepsilon, p + \varepsilon q, z\right) \exp\left\{ i(k + \sigma p/2) \cdot [(u/\varepsilon)(p + \varepsilon q) + z] \right\} \mathrm{d}q \mathrm{d}z.$$

Hence, changing variable $u_{new} := u/\varepsilon$, we obtain that the covariance is

$$\mathcal{C}_\varepsilon(t,s;\theta,\theta') = \sum_{\sigma,\sigma'=\pm 1} \sigma\sigma' \int_0^{s/\varepsilon} \mathrm{d}u \int_{\mathbb{R}^{5d}} \mathrm{d}q\mathrm{d}z\mathrm{d}q'\mathrm{d}z'\mu(\mathrm{d}p) \exp\left\{ip\left(\sigma z + \sigma' z'\right)/2\right\}$$
$$\times \exp\left\{\varepsilon\left[\Theta\left(u, p+\varepsilon q, -z - u(p+\varepsilon q)\right) + \Theta\left(u, -p+\varepsilon q', -z' - (-p+\varepsilon q')u\right)\right]\right\}$$
$$\times \hat{\theta}\left(q,-z\right)\hat{f}(-z-(p+\varepsilon q)u)\hat{\theta}'\left(q',-z'\right)\hat{f}(-z'-(-p+\varepsilon q')u).$$

Passing to the limit $\varepsilon \to 0+$, which can easily be justified via Lebesgue dominated convergence theorem, we obtain

$$\lim_{\varepsilon\to 0+} \mathcal{C}_\varepsilon(t,s;\theta,\theta') = \sum_{\sigma,\sigma'=\pm 1} \sigma\sigma' \int_0^{+\infty} \mathrm{d}u \int_{\mathbb{R}^{3d}} \mathrm{d}z\mathrm{d}z'\mu(\mathrm{d}p) \exp\left\{ip\left(\sigma z + \sigma' z'\right)/2\right\}$$
$$\times \mathcal{F}_2\theta\left(0,z\right)\hat{f}(z-pu)\mathcal{F}_2\theta'\left(0,z'\right)\hat{f}(z'+pu) = \mathbb{E}\left[\langle X,\theta\rangle\langle X,\theta'\rangle\right],$$

cf. formula (4.6).

Suppose that $\{(\bar{Z}(t),\mathcal{G}(t)),\, t\ge 0\}$ is a limiting point of $\{(\bar{Z}_\varepsilon(t),\mathcal{G}_\varepsilon(t))\, t\ge 0\}$, $\varepsilon\in(0,1]$. Then, for any $\theta\in\mathcal{S}(\mathbb{R}^{2d})$ we have

$$\langle \bar{Z}(t),\theta\rangle = \int_0^t \langle S_0(t-s)M\bar{Z}(s),\theta\rangle \mathrm{d}s + \langle \mathcal{G}(t),\theta\rangle,$$

which is equivalent to (4.5). Thus, Theorem 2 follows.

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