Discrete Mathematics 110 (1992) 43–59
North-Holland

Universal cycles for combinatorial structures

Fan Chung
Bell Communications Research, Morristown, NJ 07960, USA

Persi Diaconis
Harvard University, Cambridge, MA 02139, USA

Ron Graham
AT&T Bell Laboratories, Murray Hill, NJ 07974, USA

Received 2 July 1990
Revised 12 April 1991

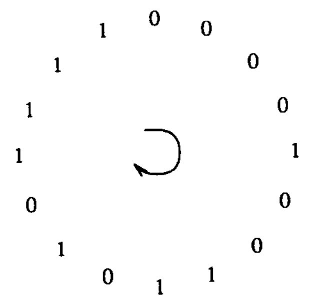
Abstract

Chung, F., P. Diaconis and R. Graham, Universal cycles for combinatorial structures, Discrete Mathematics 110 (1992) 43–59

In this paper, we explore generalizations of de Bruijn cycles for a variety of families of combinatorial structures, including permutations, partitions and subsets of a finite set.

1. Introduction

The cyclic sequence C of 16 0's and 1's shown in Fig. 0 has the following unlikely property. If we list each of the 16 possible blocks of 4 consecutive symbols of C, it turns out that they are all different. As a consequence, it follows that *every possible* 0–1 sequence of length 4 occurs this way (uniquely). The cycle C is an example of what has come to be known as a *de Bruijn cycle.* More generally, a (binary) de Bruijn cycle C_n of order n is defined to be a cyclic sequence $(x_0, x_1, \ldots, x_{2^n-1})$ where $x_i = 0$ or 1, and each possible binary sequence of length n occurs uniquely as $(x_{i+1}, \ldots, x_{i+n})$ for some i, where index addition is performed modulo 2^n. The study of such cycles has had a long and distinguished history, and has arisen in a variety of contexts, such as design of Sanskrit memory wheels, digital fault testing, pseudo-random number generation, modern public-key cryptographic schemes, and even for use by illusionists in various mind-reading effects, to mention a few. (For an overview of this history, and indeed, the whole topic of de Bruijn cycles, the reader can consult [1, 5, 21, 14].



C

Fig. 0. A de Bruijn cycle of order 4.

Among the fundamental questions one might ask concerning de Bruijn cycles are:

(i) Do de Bruijn cycles *always exist* for each n?

(ii) If so, *how many* are there?

(iii) How does one *construct* them?

(iv) In a given de Bruijn cycle C, is there an easy way of determining the ith block as a function of i?

(v) How can one 'invert' this process in C. That is, for each given block, *where* is it in C?

(vi) How can one 'cut down' a de Bruijn cycle C. That is, when is it possible to remove elements from C so that the resulting contracted cycle C' still has *distinct* blocks of length n (although some now will be missing). In the same spirit, how can one 'build up' or 'combine' de Bruijn cycles?

(vii) What are the analogues for larger alphabets (k symbols rather than 2), or more dimensions (e.g., a de Bruijn 'torus' rather than a cycle), etc.

We will summarize some of the known answers to some of these questions in Section 3.

The thrust of this paper will be to consider the analogous situation for a variety of other combinatorial structures, rather than binary n-tuples. In particular, we will outline what is known for *permutations* of an n-set (Section 4), *partitions* of an n-set (Section 5), and k-sets of an n-set (Section 6). In Section 2, we formulate our problem in a general setting, and in Section 3, we interpret de Bruijn cycles in this formulation. Finally, in Section 7, we describe possible future directions.

2. A general formulation

We begin by being given some family $\mathscr{F}_n$ of combinatorial objects of 'rank n'. We denote their number by $m := |\mathscr{F}_n|$. We assume that each $F \in \mathscr{F}_n$ is 'generated' or specified by some sequence $\langle x_1, \ldots, x_n \rangle$, where $x_i \in A$, for some fixed alphabet A.

We will say that $U = (a_0, a_1, \ldots, a_{m-1})$ is a *universal cycle* for $\mathcal{F}_n$ (or U-cycle, for short) if $\langle a_{i+1}, \ldots, a_{i+n} \rangle$, $0 \leq i < m$, runs through each element of $\mathcal{F}_n$ exactly once, where index addition is performed modulo n.

Now we can ask the standard questions: do U-cycles for $\mathcal{F}_n$ exist, if so how many, how do you construct them, invert them, combine them, extend them, etc. Of course, it is clear that some U-cycles might be better than others for some of these purposes. When this is so, how do we find 'good' ones.

In addition to their inherent combinatorial interest, one might also ask how one might use these U-cycles.

3. de Bruijn cycles

We next sketch the standard approach used for treating de Bruijn cycles. In this case,

$$\mathcal{F}_n = B_n = \{0, 1\}^n = \{(x_1, \ldots, x_n) \mid x_i \in \{0, 1\}, 1 \leq i \leq n\}, \qquad m = 2^n$$

and each binary n-tuple $(x_1, \ldots, x_n)$ is just represented by itself, i.e.,

$$\langle x_1, \ldots, x_n \rangle \leftrightarrow (x_1, \ldots, x_n).$$

(This will not be the case in most of the later situations.)

The first step in constructing potential U-cycles for B_n is to construct the (directed) *transition graph* G_n for B_n. The *vertices* of G_n are all the n-tuples $\{0, 1\}^n$. There is a *directed edge* (= arc) from $(x_1, \ldots, x_n)$ to $(y_1, \ldots, y_n)$ provided $x_2 = y_1$, $x_3 = y_2, \ldots, x_n = y_{n-1}$. Thus, arcs look like $((x_1, \ldots, x_n), (x_2, \ldots, x_n, x_{n+1}))$. What this indicates is that it is possible to go from $(x_1, \ldots, x_n)$ to $(x_2, \ldots, x_{n+1})$ in a potential U-cycle, namely, when the block $\ldots x_1, x_2, \ldots, x_n, x_{n+1} \ldots$ occurs.

We illustrate the graphs G_2 and G_3 in Fig. 1.

From this point of view, a U-cycle for B_n corresponds exactly to a directed circuit in G_n going through each vertex exactly once, i.e., a *Hamiltonian circuit* for G_n. This is both good news and bad news. The good news is that our problem has been reduced to finding a very familiar object in graph theory, namely, Hamiltonian circuits. The bad news is that these objects are well known to be difficult to find! In fact, it is an NP-complete problem to decide if a graph in general even has a Hamiltonian circuit.

Fortunately, we have a way around this problem in this case. What we can do is to define another digraph G_n^*, called the *arc digraph* of G_n, as follows. The *vertices* of G_n^* will just be the arcs of G_n. In particular the arc $((x_1, \ldots, x_{n-1}, x_n), (x_2, \ldots, x_n, x_{n+1}))$ will correspond to the vertex labelled with the $(n-1)$-tuple $(x_2, \ldots, x_n)$ in G_n^*. The *arcs* of G_n^* will be all pairs of vertices $((y_1, \ldots, y_{n-1})\,(y_2, \ldots, y_n))$ in G_n^*, i.e., so that the 'head' of the first vertex label is equal to the 'tail' of the second vertex label. In Fig. 2, we show G_2^*

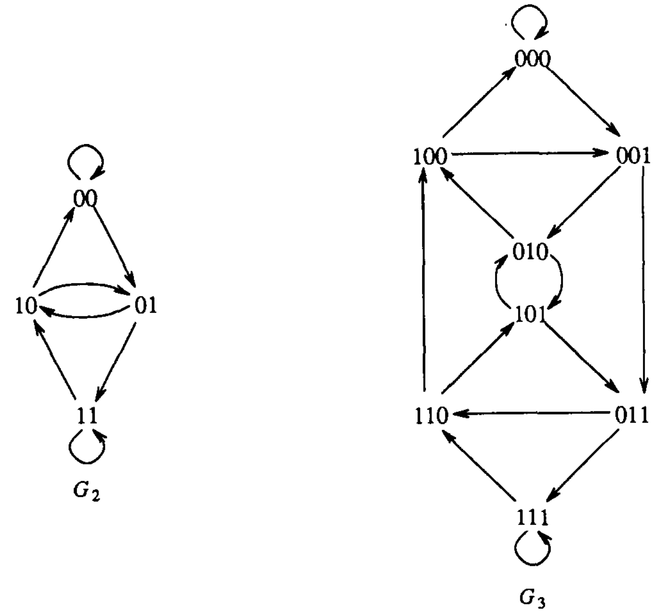


Fig. 1. The graphs G_2 and G_3.

and G_3^*. It is clear now that a Hamiltonian circuit in G_n corresponds exactly to an 'Eulerian' circuit in G_n^*, i.e., a (directed) circuit passing through each arc exactly once. The advantage of this transformation is that Eulerian circuits in digraphs are easy to detect. To state this precisely, let us call a digraph G *balanced* if for every vertex v of G, indegree(v) = outdegree(v). Also, call G *strongly connected* if for any vertices u and v of G, there is a directed path in G from u to v.

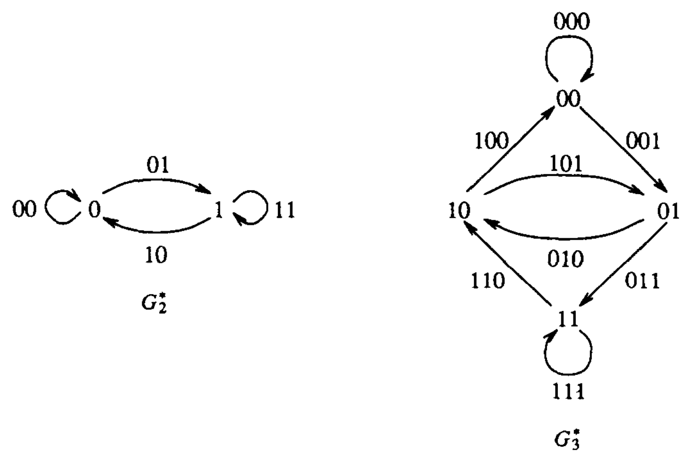


Fig. 2. The arc digraphs G_2^* and G_3^*.

Fact. *G has an Eulerian circuit if and only if G is balanced and strongly connected.*

It is not difficult to see that G_n^* is balanced and strongly connected, and so is Eulerian. This in turn shows that G_n is Hamiltonian, i.e., has a U-cycle. Notice that G_n^* is isomorphic to G_{n-1}. A more careful analysis shows that in fact G_n^* has exactly 2^{2^n-n} different Eulerian cycles. For a good discussion of this topic as well as various generalizations such as k-symbol alphabets, the reader is referred to [14, 15, 18].

In the next three sections we will attempt to apply the same analysis (with decreasing success) to permutations, partitions and k-set of an n-set, respectively.

4. Permutations

Let us denote by S_n the set of all $n!$ permutations (or arrangments) of $\{1, 2, \ldots, n\}$. If $\bar{a} = (a_1, a_2, \ldots, a_n)$ and $\bar{b} = (b_1, b_2, \ldots, b_n)$ each are n-tuples of distinct integers we will say that $\bar{a}$ and $\bar{b}$ are *order-isomorphic*, written $\bar{a} \sim \bar{b}$, if

$$a_i < a_j \Leftrightarrow b_i < b_j.$$

A U-cycle $U_n = (a_0, a_1, \ldots, a_{n!-1})$, $a_i \in \{1, 2, \ldots, N\}$, for S_n will be $n!$-tuple such that each $\sigma \in S_n$ is *order-isomorphic* to exactly one block $(a_{i+1}, \ldots, a_{i+n})$, where, of course, index addition is performed modulo $n!$ It is clear why we must in general take $N > n$ since blocks of length n must always consist of n distinct symbols. An example of U-cycle for S_3 is

$$1\ 4\ 5\ 2\ 4\ 3.$$

To begin the process of constructing U-cycles of S_n we imitate the analysis used for de Bruijn cycles and construct the transition graph G_n for S_n. We illustrate this for $N = 3$ in Fig. 3.

The arcs of G_n are defined as follows. Suppose (for $n = 3$) we have the sequence $\cdots 4\,5\,2\,x \cdots$ where we are suppressing commas. Now $452 \sim 231$. The next 3-block $52x$ could have three possibilities. If $x = 1$ then $521 \sim 321$ so that we get the arc $231 \to 321$. If $x = 3$ then $523 \sim 312$ and we have the arc $231 \to 312$.

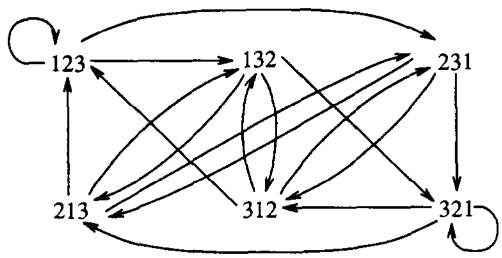


Fig. 3. G_3.

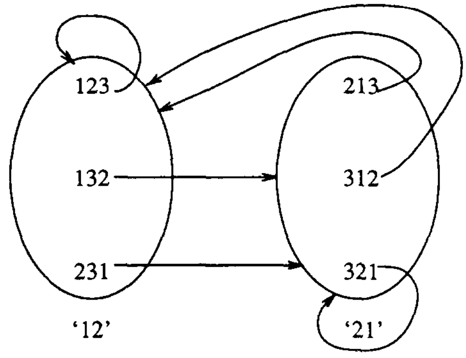


Fig. 4. $\bar{G}_3$.

Finally, if $x = 6$ then we have $526 \sim 213$ and $231 \to 213$. So, even after we find a Hamiltonian cycle in G_n, we still have to assign values a_i to realize (order-isomorphically) the appropriate elements of S_n. We will have more to say about this latter. The structure of G_3 can be simplified if we regroup the vertices as in Fig. 4.

We have grouped permutations according to the order type of the first two elements, which are '12' and '21'. An arc in $\bar{G}_3$ from 213, for example to the *group* '12' denotes that there are really *three* arcs, one from 213 to each of the elements 123, 132 and 231 in the group '12'. Since each permutation now has exactly one arc leaving it, it suffices to find an Eulerian circuit in $\bar{G}_3$ in order to produce a Hamiltonian circuit in G_3. We show such an Eulerian circuit for $\bar{G}_3$ in Fig. 5. The corresponding Hamiltonian circuit in G_3 is

$$132 \to 312 \to 123 \to 231 \to 321 \to 213$$

The key question is now this. How does such a cycle correspond to a U-cycle for S_3?

Suppose we assign (as of yet) undetermined values for the potential U-cycle as follows:

U: $a\ b\ c\ d\ e\ f$.

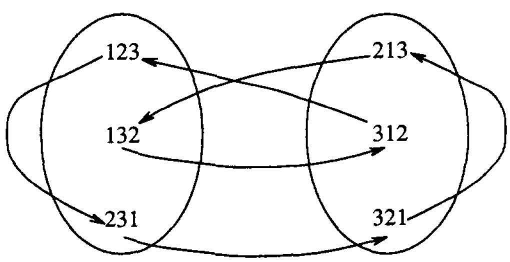


Fig. 5. An Eulerian circuit in $\bar{G}_3$.

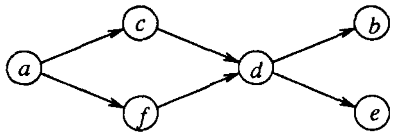


Fig. 6. P_3.

We want the first 3-block abc to be order-isomorphic to the first permutation 132 in our Hamiltonian circuit, i.e., $abc \sim 132$ which just means $a < c < b$. Similarly, we want $bcd \sim 312$ which implies $c < d < b$, $cde \sim 123$ which implies $c < d < e$, etc.

We can represent the implied inequalities among $a, b, \ldots, f$ by means of a *partial order* (which itself is just an acyclic digraph), where $i \rightarrow j$ will denote the requirement that $i < j$. We show this partial order P_3 in Fig. 6.

What we now require is a mapping of $\{a, b, \ldots, f\}$ into $\{1, 2, \ldots, N\}$ which *preserves order*, i.e., a *linear extension* λ of P_3 into $\{1, 2, \ldots, N\}$ for a suitable

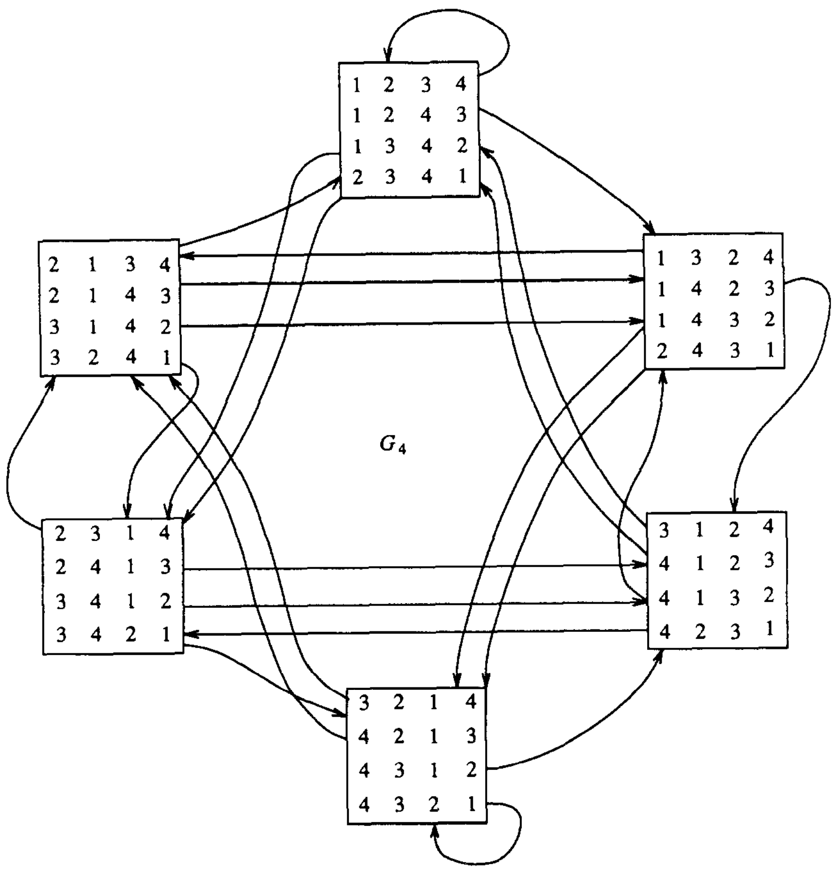


Fig. 7. The clustered transition graph $\bar{G}_4$ for S_4.

N. In particular, it is natural to make N as small as possible (so that in particular the mapping should be onto). In this case, we can choose $N = 4$ and take $\lambda(a) = 1$, $\lambda(c) = \lambda(f) = 2$, $\lambda(d) = 3$, $\lambda(b) = \lambda(e) = 4$, which results in the U-cycle 1 4 2 3 4 2 for S_3.

In Fig. 7 we show the 'clustered' transition graph $\bar{G}_4$ for S_4. A particularly nice Eulerian circuit for $\bar{G}_4$ is given in Fig. 8.

If we assume that $U_4 = abc \cdots x$ is a U-cycle which realizes this ordering of S_4 then we can construct as we did for S_3 the implied partial order P_4 (shown in Fig. 8). This we show in Fig. 9.

The main point is that P_4 has height (= length of longest chain) 5. Thus, we can define the linear extension $\lambda: \{a, \ldots, x\} \to \{1, 2, 3, 4, 5\}$ by $\lambda(z) :=$ length of longest chain ending in z, to produce the U-cycle

$$1\ 2\ 3\ 4\ 1\ 2\ 5\ 3\ 4\ 1\ 5\ 3\ 2\ 1\ 4\ 5\ 3\ 2\ 4\ 1\ 3\ 2\ 5\ 4.$$

In general, we can cluster vertices of the transition graph G_n to form $\bar{G}_n$ (by grouping together those n permutations for which the initial $(n-1)$ blocks are order-isomorphic), which is easily checked to be balanced and strongly connected, and hence Eulerian. It is shown in Hurlbert [10] that by appropriately

a b c d
1 2 3 4 *e*
2 3 4 1 *f*
2 3 1 4 *g*
3 1 4 2 *h*
1 4 2 3 *i*
4 2 3 1 *j*
3 4 2 1 *k*
3 2 1 4 *l*
2 1 4 3 *m*
1 4 3 2 *n*
4 3 2 1 *o*
4 2 1 3 *p*
2 1 3 4 *q*
1 2 4 3 *r*
2 4 3 1 *s*
4 3 1 2 *t*
3 1 2 4 *u*
1 3 4 2 *v*
2 4 1 3 *w*
4 1 3 2 *x*
1 3 2 4 *a*
3 2 4 1 *b*
3 4 1 2 *c*
4 1 2 3

Fig. 8. An Eulerian circuit for $\bar{G}_4$.

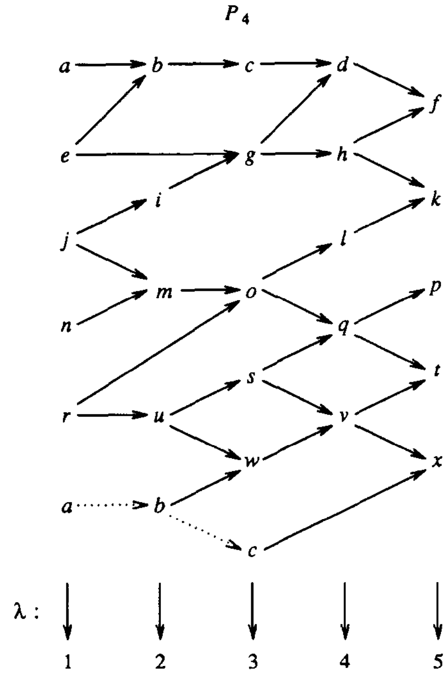


Fig. 9. A linear extension.

restricting C, the implied ordering on the values in the 'lifted' U-cycle is in fact a partial order $P_n = P_n(C)$, i.e., has no cycles. (In fact, we believe this to be the case for *any* Eulerian circuit C.) If $h(P_n)$ denotes the height of P_n then there is a linear extension of P_n into $\{1, 2, \ldots, h(P_n)\}$, and consequently there is a U-cycle for S_n from symbols in $\{1, 2, \ldots, h(P_n)\}$.

Suppose we define $N(n) := \min_C h(P_n(C))$ where C ranges over all Eulerian circuits in $\bar{G}_n$. Then any U-cycle for S_n must use at least $N(n)$ different symbols. The best bounds we currently have for $N(n)$ are

$$N(2) = 2, \quad N(3) = 4, \quad N(4) = 5 \quad \text{and} \quad n + 1 \leqslant N(n) \leqslant 6n \quad \text{for} \quad n \geqslant 5.$$

However, we believe the following.

Conjecture. $N(n) = n + 1$, $n \geqslant 3$.

We close this section with several questions. How many U-cycles for S_n are there with exactly $N(n)$ different vertices? What about with at most $N(n) + c$ entries for a fixed constant c? Exponentially many? Can we find U-cycles which are easy to invert? Suppose we just want a specified subset $X \subseteq S_n$ to be represented by U_n. For which X is this possible?

$\left\{ {4 \atop 1} \right\} = 1$	$\left\{ {4 \atop 2} \right\} = 7$	$\left\{ {4 \atop 3} \right\} = 6$	$\left\{ {4 \atop 4} \right\} = 1$
1234	1 | 234	1 | 2 | 34	1 | 2 | 3 | 4
	2 | 134	1 | 3 | 24	
	3 | 124	1 | 4 | 23	
	4 | 123	2 | 3 | 14	
	12 | 34	2 | 4 | 13	
	13 | 24	3 | 4 | 12	
	14 | 23		

Fig. 10. Partitions of $\{1, 2, 3, 4\}$, U_4: $a\,b\,c\,b\,c\,c\,c\,c\,d\,d\,c\,d\,e\,e\,c$.

5. Partitions

The next class of objects we consider is the set of P_n of *partitions* of the n-element set $\{1, 2, \ldots, n\}$. The number of such partitions is just $\sum_{k=1}^{n} \{{n \atop k}\}$, where $\{{n \atop k}\}$ denotes the Stirling number of the second kind, and satisfies the recurrence

$$\left\{ {n \atop k} \right\} = k \left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\}$$

(e.g., see [8]).

How will we represent partitions? We will do the following. We illustrate the idea for $n = 8$. A U-cycle for P_n will be a sequence composed of symbols from the set $A = \{a, b, c, \ldots\}$. A block, for example, $a\,b\,a\,c\,b\,c\,c\,d$, will represent a partition, in this case $13 \mid 25 \mid 467 \mid 8$, by putting i and j in the same group of the partition if and only if the ith and jth symbols of the block are the same. In Fig. 10, we list the 15 partitions of $\{1, 2, 3, 4\}$ and a U-cycle U_4 for P_4.

We can proceed in the canonical way in searching for U-cycles by first considering the corresponding transition graph G_n. In Fig. 11(a) we show G_3. In Fig. 11(b) we redraw G_3 by clustering certain partitions together as shown, to form $\bar{G}_3$.

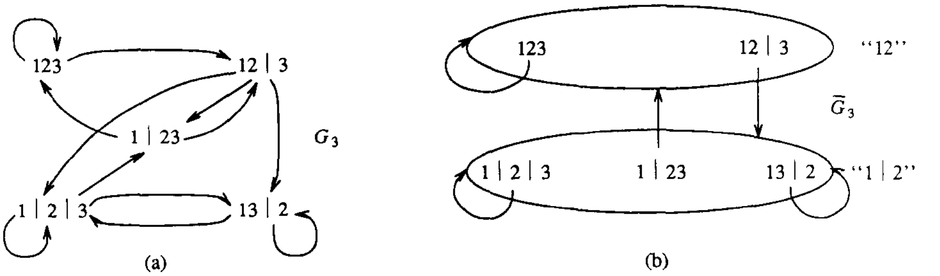


Fig. 11. The graphs G_3 and $\bar{G}_3$.

$$
\begin{array}{ll|ccccccc|l}
? & U_4: & x_1 & x_2 & x_3 & x_4 & x_5 & & & \\
\hline
 & & 1 & 2 & 3 & & & & & \Rightarrow\ x_1 = x_2 = x_3 \\
 & & & 1 & 2\mid & 3 & & & & \Rightarrow\ x_4 \neq x_3 \\
 & & & & 1 & 3\mid & 2 & & & \Rightarrow\ x_5 = x_3 \\
 & & & & & 1\mid & 2\mid & 3 & & \Rightarrow\ x_1 \neq x_5 \\
 & & & & & & 1\mid & 2 & 3 &
\end{array}
$$

Fig. 12.

We use the same convention as in the preceding section, namely, an arc from a partition π to a cluster means that arcs go from π to *all* partitions of the custer. This reduced graph $\bar{G}_3$ is Eulerian, with the only Eulerian circuit being

$$
\begin{tikzcd}
123 \arrow[r] & 12\mid 3 \arrow[r] & 13\mid 2 \arrow[r] & 1\mid 2\mid 3 \arrow[r] & 1\mid 23 \arrow[llll, bend right]
\end{tikzcd}
$$

The final step is to 'lift' this circuit to an actual U-cycle by assigning appropriate symbols in order to realize the corresponding partitions. We show the set-up in Fig. 12.

However, we now get a contradiction since we can deduce $x_5 \neq x_1 = x_3 = x_5$. Thus, we have an example of a Hamiltonian circuit in G_n which cannot be 'lifted' to a U-cycle. In fact, there are no U-cycles for P_3.

Undaunted, we move on to P_4. In Fig. 13, we show $\bar{G}_4$.

As before, if we imagine contracting clusters to points, this graph is Eulerian. The reader may wish to test his or her understanding up to this point by finding an Eulerian circuit in $\bar{G}_4$ and extending it to a U-cycle for P_4 (there is more than one way to do this).

For the general case of P_n, this procedure works quite well. It is not difficult to see that the clustered graph $\bar{G}_n$ is always Eulerian (for $n \geq 3$). The only problem we have to worry about is that some Eulerian circuits might not be able to be converted to U-cycles. This can only happen if the implied (in)equalities in the symbols of the U-cycle end up with forcing $x \neq x$ for some symbol x (as happened for $n = 3$). To prevent this, it is enough to require that a specific sequence W of partitions occur in the Eulerian circuit C. The purpose of W is to prevent a sequence of equalities (or unequalities) from going across the corresponding portion of the U-cycle. For example, take $n = 4$ and let W be

$$1\mid 23\mid 4, \quad 12\mid 34, \quad 1\mid 234, \quad 1234, \quad 123\mid 4.$$

When this portion of C is 'lifted' we get the situation shown in Fig. 14. Thus, we must have

$$a_{i+1} \neq a_{i+2} = a_{i+3} \neq a_{i+5} = a_{i+6} = a_{i+7} \neq a_{i+8}.$$

We can think of W as a 'breaker' since if $r \leq i$ and $s \geq i + 8$ then neither $a_r = a_s$ nor $a_r \neq a_s$ can be forced. In particular, if C has a 'breaker' which does not

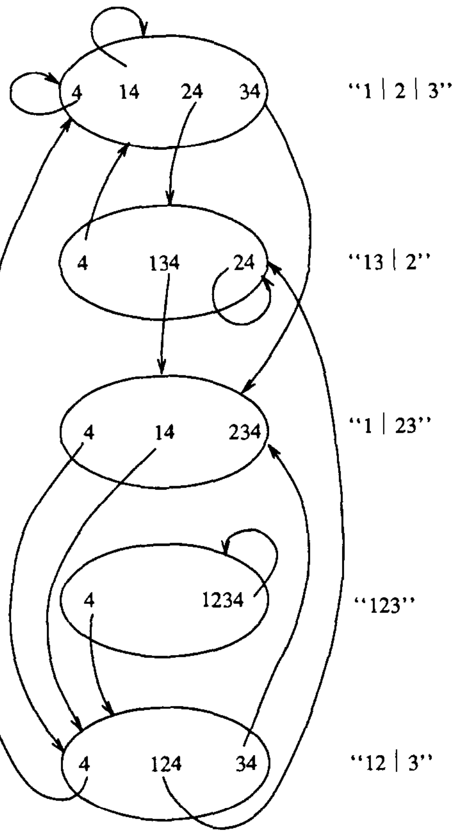


Fig. 13. The reduced graph $\bar{G}_4$.

include $1\,|\,2\,|\,3\,|\,4$ then C can always be lifted to a U-cycle. It is not difficult to show that for $n \geqslant 4$ this can always be done.

It is amusing to note that there are exactly 52 partitions of $\{1, 2, 3, 4, 5\}$. In fact, a U-cycle for P_5 can be constructed with the alphabet $A = \{D, C, H, S, J\}$ so that the symbol J occurs just once, and each of the other symbols occur at most 13 times. For example, one such cycle is

DDDDDCHHHCCDDCCCHCHCSHHSDSSDSSHSDDCHSSCHSHDHSCHSJCDC.

U_4 : ...	a_{i+1}	a_{i+2}	a_{i+3}	a_{i+4}	a_{i+5}	a_{i+6}	a_{i+7}	a_{i+8} ...
	1 $\mid$	2	3 $\mid$	4				
		1	2 $\mid$	3	4			
			1 $\mid$	2	3	4		
				1	2	3	4	
					1	2	3 $\mid$	4

Fig. 14.

In particular, this cycle can be realized with an ordinary deck of playing cards with one spade ($=S$) replaced by a joker ($=J$). It is not hard to see that for P_n, we must have an alphabet $|A| \geq n$. For $N \geq n$, how many U-cycles for P_n are there with $|A| = N$? How do you invert *any* of these U-cycles?

6. k-Sets of an n-set

The final class of objects we consider is the family $[{n \atop k}]$ of all k-element subsets ($= k$-sets) of an n-element set $\{0, 1, \ldots, n-1\}$. As an example of a U-cycle for this situation, we have for $n = 8$, $k = 3$, the following cycle U:

$$02456145712361246703671345034601250135672560234723570147.$$

A distinguishing feature of this situation is that each 3-set might occur in any of 6 possible orders in U, but it is only allowed to occur once. That is, since the first 3-block 024 represents the 3-set $\{0, 2, 4\}$ then none of the five other 3-blocks 042, 204, 240, 402 and 420 can occur in U. One consequence of this fact is that we cannot even define a transition graph G for $[{n \atop k}]$! For if $\{1, 2, 3\}$ is represented by the block 123, for example, then the arc $\{1, 2, 3\} \to \{2, 3, 4\}$ is possible in G (by having the block continue $1234 \cdots$). However, if $\{1, 2, 3\}$ is represented by 213 then $\{1, 2, 3\} \to \{2, 3, 4\}$ cannot be an arc in G. Since we do not know which way $\{1, 2, 3\}$ will be represented then we cannot give a meaningful definition of G.

There is a simple modular condition which is *necessary* for the existence of U-cycles for $[{n \atop k}]$.

Fact. *If* $[{n \atop k}]$ *has a U-cycle then k divides* $\binom{n-1}{k-1}$.

Proof. Consider a fixed symbol $a_i = x$ in a U-cycle C. Since all symbols a_{i+j}, $-k < j < k$, must be distinct from x, then each copy of x occurs in exactly k k-blocks of C. Since these k-blocks represent k-sets of $\{0, \ldots, n-1\}$ which contain x, and there are exactly $\binom{n-1}{k-1}$ if these, the conclusion follows. □

It is easy to see that U-cycles exist for $[{n \atop 2}]$ whenever this necessary condition is satisfied, i.e., n is odd.

It has been shown by Jackson [12] that this necessary condition is in fact sufficient for $k = 3$ if n is large enough.

Theorem [12]. *U-cycles exist for* $[{n \atop 3}]$, $n \geq 8$, *provided* $\binom{n-1}{2} \equiv 0 \pmod 3$.

Idea of proof. We illustrate the idea for $n = 8$. We first tabulate all possible different ways of selecting 3 elements from an 8-cycle where we identify two choices if they only differ by a rotation. We describe these by their sequences of differences between consecutive elements (modulo 8) (see Fig. 15). We next

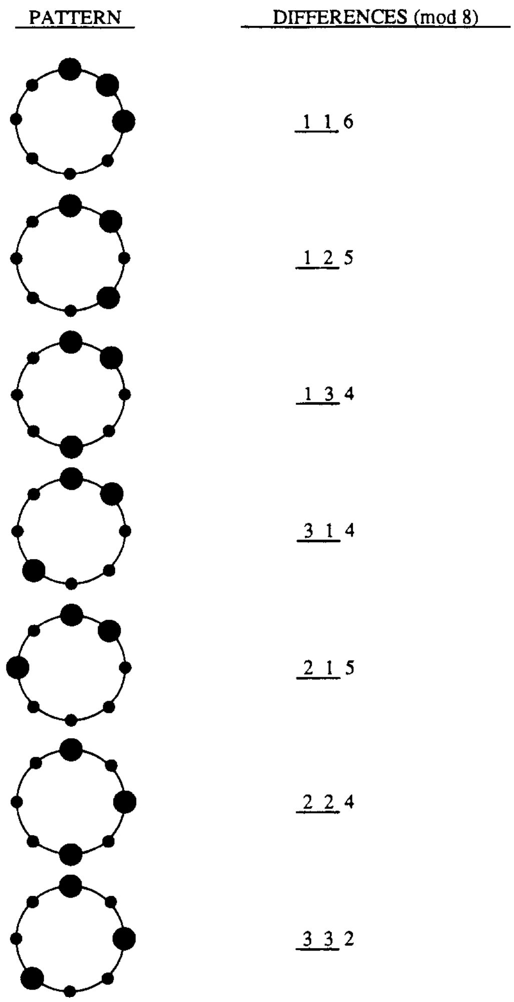


Fig. 15. Possible cyclic patterns for 3-sets of an 8-set.

select for each (ordered) pattern *two* of the three differences (underlined in Fig. 15).

Now we construct a digraph G with vertices labeled by 1, 2 and 3, and arcs from i to j if ij is an (ordered) pair of differences selected in the previous stage. We show G in Fig. 16.

For the next step we look for an Eulerian circuit C in G. In this case we take

$$2\ 2\ 1\ 1\ 3\ 3\ 1.$$

Finally we check that the sum Σ of the elements of C is relatively prime to $n = 8$. Since $\Sigma = 5$ in this case, then this stage passes. If we have managed to succeed up to this point then we can now construct our U-cycle U as follows. We take the

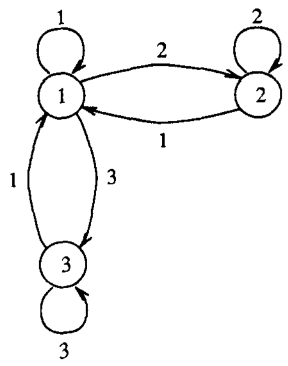


Fig. 16. The graph for 3-sets of an 8-set.

'template' of differences 2 2 1 1 3 3 1 formed by C repeated 8 times, and construct the sequence of length $7 \cdot 8 = 56$ having these differences (mod 8) between consecutive elements. (It does not matter what the first element of U is). Thus, U (starting with 0) is

$$\begin{array}{lcccccccccccccccc} \Delta: & 2 & 2 & 1 & 1 & 3 & 3 & 1 & 2 & 2 & 1 & 1 & 3 & 3 & 1 & 2 & \cdots \\ U: & 0 & 2 & 4 & 5 & 6 & 1 & 4 & 5 & 7 & 1 & 2 & 3 & 6 & 1 & 2 & 4 \quad \cdots \end{array}.$$

What Jackson shows is that it is always possible to construct a U-cycle for $[{n \atop 3}]$ this way, provided $3 \mid \binom{n-1}{2}$, i.e., $n \not\equiv 0 \pmod 3$, and $n \geqslant 8$.

These techniques can be extended to show the following.

Theorem. *U-cycles exist for $[{n \atop 4}]$ provided $\binom{n-1}{3} \equiv 0 \pmod 4$, $(n, 4) = 1$ and n is sufficiently large.*

It has very recently been shown by Hurlbert [10] that the necessary condition $\binom{n-1}{5} \equiv 0 \pmod 6$ is also sufficient for the existence of U-cycles for $[{n \atop 6}]$. However, for $k = 5$ or $k \geqslant 7$ we are still completely baffled.

We are willing to make the following conjecture though.

Conjecture (\$100). U-cycles exist for $[{n \atop k}]$ always exist provided k divides $\binom{n-1}{k-1}$ and $n \geqslant n_0(k)$.

7. Future directions

There are of course many other combinatorial structures for which these and similar questions can be raised. Thus include, for example, permutations with ties, ordered k-sets of an n-set, k-sets of an n-element multi-set, k-dimensional subspaces of an n-dimensional vector space over GF(q), combinatorial k-spaces

$$\begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{array}$$

Fig. 17. A de Bruijn torus for 2×2 arrays.

of an n-space (a la Hales–Jewett; see [9]), etc. One could also ask for higher-dimensional analogues of these questions. For example, is it always possible to construct a *universal torus* T for every $2k$-by-$2k$ binary array? In other words, we are asking for a (square) 2^{2k^2}-by-2^{2k^2} binary array T, with horizonal and vertical sides, respectively, identified, so that all $2k$-by-$2k$ binary arrays occur in T exactly once. The simplest example of such a T is shown in Fig. 17. In fact, such T always exist (see [6]) although their number for each size is not known.

Non-square toruses have been investigated in [2–4, 7, 11, 16, 18–19]. and in particular in [20], where they arise in connection with robot self-location problems.

Clearly we have barely scratched the surface of this subject, with the vast bulk of the interesting results remaining yet to be discovered. An excellent start in some of these directions can be found in [10].

References

[1] N.G. de Bruijn, A combinatorial problem, Proc. Nederl. Akad. Wetensch. 49 (1946) 758–764.
[2] C.R.J. Clapham, Universal tilings and universal (0, 1)-matrices, Discrete Math. 58 (1986) 87–92.
[3] J.C. Cook, Toroidal tilings from de Bruijn—Good cyclic sequences, Discrete Math. 70 (1988) 209–210.
[4] K. Dehnhardt and H. Harborth, Universal tilings of the plane by 0–1 matrices, Discrete Math. 73 (1988/89) 65–70.
[5] H. Fredericksen, A survey of full length nonlinear shift register cycle algorithms, SIAM Rev. 24 (1982) 195–221.
[6] C.T. Fan, S.M. Fan, S.L. Ma and M.K. Sin, On de Bruijn arrays, Ars Combin. 19A (1985) 205–213.
[7] B. Gordon, On the existence of perfect maps, IEEE Trans. Inform. Theory 12 (1966) 486–487.
[8] R.L. Graham, D.E. Knuth and O. Patashnik, Concrete Mathematics (Addison-Wesley, Reading, MA, 1989).
[9] R.L. Graham, B.L. Rothschild and J.H. Spencer, Ramsey Theory (Wiley, New York, 1980).
[10] G. Hurlbert, Ph.D. Thesis, Rutgers Univ., 1990.
[11] A. Iványi, Construction of infinite de Bruijn arrays, Discrete Appl. Math. 22 (1988/89) 289–293.
[12] Brad Jackson, personal communication.
[13] D.E. Knuth, Oriented subtrees of an arc digraph, J. Combin. Theory 3 (1967) 309–314.
[14] D.E. Knuth, Fundamental Algorithms, The Art of Computer Programming, Vol. 1 (Addison-Wesley, Reading, MA, 2nd ed., 1973) 371–381, 576–581.
[15] A. Lempel, m-Ary closed sequences, J. Combin. Theory 10 (1971) 253–258.

[16] J.H. van Lint, F.J. MacWilliams and N.J.A. Sloane, On pseudo-random arrays, SIAM J. Appl. Math. 36 (1979) 62–72.
[17] M.H. Martin, A problem in arrangements, Bull. Amer. Math. Soc. 40 (1934) 859–864.
[18] F.J. MacWilliams and N.J.A. Sloane, Pseudo-random sequences and arrays, Proc. IEEE 64 (1976) 1715–1729.
[19] T. Normura, H. Miyakawa, H. Imai and A. Fukuda, A theory of two-dimensional linear arrays, IEEE Trans. Inform. Theory 18 (1972) 775–785.
[20] F.W. Sinden, Sliding window codes, AT&T Bell Labs Tech. Memorandum, 1985.
[21] S. Stein, The mathematician as an explorer, Sci. Amer. May (1961) 149–158.
[22] M. Yoeli, Binary ring sequences, Amer. Math. Monthly 69 (1962) 852–855.