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Quasi-linear preferences in the macroeconomy: Indeterminacy, heterogeneity and the representative consumer

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Abstract. We use aggregation theory to investigate the link between one-consumer and multi-consumer economies under a quasi-linear class of preferences. Our study is carried out in the context of the neoclassical growth model. The quasi-linear preferences considered are additive in consumption and leisure and linear in leisure. We first show that in a homogeneous agents economy, the individual hours worked are not uniquely determined. We then demonstrate that the indeterminacy can be resolved by introducing heterogeneity. For example, idiosyncratic shocks to productivities or imperfect substitutability of labor restore the uniqueness of equilibrium. As a special case, our analysis includes the indivisible labor model by Hansen (1985).

JEL Classification C73, D90, E21

Key words: Quasi-linear preferences, heterogeneity, representative consumer, indeterminacy, idiosyncratic shocks, indivisible labor

1 Introduction

Much of the research in macroeconomics is built around the representative agent abstraction. The hope is that the replacement of many heterogeneous consumers with just one consumer will not be essential for the results. In some cases, the effect of heterogeneity on the aggregate predictions of the models is, indeed, either absent, as is under Gorman's (1953) exact aggregation, or quantitatively unimportant, as in,

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e.g., Krusell and Smith (1998). However, there are also examples in the literature in which the implications of one-consumer models do not carry over to heterogeneous agents economies, see Atkeson and Ogaki (1996), Maliar and Maliar (2001, 2003). To identify the cases in which the representative agent assumption can be safely used, one needs to gain some understanding of how heterogeneity can affect aggregate dynamics.

Apart from the aggregate behavior of the actual economies, economists are interested in distributive issues. Several papers were able to gain useful insights into the determinants of the distributional dynamics by exploiting the link between one-consumer and multi-consumer economies, see Chatterjee (1994), Caselli and Ventura (2000), Maliar and Maliar (2000, 2001). Such literature proceeds by constructing a representative consumer and uses the knowledge of the aggregate dynamics to restore the evolution of distributions.

This paper employes aggregation theory to investigate the connection between one-consumer and multi-consumer economies under one particular class of preferences, namely, the quasi-linear one. Quasi-linear preferences are those that are additive in all commodities and linear in some commodities. The assumption of quasi-linear preferences is common in incentive literature and in public finance. Furthermore, it is essentially the basis of the indivisible labor model by Hansen (1985) and Rogerson (1988), which is one of the benchmark macroeconomic models. A characteristic feature of quasi-linear preferences is that they are not strictly convex. Under such preferences, the existence and uniqueness of an interior optimal allocation is not, in general, guaranteed.

Our analysis is carried out in a general equilibrium framework. We place ourselves in a production economy with infinitely lived agents, who derive utility from two commodities that we interpret as consumption and leisure. The momentary utility function of each agent is quasi-linear: it is additive in consumption and leisure, and linear in leisure.

We begin by considering the economy with homogeneous agents. We show that such an economy has an indeterminacy of equilibrium, in the sense that one can construct infinitely many allocations for individual hours worked, satisfying all the equilibrium conditions. It is therefore possible that agents who have identical fundamentals (i.e., endowments, preferences and productivities), and who are placed in the same environment, make different labor-leisure decisions. The assumption of multiple consumers is important for the indeterminacy result. What distinguishes a multi-consumer from a one-consumer economy is essentially that, in the former case, the production possibility frontier is linear from the private perspective, whereas in the latter case, it is strictly convex. We shall also emphasize that the indeterminacy described above arises in an optimal economy.[1] Furthermore, the indeterminacy occurs in our model, even if the number of agents is finite.[2]

[1] In this respect, our paper differs from the existing literature, where indeterminacy occurs as a result of market imperfections, e.g., increasing returns, incomplete markets, asymmetric information (see Benhabib and Farmer, 1999, for a survey).

[2] Kehoe and Levine (1985) argue that the indeterminacy of equilibrium may arise because of the assumption of infinite number of an infinitely lived consumers.

Our main finding is that indeterminacy can be ruled out by the introduction of heterogeneity in labor. The role of heterogeneity is, perhaps, best understood by looking at the following two examples: First, we assume that individual labor productivities are subject to shocks. If markets are complete, then the optimal allocations for individual working hours are at the corners: an agent works the maximum number of hours possible when productivity (and, consequently, the wage) is high, and she spends all of her time endowment on leisure when it is low. As a result, the individual hours worked are uniquely determined. Secondly, we assume that labor of different individuals is not perfectly substitutable in production. The individual budget constraints, which are linear under the assumption of perfect substitutes, then become strictly convex. Thus, the indeterminacy is again resolved. It is worth emphasizing that not every type of heterogeneity restores the uniqueness of equilibrium. For example, indeterminacy is present in an economy where agents are heterogeneous in initial endowments, preference parameters and productivities, if the relative productivities do not change over time.

The property that is common to all heterogeneous agents economies considered is that their aggregate dynamics can be characterized by representative consumer models. However, this is not aggregation in the sense of Gorman (1953): the structure of the representative consumer models, which we construct in the paper, depends on specific assumptions about heterogeneity and does not, in general, coincide with the structure of the corresponding homogeneous agents counterparts.[3] In particular, it is possible that the preferences of the representative consumer are strictly convex even though the preferences of all heterogeneous agents are quasi-linear (i.e., not strictly convex). The aggregation results presented in this paper can be viewed as further examples of "imperfect aggregation", described in Maliar and Maliar (2003).[4]

The paper is organized as follows: Section 2 describes a competitive equilibrium economy in which agents have a quasi-linear type of preferences. Section 3 presents the corresponding planner's economy. Section 4 establishes indeterminacy of equilibrium under the assumption of homogeneous agents. Section 5 discusses two examples in which indeterminacy is ruled out by introducing heterogeneity, and finally, Section 6 concludes.

2 The model

Time t is discrete and the horizon is infinite, $t \in T$, where $T = \{0, 1, 2, ...\}$. The economy consists of a representative firm and infinitely-lived agents with names in the set S, which is normalized by $\int_S ds = 1$. The agents differ in three dimensions: endowments, preferences and productivities (skills).

[3] As is known from Blackorby and Schworm (1993), the assumption of a quasi-linear utility function is inconsistent with Gorman's (1953) type of aggregation, unless the utility function is linear in all commodities.

[4] This type of aggregation is based on a representation of the social utility function proposed by Constantinides (1982); and it is employed in Atkeson and Ogaki (1996), and in Maliar and Maliar (2001, 2003).

We denote the skills of agent s in period t by β_t^s. The distribution of skills in period t is $B_t = \{\beta_t^s\}^{s\in S} \in \Im \subset R_+^S$. We assume that B_t follows a stationary first-order Markov process. Specifically, let $\Re$ be the Borel σ-algebra on $\Im$. Define a transition function for the distribution of skills $\varPi : \Im \times \Re \to [0,1]$ on the measurable space $(\Im, \Re)$ such that: for each $z \in \Im$, $\varPi(z, \cdot)$ is a probability measure on $(\Im, \Re)$, and for each $Z \in \Re$, $\varPi(\cdot, Z)$ is a $\Re$-measurable function. We shall interpret the function $\varPi(z, Z)$ as the probability that the next period's distribution of skills lies in the set Z given that the current distribution of skills is z, i.e., $\varPi(z, Z) = \Pr\{B_{t+1} \in Z \mid B_t = z\}$. The initial distribution of skills $B_0 \in \Im$ is given.[5]

The distribution of skills B_t is the only exogenous state variable of our economy. We assume that there is a complete set of markets, i.e., that the agents can trade Arrow securities, contingent on all possible realizations of the distribution of skills.

An agent $s \in S$ derives utility from consumption, c_t^s, and leisure, l_t^s. We focus exclusively on the case in which the momentary utility functions of all agents are quasi-linear: additive in consumption and leisure, and linear in leisure. That is, for each agent $s \in S$, we assume:

$$u^s(c_t^s, l_t^s) = u^s(c_t^s, \overline{n} - n_t^s) \sim v^s(c_t^s) + A^s n_t^s,$$

where the variables n_t^s and $\overline{n}$ are working hours and the total time endowment, respectively.[6] We assume that v^s is strictly increasing and strictly concave and that $A^s < 0$.

The agent s solves the following utility maximization problem:

$$\max_{\{c_t^s, n_t^s, k_{t+1}^s\}_{t\in T}} E_0 \left[\sum_{t=0}^{\infty} \delta^t \left(v^s(c_t^s) + A^s n_t^s\right)\right] \tag{1}$$

$$\text{s.t.} \quad c_t^s + k_{t+1}^s + \int_{\Re} p_t(Z)\, m_{t+1}^s(Z)\, dZ = w_t^s n_t^s + (1 - d + r_t)k_t^s + m_t^s(B_t), \tag{2}$$

where E_0 is the operator of conditional expectation. The discount factor is $\delta \in (0,1)$. The wage per unit of labor supplied by agent s is w_t^s. The agent owns capital stock k_t^s and rents it to the firm at the rental price r_t. Capital depreciates at the rate $d \in (0,1]$. The portfolio of Arrow securities bought by the agent in period t is $\{m_{t+1}^s(Z)\}_{Z\in\Re}$. The price of security $p_t(Z)$ is to be paid in period t for the delivery of one unit of the consumption good in period $t+1$ if $B_{t+1} \in Z$. Initial endowments of capital, k_0^s, and Arrow securities, $m_0^s(B_0)$, are given. The total endowment is $(1 - d + r_0)k_0^s + m_0^s(B_0) \equiv \kappa_0^s \subset R_+$.

The representative firm rents capital and hires labor to maximize period-by-period profits. Capital input is given by $k_t = \int_S k_t^s ds$. Labor input is determined

[5] Note that the above formulation allows for aggregate uncertainty in the distribution of skills, as idiosyncratic disturbances can be correlated across agents.

[6] Here, and further on in the paper, $\sim$ means "is identical up to an additive constant".

by a function h, which depends on both, individual hours worked and skills. Consequently, the problem solved by the firm is

$$\max_{k_t,\, \{n_t^s\}^{s\in S}} \pi_t = f\left(k_t, h_t\right) - r_t k_t - \int_S w_t^s n_t^s ds \tag{3}$$

$$\text{s.t.} \quad h_t = h\left(\{n_t^s, \beta_t^s\}^{s\in S}\right). \tag{4}$$

The production function, f, has constant returns to scale, is strictly increasing in both arguments, continuously differentiable, strictly concave, and satisfies the appropriate Inada conditions. The labor input function, h, has constant returns to scale, is strictly increasing in all arguments, continuously differentiable and concave.

Definition. A competitive equilibrium in the economy $(1)-(4)$ is a sequence of contingency plans for the consumers' allocation, the firm's allocation and the prices, such that, given the prices, the allocation of each consumer solves the utility maximization problem (1), (2), the allocation of the firm solves the profit maximization problem (3), (4); also, capital and labor markets clear and the economy's resource constraint is satisfied. The equilibrium quantities are to be such that c_t^s, w_t^s, k_t, $r_t \geq 0$ and $0 \leq n_t^s \leq \overline{n}$ for all t, s.[7]

3 Planner's economy

According to the First Welfare Theorem, in an economy with complete markets and without distortions, like ours, a competitive equilibrium is Pareto optimal and can be restored as the optimal choice of a social planner, who maximizes social welfare. Specifically, let us define the social utility function[8]

$$U\left(c_t, h_t, \{\lambda^s, \beta_t^s\}^{s\in S}\right) \equiv \max_{\{c_t^s, n_t^s\}^{s\in S}} \left\{ \int_S \lambda^s \left(v^s\left(c_t^s\right) + A^s n_t^s\right) ds \;\middle|\; \begin{array}{c} \int_S c_t^s ds = c_t \\ h\left(\{n_t^s, \beta_t^s\}^{s\in S}\right) = h_t \\ 0 \leq n_t^s \leq \overline{n} \end{array} \right\}, \tag{5}$$

where c_t is the aggregate consumption, and $\{\lambda^s\}^{s\in S} \subset R_+^S$ is the distribution of welfare weights, normalized to one, $\int_S \lambda^s ds = 1$.

A competitive equilibrium in the heterogeneous agents economy $(1)-(4)$ can therefore be restored by solving the following planner's problem

$$\max_{\{c_t, h_t, k_{t+1}\}_{t\in T}} E_0 \sum_{t=0}^{\infty} \delta^t U\left(c_t, h_t, \{\lambda^s, \beta_t^s\}^{s\in S}\right) \tag{6}$$

[7] The formulated setup is a heterogeneous agents variant of Kydland and Prescott's (1982) model. Under the assumption of strictly convex preferences, similar heterogeneous agents settings have been studied in, e.g., Kydland (1984, 1995), Atkeson and Ogaki (1996), Maliar and Maliar (2001, 2003).

[8] The constraint $0 \leq n_t^s \leq \overline{n}$ plays an important role in our analysis because the model has, in general, corner solutions for hours worked.

$$\text{s.t.} \quad c_t + k_{t+1} = (1-d)\, k_t + f\,(k_t, h_t)\,. \tag{7}$$

To be precise, for any given distribution of endowments in the decentralized economy $(1) - (4)$, there exists a set of welfare weights in the planner's economy $(5) - (7)$ such that a competitive equilibrium allocation is a solution to the planner's problem. We should point out that the notion of the social utility function (5) does not, in general, imply aggregation in the sense of Gorman (1953), since U is allowed to depend not only on aggregate quantities but also on the distributions of welfare weights and skills.[9]

With the assumptions of convex preferences and production sets, we have the Second Welfare Theorem, which is the converse of the First Welfare Theorem. Specifically, any Pareto optimal allocation can be reached as a competitive equilibrium with appropriate transfers of wealth (i.e., initial endowments). In our model, the correspondence between Pareto optimal and competitive equilibrium allocations is identified by the expected lifetime budget constraints. Such constraints are obtained by applying forward recursion to individual budget constraints (2) and by imposing the transversality condition,[10]

$$E_0 \left[\sum_{\tau=0}^{\infty} \delta^{\tau} \frac{(v^s)'\,(c_\tau^s)}{(v^s)'\,(c_0^s)} \left(c_\tau^s - n_\tau^s w_\tau^s\right) \right] = \kappa_0^s. \tag{8}$$

Given any Pareto optimal allocation, we can compute the left side of (8) for each s. As a result, we obtain uniquely determined transfers of wealth supporting a given Pareto optimal allocation.

It is convenient to characterize the social utility function U by using the Kuhn-Tucker conditions.

Proposition 1. *For the economy* $(1) - (4)$, *U takes the form*

$$U\left(c_t, h_t, \{\lambda^s, \beta_t^s\}^{s\in S}\right) = V\left(c_t, \{\lambda^s\}^{s\in S}\right) + W\left(h_t, \{\lambda^s, \beta_t^s\}^{s\in S}\right), \tag{9}$$

where V is defined by $\int_S c_t^s ds = c_t$ and

$$c_t^s = \left((v^s)'\right)^{-1} \left(\frac{1}{\lambda^s} V'\left(c_t, \{\lambda^s\}^{s\in S}\right)\right); \tag{10}$$

W is defined by $h\left(\{n_t^s, \beta_t^s\}^{s\in S}\right) = h_t$ and

$$A^s\lambda^s - \frac{\partial h}{\partial n_t^s} W_1\left(h_t, \{\lambda^s, \beta_t^s\}^{s\in S}\right) \begin{cases} < 0 \Rightarrow & n_t^s = 0 \\ > 0 \Rightarrow & n_t^s = \overline{n} \\ = 0 \Rightarrow & 0 \le n_t^s \le \overline{n} \end{cases}; \tag{11}$$

and where V_1 and W_1 denote the first-order partial derivatives of V and W with respect to c_t and h_t, correspondingly.

[9] See Maliar and Maliar (2003) for a discussion.

[10] For the derivation of this constraint, see Maliar and Maliar (2001).

Proof. See Appendix. □

As follows from Proposition 1, the optimal allocation for the individual working hours in the planner's economy can be either interior or at corners. With the following proposition, we establish that the interiority of the planner's allocation for working hours implies the interiority of the corresponding allocation in the decentralized economy, and vice versa.

Proposition 2. *The optimal allocation for individual hours worked is interior in the decentralized economy (1)–(4) if, and only if, it is interior in the associated planner's problem (5)–(7).*

Proof. See Appendix. □

4 Homogeneous agents: Indeterminacy

We first consider the economy $(1)-(4)$ populated by agents with identical fundamentals (i.e., endowments, preferences and skills) such that $v^s = v$, $A^s = A$, $\kappa_0^s = k_0$ and $\beta_t^s = 1$ for all s. We assume that the labor input is given by the sum of individual hours worked, $h_t = \int_S n_t^s ds$, and that the economy consists of more than one agent.

As follows from Proposition 1, if an equilibrium is interior, then (11) holds with equality. Given that $\int_S \lambda^s ds = 1$, we have $\lambda^s = 1$ for all s. From (10), we obtain that $c_t^s = c_t$ for all s. Moreover, the economy admits a representative consumer with a quasi-linear utility function

$$U\left(c_t, h_t, \{1,1\}^{s\in S}\right) \sim v\left(c_t\right) + Ah_t. \tag{12}$$

As has already been shown in Hansen (1985), the aggregate equilibrium allocation $\{c_t, h_t, k_{t+1}\}_{t\in T}$ is uniquely determined by (6), (7), (12).

Nevertheless, individual hours worked, $\{n_t^s\}_{t\in T}^{s\in S}$, are not uniquely determined. Indeed, optimality condition (11) does not contain n_t^s and, thus, imposes no restriction on the choice of working hours except for $0 \leq n_t^s \leq \overline{n}$ for all s and t. The other conditions to be satisfied in equilibrium are market clearing in labor, $h_t = \int_S n_t^s ds$, and expected lifetime budget constraint (8). The former restricts the sum of individual working hours within each period, whereas the latter does so across periods. We demonstrate that these two conditions are not sufficient to identify individual labor choice.

Proposition 3. *If an interior equilibrium in the homogeneous agents economy* $(1)-(4)$ *exists, then an infinite number of equilibrium allocations exists for individual hours worked.*

Proof. See Appendix. □

What is the source of the indeterminacy of the distribution of working hours? At the individual level, working hours enter linearly in both the preferences and the budget constraint. From the perspective of the agent, hours worked in different periods are perfect substitutes. Consequently, the agent is indifferent between any sequences of working hours leading to the same lifetime disutility from working. At the same time, the working hours of different individuals are perfectly substitutable in production. From the point of view of the economy as a whole, any subdivision of working hours among agents is optimal as long as it leads to the optimal aggregate labor input in all periods.

The assumptions that are essential for the existence of indeterminacy are that the economy lasts for more than one period and that it consists of more than one agent. In a one-period economy, individual working hours are uniquely determined by budget constraint (8): all agents consume the same amount, $c_t^s = c_t$, and spend the rest of their endowment on leisure. In turn, in a one-agent economy, the equilibrium is unique because the individual working hours suppled by the agent coincide with the labor input demanded by the firm, $n_t^s = h_t$ for all t.[11] The individual budget constraint, which is linear in the multi-consumer model, becomes strictly convex in the one-consumer setup.

Our analysis has a direct implication for the indivisible labor model by Hansen (1985). In such a model, the time allocated to the market job can only have one of two values, i.e., some fixed number of hours or zero hours. Instead of choosing hours worked, an agent decides on the probability of being employed. Whether the agent gets the job or not is determined by a lottery, which is won with the probability chosen by the agent. Regardless of the realization of the employment lottery, the agent is paid the expected labor income (the one corresponding to the probability of employment chosen). It is assumed that there is a continuum of ex-ante identical agents whose utility functions are additive in consumption and leisure and logarithmic in both consumption and leisure. As formulated in Hansen (1985), the problem of an individual is

$$\max_{\{c_t,\alpha_t,k_{t+1}\}_{t\in T}} E_0 \sum_{t=0}^{\infty} \delta^t \left[\log(c_t) + A\alpha_t \log(1-h_0)\right] \tag{13}$$

$$\text{s.t.} \quad c_t + k_{t+1} \le (1-d+r_t)\, k_t + w_t \alpha_t h_0, \tag{14}$$

where α_t is the probability of being employed, the total time endowment of the agent is equal to one, and h_0 is the indivisible number of hours supplied to the firm by an employed agent. As shown in Hansen (1985), the aggregate equilibrium allocation in the indivisible labor model (13), (14) is described by the planner's problem (6), (7), (12) and it is uniquely determined. Regarding the individual optimal allocation, however, our analysis suggests that the optimal choice for individual probability of employment, α_t, is not uniquely determined. At the individual level, the probability enters linearly in both the preferences (13) and the budget constraint (14). This feature gives rise to an indeterminate equilibrium.

[11] Indeed, consider the First Order Condition (FOC) of (1), (2) with respect to working hours. With identical agents, we have $v'(c_t^s)\, w_t = A$, where $w_t = \partial f(k_t, h_t)/\partial k_t$. If there is only one agent, then $h_t = n_t^s$, which allows us to identify n_t^s. However, if there is more than one agent, we only know $h_t = \int_S n_t^s ds$, which is not sufficient to identify the distribution $\{n_t^s\}^{s\in S}$.

5 Ruling out indeterminacy by introducing heterogeneity in labor

In this section, we analyze the implications of heterogeneous agents versions of the model. We first show that, similarly to the case of identical agents, the optimal consumption allocations are uniquely determined in heterogeneous agents economies. We subsequently demonstrate that introducing labor heterogeneity can rule out the indeterminacy of the individual labor choice.

5.1 Consumption choice

We first study the consumption choice. Integrating (10) across agents yields

$$c_t = \int_S \left((v^s)'\right)^{-1} \left(\frac{1}{\lambda^s} V_1 \left(c_t, \{\lambda^s\}^{s\in S}\right)\right) ds. \tag{15}$$

For a given set of weights $\{\lambda^s\}^{s\in S}$ and aggregate consumption c_t, there is a unique value of $V_1\left(c_t, \{\lambda^s\}^{s\in S}\right)$ solving (15). Given the function V_1, the optimal consumption of each agent is uniquely determined by (10). Below, we show an example in which V can be derived explicitly.

Example 1. Assume that $v^s\left(c_t^s\right) = \left(c_t^s\right)^\gamma$, where $\gamma \in (0,1)$. Equations (10) and (15), respectively, become

$$c_t^s = \left(\frac{1}{\lambda^s\gamma} V_1 \left(c_t, \{\lambda^s\}^{s\in S}\right)\right)^{1/(\gamma-1)}, \text{ and } c_t = \left(\frac{1}{\gamma\xi} V_1 \left(c_t, \{\lambda^s\}^{s\in S}\right)\right)^{1/(\gamma-1)},$$

where $\xi = \left(\int_S (\lambda^s)^{1/(1-\gamma)} ds\right)^{1-\gamma}$. Hence, we have $V\left(c_t, \{\lambda^s\}^{s\in S}\right) \sim \xi c_t^\gamma$. Furthermore, combining the above expressions for c_t^s and c_t, we get

$$c_t^s = (\lambda^s/\xi)^{1/(1-\gamma)} \cdot c_t.$$

This formula determines the optimal distribution of consumption. □

5.2 Labor choice

To characterize the optimal labor choice, we are to make specific assumptions on how working efforts of different individuals are aggregated into the labor input. We begin by discussing the case in which the agents' efforts are perfectly substitutable in production. Later in the section, we will consider the case of imperfect substitutes.

5.2.1 Perfect substitutes

Assume that the working hours of different agents are perfect substitutes[12]

$$h_t = \int_S n_t^s \beta_t^s ds. \quad (16)$$

We first consider the case in which the decentralized economy $(1)-(4)$, (16) has an interior equilibrium, or equivalently, the planner's problem $(5)-(7)$, (16) has an interior solution (see Proposition 2). In such a case, we have the following result.

Proposition 4. *a). In order an interior solution to the planner's problem (5)–(7), (16) exists, it is necessary that $\beta_t^s = \beta_t b^s$ for all t and s.*
b). The welfare weights in (5) satisfy $\lambda^s = \frac{b^s/A^s}{\int_S b^s/A^s ds}$ for all s.
c). $W\left(h_t, \{\lambda^s, \beta_t^s\}^{s\in S}\right) \sim A_t h_t$, where $A_t = \left(\beta_t \int_S b^s/A^s ds\right)^{-1}$.
d). There exists an infinite number of optimal allocations for individual hours worked.

Proof. See Appendix. □

The result that only one set of weights is consistent with an interior equilibrium is related to the well-known property of quasi-linear utility function that a change in wealth affects the demand for commodities that enter the utility function linearly but does not affect the demand for any other commodities. In the context of our model, this property means that the optimal choice of individual consumption is independent of the distribution of initial endowments. In the one-period version of our model, for example, the Engel curves are parallel zero-sloped straight lines with the origin (consumption intercept) being determined by v^s, A^s and b^s.[13] Given that the optimal consumption of agents is the same for all distributions of initial endowments, we always have the same set of welfare weights.

The existence of the representative consumer may seem surprising since, as it is known from Blackorby and Schworm (1993), the assumption of quasi-linear utility function is inconsistent with aggregation unless the utility function is linear in all commodities. We are able to escape this negative implication of Blackorby and Schworm's (1993) analysis because our aggregation requirement is weaker than the one they use. Specifically, we allow for the case in which the social utility function depends not only on aggregate quantities but also on the distribution of welfare weights. We therefore construct a representative consumer, not for all, but only for the unique set of welfare weights that corresponds to an interior equilibrium. The

[12] This assumption is common in the heterogeneous agents literature, e.g., Kydland (1984, 1995), Aiyagari (1994), Krusell and Smith (1998), Castañeda, Díaz-Giménez and Ríos-Rull (1998), Maliar and Maliar (2001).

[13] A class of utility functions leading to parallel Engel curves is called similarly quasi-homothetic. The property of quasi-homotheticity is known to be both necessary and sufficient for aggregation in the sense of Gorman (1953). Therefore, if the economy studied in this section has an interior equilibrium, then it admits a representative consumer, independently of specific assumptions about individual characteristics $\{v^s, A^s, b^s\}^{s\in S}$.

definition of a representative consumer used in Blackorby and Schworm (1993) is more restrictive as it requires the preferences of the representative consumer to be the same for all sets of welfare weights.

As follows from Proposition 4, if we restrict our attention to interior equilibrium, the indeterminacy does not disappear after the introduction of heterogeneity.[14] We shall point out that the existence of an interior equilibrium requires imposing the specific restriction that the relative skills of agents do not change over time. If such a restriction is not satisfied for some s and t, then the corresponding optimal allocations are not interior. Below, we argue that having the optimal allocations for working hours at corners can help to restore the uniqueness of equilibrium.

According to (11), the agent's optimal labor choice in the presence of the corner solutions is as follows: to work the maximum possible number of hours $\overline{n}$ if the current labor productivity β_t^s is strictly higher than $A^s\lambda^s/W_1$, to work zero hours if β_t^s is strictly lower than $A^s\lambda^s/W_1$, and to work a certain number of hours $n_t^s \in [0, \overline{n}]$, if β_t^s is equal to $A^s\lambda^s/W_1$. The aggregate labor input is therefore given by

$$h_t = \overline{n} \int_{\beta_t^s > \frac{A^s\lambda^s}{W_1}} \beta_t^s ds + \int_{\beta_t^s = \frac{A^s\lambda^s}{W_1}} n_t^s \beta_t^s ds. \tag{17}$$

In order to have a unique equilibrium, it is necessary to insure that the set of agents, who have interior equilibrium allocations, has a measure zero. This can be done by assuming that in each period, individual skills are randomly drawn from some distribution with a continuous density.

Example 2. Consider an economy populated by ex-ante identical agents with names in the interval $S = [0, 1]$. Suppose that the individual skills are $\beta_t^s = \beta_t b_t^s$, where β_t is the aggregate component, and b_t^s is the idiosyncratic component drawn from the uniform distribution $\left[0, \sqrt{2}\right]$.[15]

Note first that ex-ante identical agents have identical welfare weights, $\lambda^s = 1$ for all s. Denote by b_t the value of idiosyncratic component that makes the agents indifferent between working or not, i.e., $b_t = A/\left(\beta_t W_1\right)$, where $A^s = A$ for all s. Aggregate labor input (17) then becomes:

$$h_t = \overline{n}\beta_t \int_{b_t}^{\sqrt{2}} b_t^s db_t^s = \overline{n}\beta_t \left(1 - \frac{b_t^2}{2}\right). \tag{18}$$

Note that the set of agents whose optimal allocations for working hours are interior (i.e., whose skills are $b_t^s = b_t$) has a measure zero.

Combining (18) with the threshold condition $b_t = A/\left(\beta_t W_1\right)$ yields

$$W_1\left(h_t, \{\lambda^s, \beta_t^s\}^{s\in S}\right) = \frac{A}{\beta_t\sqrt{2\left(1 - \frac{h_t}{\overline{n}\beta_t}\right)}}.$$

[14] Kydland (1984) studies a similar two-agent model under the Cobb-Douglas (strictly concave) utility function. He finds that the model reproduces empirical observations that high-productivity agents work more and experience less volatility of labor than the low-productivity. Because of the indeterminacy, similar questions cannot be asked if the utility function is quasi-linear. Neither can they be addressed in the indivisible labor framework.

[15] A similar example is considered in Cho (1995).

Therefore, the function W is

$$W\left(h_t, \{\lambda^s, \beta_t^s\}^{s\in S}\right) \sim -A\overline{n}\sqrt{\frac{1}{2}\left(1-\frac{h_t}{\overline{n}\beta_t}\right)}. \tag{19}$$

The planner's problem (6), (7) with the social utility function, defined by (15), (19), yields the aggregate equilibrium allocation. If such an allocation is given, the threshold condition uniquely determines who works and who does not. Note that the function W is strictly concave, even though the utility functions of all heterogeneous agents are quasi-linear. □

We should also mention that the assumption that agents are ex-ante identical is not essential for resolving the indeterminacy. The optimal allocations for individual hours worked will be unique, independently of a specific distribution of welfare weights across agents.

5.2.2 Imperfect substitutes

Suppose now that the individual working efforts are not perfectly substitutable in production so that the marginal labor input of an agent s can depend on both skills and quantities of labor supplied by all agents in the economy:

$$\frac{\partial h_t}{\partial n_t^s} = \frac{\partial}{\partial n_t^s} h\left(\{n_t^s, \beta_t^s\}^{s\in S}\right). \tag{20}$$

In this case, optimality condition (11) provides S equations containing S unknown individual hours worked, $\{n_t^s\}^{s\in S}$. Below, we show an example in which this system of equations (11) has a unique solution. Thus, the assumption of imperfect substitutes can help us to resolve the indeterminacy of the individual labor decisions.[16]

Example 3. Assume that h is given by the CES function

$$h_t = \left(\int_S \beta_t^s \left(n_t^s\right)^{\varepsilon} ds\right)^{1/\varepsilon}, \qquad \varepsilon \in (-\infty, 1).$$

The parameter ε determines the degree of substitutability and complementarity among the labor inputs of different agents. In the limits, $\varepsilon = 1$ and $\varepsilon \to -\infty$, we have perfect substitutes (16) and perfect compliments, respectively.

Suppose that equilibrium is interior, i.e., condition (11) holds with equality. By finding the derivative $\frac{\partial h_t}{\partial n_t^s}$ of the CES function, substituting it into (11) and solving with respect to n_t^s, we get

$$n_t^s = \left(\frac{\beta_t^s}{A^s\lambda^s} W_1\left(h_t, \{\lambda^s, \beta_t^s\}^{s\in S}\right)\right)^{1/(1-\varepsilon)} h_t. \tag{21}$$

[16] If the economy is populated by an infinite number of agents, then the equilibrium is unique up to the allocation of a zero-measure subset of agents. This is because such a subset of agents has no effect on the aggregate equilibrium allocation, see Kehoe and Levine (1985).

Integrating $\beta_t^s\,(n_t^s)^\varepsilon$ across agents and rearranging the terms yields

$$W_1\left(h_t,\{\lambda^s,\beta_t^s\}^{s\in S}\right)=\left[\int_S \beta_t^s\left(\frac{\beta_t^s}{A^s\lambda^s}\right)^{\varepsilon/(1-\varepsilon)}ds\right]^{(1-\varepsilon)/\varepsilon}\equiv A_t.$$

Therefore, we have $W\left(h_t,\{\lambda^s,\beta_t^s\}^{s\in S}\right)\sim A_t h_t$. The aggregate equilibrium allocation can again be computed by solving the planner's problem (6), (7). The individual hours worked are uniquely determined by (21). □

The property that helps us restore the uniqueness of equilibrium in this case is the strict convexity of individual budget constraints.

6 Concluding comments

This paper investigates the consequences of quasi-linear preferences (additive in consumption and leisure, and linear in leisure) in a dynamic general equilibrium model with multiple consumers. To characterize the distributional dynamics, we employ the aggregation theory. Our main findings are as follows: If consumers are homogeneous, there is indeterminacy of equilibrium: the model does not produce sharp predictions about the individual hours worked. However, the indeterminacy does not, in general, survive the introduction of heterogeneity in labor. Two examples of labor heterogeneity that restore the uniqueness of equilibrium are idiosyncratic shocks to skills and imperfect substitutability of labor in production. All the above results also apply to the indivisible labor model by Hansen (1985).

Appendix

Proof of Proposition 1

The definition of the social utility function (5) implies

$$\lambda^s\,(v^s)'\,(c_t^s)+\mu_t=0, \tag{22}$$

$$A^s\lambda^s+\eta_t\frac{\partial h_t}{\partial n_t^s}+\varsigma_t^s-\zeta_t^s=0, \tag{23}$$

$$\varsigma_t^s\geq 0 \quad\text{and}\quad \varsigma_t^s n_t^s=0, \tag{24}$$

$$\zeta_t^s\geq 0 \quad\text{and}\quad \zeta_t^s\,(\overline{n}-n_t^s)=0, \tag{25}$$

$$\frac{\partial U}{\partial c_t}+\mu_t=0, \tag{26}$$

$$\frac{\partial U}{\partial h_t}+\eta_t=0, \tag{27}$$

where μ_t, η_t, ς_t^s, ζ_t^s are the Lagrange multipliers associated with the constraints $\int_S c_t^s ds=c_t$, $h\left(\{n_t^s,\beta_t^s\}\right)=h_t$, $n_t^s\geq 0$ and $n_t^s\leq\overline{n}$, respectively. By substituting (22) into (26) and solving with respect to c_t^s, we obtain (10). Conditions (23)–(25) together with (27) yield (11). The fact that the individual utility functions are additive in consumption and working hours implies that the social utility function, U, is additive in aggregate consumption and labor, i.e., U can be written as the sum of two subfunctions, V and W, as is in (9). This can be shown by substituting (10) and (11) into the objective function in (5). □

Proof of Proposition 2

The FOCs of the consumer's utility maximization problem (1) , (2) with respect to Arrow securities, capital, consumption and hours worked, respectively, are

$$\lambda_t^s p_t(Z) = \delta \lambda_{t+1}^s \Pr\{B_{t+1} \in Z \mid B_t = z\}_{Z \in \Re, z \in \Im}, \quad (28)$$

$$\lambda_t^s = \delta E_t \left[\lambda_{t+1}^s (1 - d + r_{t+1}) \right], \quad (29)$$

$$(v^s)'(c_t^s) + \lambda_t^s = 0, \quad (30)$$

$$A^s + \lambda_t^s w_t \frac{\partial h_t}{\partial n_t^s} + \widetilde{\varsigma}_t^s - \widetilde{\zeta}_t^s = 0, \quad (31)$$

with $\widetilde{\varsigma}_t^s$ and $\widetilde{\zeta}_t^s$ satisfying

$$\widetilde{\varsigma}_t^s \geq 0 \quad \text{and} \quad \widetilde{\varsigma}_t^s n_t^s = 0, \quad (32)$$

$$\widetilde{\zeta}_t^s \geq 0 \quad \text{and} \quad \widetilde{\zeta}_t^s (\overline{n} - n_t^s) = 0. \quad (33)$$

Here, λ_t^s, $\widetilde{\varsigma}_t^s$, $\widetilde{\zeta}_t^s$ are the Lagrange multipliers corresponding to budget constraint (2) and restrictions $n_t^s \geq 0$ and $n_t^s \leq \overline{n}$, respectively, and $w_t^s = w_t \frac{\partial h_t}{\partial n_t^s}$ with $w_t \equiv \frac{\partial f(k_t, h_t)}{\partial h_t}$. Note that equation (28) implies that for any two agents $s', s'' \in S$, we have

$$\frac{\lambda_t^{s'}}{\lambda_t^{s''}} = \frac{\lambda_{t+1}^{s'}}{\lambda_{t+1}^{s''}} \text{ for all } Z \in \Re. \quad (34)$$

Therefore, we can rewrite the Lagrange multiplier λ_t^s as $\lambda_t^s = \lambda_t / \lambda^s$. This is the standard implication of the complete markets assumption that the ratio of marginal utilities of any two agents remains constant in all periods and states of nature. Without loss of generality, we normalize the multipliers by $\int_S \lambda^s ds = 1$.

Assume that the equilibrium allocation for individual working hours in the decentralized economy is interior, i.e., $\widetilde{\varsigma}_t^s = 0$ and $\widetilde{\zeta}_t^s = 0$ for all s. Using the result $\lambda_t^s = \lambda_t / \lambda^s$, we have that for any two agents $s', s'' \in S$,

$$\frac{A^{s'} \lambda^{s'}}{A^{s''} \lambda^{s''}} = \frac{\partial h_t / \partial n_t^{s'}}{\partial h_t / \partial n_t^{s''}}. \quad (35)$$

For the planner's solution for individual working hours to satisfy (35), we must have that, in (23), $\varsigma_t^s = 0$ and $\zeta_t^s = 0$ for all s. Hence, the planner's allocation for individual hours worked is interior.

The same type of arguments can be used to prove the converse statement: assuming that the planner's solution is interior, $\varsigma_t^s = 0$ and $\zeta_t^s = 0$ for all s, we obtain that the individual working hours satisfy (35), which implies that for all s, $\widetilde{\varsigma}_t^s = 0$ and $\widetilde{\zeta}_t^s = 0$ in (31), i.e., the equilibrium allocation for working hours in the decentralized economy is interior. □

Proof of Proposition 3

We assume that the economy consists of more than one agent. Let $\{c_t, h_t, k_{t+1}\}_{t\in T}$ be the aggregate optimal allocation. First of all, note that the symmetric allocation such that all agents behave identically, $c_t^s = c_t$ and $n_t^s = h_t$ for all s, satisfies all the optimality conditions. Let us show that there exists also an infinite number of optimal allocations with asymmetric labor choices.

Consider an allocation $\{\widetilde{c}_t^s, \widetilde{n}_t^s\}_{t\in T}^{s\in S}$ such that $\widetilde{c}_t^s = c_t$ for all s and $\widetilde{n}_t^s = h_t$ for all s except of agents s' and s'', whose labor decisions are as follows:

$$\left\{\widetilde{n}_t^i\right\}_{t\in[0,\ldots,\tau-1,\tau+2,\ldots,\infty)} = \{h_t\}_{t\in[0,\ldots,\tau-1,\tau+2,\ldots,\infty)}, \qquad i\in\{s', s''\},$$
$$\widetilde{n}_\tau^{s'} = h_\tau + \varepsilon, \qquad \widetilde{n}_\tau^{s''} = h_\tau - \varepsilon,$$
$$\widetilde{n}_{\tau+1}^{s'} = h_{\tau+1} - \frac{\varepsilon}{\delta}, \qquad \widetilde{n}_{\tau+1}^{s''} = h_{\tau+1} + \frac{\varepsilon}{\delta},$$

where $\varepsilon > 0$ is any number such that $\widetilde{n}_\tau^{s'}, \widetilde{n}_\tau^{s''}, \widetilde{n}_{\tau+1}^{s'}, \widetilde{n}_{\tau+1}^{s''} \in [0, \overline{n}]$. The above construction is always possible under our assumption of interior equilibrium, which guarantees that the equilibrium allocations for the aggregate labor input, h_t, are interior for all t.

Observe that the asymmetric allocation leads to the same aggregate labor input as the symmetric one, since we have $\widetilde{n}_t^{s'} + \widetilde{n}_t^{s''} = h_t + h_t$ for $t\in\{\tau, \tau+1\}$. Given that the level of output does not change, the equal consumption, $\widetilde{c}_t^s = c_t$ for all s, is feasible in the asymmetric case. Furthermore, note that in the asymmetric and symmetric cases, all agents, including $i\in\{s', s''\}$, get the same lifetime disutility from working

$$E_0\left\{\ldots + \delta^\tau A\left(\widetilde{n}_\tau^s + \delta\widetilde{n}_{\tau+1}^s\right) + \ldots\right\} = E_0\left\{\ldots + \delta^\tau A\left(h_\tau + \delta h_{\tau+1}\right) + \ldots\right\}.$$

As the asymmetric allocation implies the same expected lifetime utility for all agents as the symmetric one, this allocation is optimal. □

Proof of Proposition 4

a). According to (11), for any two agents $s', s'' \in S$, we have

$$\frac{A^{s'}\lambda^{s'}}{A^{s''}\lambda^{s''}} = \frac{\beta_t^{s'}}{\beta_t^{s''}} \qquad \text{for all } t. \tag{36}$$

Therefore, an interior equilibrium is possible only if the relative skills of agents do not change over time, i.e., $\beta_t^s = \beta_t\beta^s$ for all s.

b). The formula for λ^s follows from (36) and normalization $\int_S \lambda^s ds = 1$.

c). By substituting $\frac{\partial h}{\partial n_t^s} = \beta_t^s$ into (11), expressing λ^s and using the normalization $\int_S \lambda^s ds = 1$, we obtain $W_1 = A_t$ and, thus, $W \sim A_t h_t$.

d). The proof is parallel to that of Proposition 3. We again assume that the economy consists of more than one agent. Let $\{c_t, h_t, k_{t+1}\}_{t\in T}$ be an aggregate optimal allocation. Note that in the economy with heterogeneous agents, the symmetric allocation is, in general, not a solution. The optimal distribution of consumption across agents, $\{c_t^s\}_{t\in T}^{s\in S}$, is the one which satisfies condition (10). The optimal distribution of working hours, $\{n_t^s\}_{t\in T}^{s\in S}$, is, however, not identified by the corresponding optimality condition (11). We first construct one distribution $\{n_t^s\}_{t\in T}^{s\in S}$ satisfying all of the optimality conditions, and then show that an infinite number of other optimal distributions exists.

One optimal distribution of working hours can be constructed by assuming that each agent works a fixed share of aggregate labor input, $n_t^s = n^s h_t$ for all t and s. The optimal shares are identified by individual expected lifetime budget constraints (8)

$$n^s = \frac{\kappa_0^s - E_0 \sum\limits_{\tau=0}^{\infty} \delta^\tau \frac{(v^s)'(c_\tau^s)}{(v^s)'\left(c_0^s\right)} c_\tau^s}{\beta^s E_0 \sum\limits_{\tau=0}^{\infty} \delta^\tau \frac{(v^s)'(c_\tau^s)}{(v^s)'\left(c_0^s\right)} h_\tau w_\tau \beta_\tau}.$$

Consider, now, the allocation $\{\widetilde{c}_t^s, \widetilde{n}_t^s\}_{t\in T}^{s\in S}$ such that $\widetilde{c}_t^s = c_t^s$ for all s and $\widetilde{n}_t^s = n_t^s$ for all s except for agents s' and s'', whose labor decisions are

$$\left\{\widetilde{n}_t^i\right\}_{t\in[0,\ldots,\tau-1,\tau+2,\ldots,\infty)} = \left\{n_t^i\right\}_{t\in[0,\ldots,\tau-1,\tau+2,\ldots,\infty)}, \qquad i \in \{s', s''\},$$
$$\widetilde{n}_\tau^{s'} = n_\tau^{s'} + \frac{\varepsilon}{\beta^{s'}}, \qquad \widetilde{n}_\tau^{s''} = n_\tau^{s''} - \frac{\varepsilon}{\beta^{s''}},$$
$$\widetilde{n}_{\tau+1}^{s'} = n_{\tau+1}^{s'} - \frac{\varepsilon}{\delta\beta^{s'}}, \qquad \widetilde{n}_{\tau+1}^{s''} = n_{\tau+1}^{s''} + \frac{\varepsilon}{\delta\beta^{s''}},$$

where $\varepsilon > 0$ is a number such that $\widetilde{n}_\tau^{s'}, \widetilde{n}_\tau^{s''}, \widetilde{n}_{\tau+1}^{s'}, \widetilde{n}_{\tau+1}^{s''} \in [0, \overline{n}]$.

As in Proposition 3, the distributions $\{n_t^s\}_{t\in T}^{s\in S}$ and $\{\widetilde{n}_t^s\}_{t\in T}^{s\in S}$ lead to the same aggregate labor input in all periods, so that output and consumption are also the same. Furthermore, $\{n_t^s\}_{t\in T}^{s\in S}$ and $\{\widetilde{n}_t^s\}_{t\in T}^{s\in S}$ imply the same lifetime disutility from working for all consumers. Consequently, the allocation $\{\widetilde{c}_t^s, \widetilde{n}_t^s\}_{t\in T}^{s\in S}$ is optimal. □

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