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Bipolarity, Choice, and Entro-Field

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ABSTRACT

Until now, formal models of bipolar choice have been phenomenological and were not related to the deep principles of processing information by live organisms, which limited the applications of these models and made it difficult to generalize them to the case of multi-alternative choice. We demonstrate here how to deduce a model theoretically based on a general definition of the self-reflexive system and one assumption which we called the Axiom of the Second Choice. We show further that such a deduction of the model reveals its unexpected connection to the relations between an internal variable of the self-reflexive system, a partial derivative of the entropy of the environmental influence, and a partial derivative of the entropy of choice made by the system. This connection allows us to expand the two-alternative model of bipolar choice to the case of an arbitrary number of alternatives.

Keywords: axiom of the second choice, reflexivity, matching law, bipolarity

1. INTRODUCTION

A real choice is often entailed with a conflict between profits and ideals. To analyze such situations, a special model of binary choice has been constructed; in it, the alternatives are polarized: one identifies the positive pole and the other identifies the negative one [1], [2], [3]. This model can be represented with the following equation:

$$\frac{1-X}{X}=\frac{1-x_1}{x_1}x_2 \ , \tag{1}$$

where X is the probability of choosing the positive pole, x_1, where $0 < x_1 < 1$, is normalized attractiveness corresponding to the positive alternative, and x_2, where $0 < x_2 \leq 1$, is predetermined by factors lying beyond a given situation of choice. This parameter disturbs the proportionality between the probabilities of the alternatives' choice and their attractiveness, that is, the exact matching is broken. Although this model is widely used for simulation of decision-making processes (see for example, [4], [5], [6], [7], [8], [9], [10]), its applications are restricted by two factors. First, the model is phenomenological, that is, it is not based on the deep principles of the information processing by live organisms. Second, attempts to generalize this model to the multi-alternative choice were unsuccessful, because they were not justifiable. In this work we demonstrate that the model can be deduced from a very general definition of a self-reflexive system and an assumption which we called the Axiom of the Second Choice.

It turns out that this model corresponds to the equation connecting an internal variable of the self-reflexive system with a partial derivative of the entropy of the environmental influence and with a partial derivative of the entropy of choice. This connection allows us to generalize the model to a case with $n \geq 2$ alternatives.

2. SELF-REFLEXIVE SYSTEM

A system is called *self-reflexive* if it can be presented with the following two functions:

$$B = \Phi_p(S) \ , \tag{2}$$

$$\Phi_p^{-1}(B) = S \ , \tag{3}$$

where $B \in A_1$, $p \in A_2$, and $S \in A_3$. Variable B represents the behavior of the system, parameter p corresponds to the influence of the environment at the moment of choice, and S is an internal variable ('self') whose value does not depend on the environmental influence at the moment of choice. A_1 is a set of the system's actions, A_2 is a set of the environmental influences, and A_3 is a set of values of the internal variable. Equation (2) corresponds to the statement that under fixed environment's influence, the behavior of the system may change only as a result of the internal variable change. Equation (3) shows that a reverse correlation must exist between the behavior and the internal variable [11].

3. THE PARTIAL DERIVATIVE OF ENTROPY

Consider Shannon's function

$$H = -\sum_{i=1}^{n} u_i \ln u_i \; , \tag{4}$$

where $0<u_i<1$, $u_1 + u_2 + ... + u_n = 1$.
This function can be presented as

$$H(u_1, u_2, \ldots, u_{n-1}) = -\sum_{i=1}^{n-1} u_i \ln u_i - u_n \ln u_n \; ,$$

where $u_n = 1 - u_1 - u_2 - \ldots - u_{n-1}$. Variables $u_1, u_2, ..., u_i, ..., ,u_{n-1}$ will be called *locally independent* if for any set of their values, a change of u_i for Δu_i, which is $o(u_n)$, does not change the value of $u_j \, (j \neq i, \, j < n)$, but changes only u_n for

$$\Delta u_n = - \Delta u_i \, .$$

In this case,

$$\frac{\partial}{\partial u_j} H(u_1, u_2, \ldots, u_{n-1}) = - \sum_{i=1}^{n} \frac{\partial u_i}{\partial u_j} \ln(e u_i) = - \ln \frac{u_j}{u_n}, \tag{5}$$

$j = 1, 2, ..., n\text{-}1$, where $u_n = 1 - u_1 - u_2 - ... - u_{n-1}$. Since function $H(u_1, u_2, ..., ,u_{n-1})$ has a continuous partial derivative on every variable in the vicinity of each point in the open interval (0,1), then this function has a total differential in any point of the open interval (0,1):

$$dH = -\sum_{j=1}^{n-1} \ln \frac{u_j}{u_n} du_j \; ,$$

where

$$- \ln \frac{u_j}{u_n} = \frac{\partial}{\partial u_j} H(u_1, u_2, \ldots, u_{n-1}) \, .$$

Let us introduce (n-1)-dimensional Euclidian space. By analogy with mechanics we will consider vector

$$F = (-\ln \frac{u_1}{u_n}, -\ln \frac{u_2}{u_n}, \ldots, -\ln \frac{u_{n-1}}{u_n}) = grad \; H(u_1, u_2, \ldots, u_{n-1})$$

as a vector of "force" and vector

$$R = (du_1, du_2, \ldots, du_{n-1})$$

as a vector of an elementary shift. Now dH can be presented as a scalar product

$$dH = F \bullet R \; .$$

Therefore, function $H(u_1, u_2, ..., u_{n-1})$ can be considered as potential of "entro-force" (F) affecting a certain point, and $-\ln \frac{u_j}{u_n}$ (j = 1, 2, ..., n-1) as projection of "force" to the axis of Euclidian space. The "force" field corresponding to potential H will be called the field of *entro-forces* or *entro-field*.

4. A MODEL OF TWO-ALTERNATIVE BIPOLAR CHOICE

While describing a model of the two-alternative bipolar choice, we assume that the choice is made by a self-reflexive system. We assume also that the process of choice consists of two stages. First, the system makes a "preliminary" choice and then either realizes or cancels it. A choice which was realized will be called *actual choice*.

Let the system face a choice between positive and negative alternatives and let the probability that the positive alternative is chosen be given by

$$X(S) = \Phi_{x_1}(S) \; , \tag{6}$$

where $0<X<1$, $0<x_1<1$, and $S \geq 0$. Let

$$X(0) = x_1. \tag{7}$$

The value of x_1 is the relative probability of the environmental push toward positive alternative. Numbers x_1 and 1-x_1 are interpreted as normalized attractiveness of the positive and negative alternatives, respectively. Thus, Eq. (7) means that when S=0, the probability of the positive alternative being chosen is equal to its normalized attractiveness. Let us introduce the following assumption:

The Axiom of the Second Choice

When the value of the internal variable grows from S to $S + \Delta S$, where ΔS is $o(S)$, and the value of x_1 remains unchangeable, the procedure of choice is as follows. First, the system performs a preliminary choice with the probability of the positive alternative being chosen is equal to $X(S)$. If it is chosen, the system realizes its choice. If the negative alternative is chosen, the system cancels its choice with probability $c\Delta S$ (where $c>0$ and does not depend on S) and repeats its choice again with the probability $X(S)$ of choosing the positive alternative, after which realizes its choice independently of the chosen alternative's polarity.

It follows from this axiom that for the value of the internal variable equal to $S + \Delta S$, an actual choice is made with the probability of

$$X(S+\Delta S) = X(S) + (1 - X(S))X(S)c\Delta S \quad , \qquad (8)$$

from where

$$\Delta X(S) = c(1 - X(S))X(S)\Delta S \ .$$

When $\Delta S \to 0$, the equation for actual choice looks as follows:

$$\frac{dX(S)}{dS} = c(1 - X(S))X(S) \ . \qquad (9)$$

Equation (9) is well known and called *logistic* (see, for example, [12]). By solving (9) under condition (7) we obtain

$$X(S) = \frac{x_1}{x_1 + (1 - x_1)\exp(-cS)} \ . \qquad (10)$$

This expression can be represented as follows:

$$\frac{1 - X}{X} = \frac{1 - x_1}{x_1}\exp(-cS) \ . \qquad (11)$$

We see that (1) is equivalent to (11), if $x_2=\exp(-cS)$.

Statement 1. Reverse function of $X = \Phi_{x_1}(S)$ exists and is given by

$$\Phi_{x_1}^{-1}(X) = \frac{1}{c}\left(\frac{\partial H(x_1)}{\partial x_1} - \frac{\partial H(X)}{\partial X}\right) , \qquad (12)$$

where

$$H(x_1) = -x_1 \ln x_1 - (1 - x_1)\ln(1 - x_1) \ ,$$
$$H(X) = -X \ln X - (1 - X)\ln(1 - X) \ ,$$

and $0<x_1<1$, $0<X<1$.

Proof. By substitution $X(S)$ from (10) to (12) we obtain

$$\frac{1}{c}\left(\frac{\partial H(x_1)}{\partial x_1} - \frac{\partial H(X(S))}{\partial X(S)}\right) = S \ . \qquad (13)$$

We see that the self-reflexive system performing binary bipolar choice is described by Eqs. (10) and (12) which correspond to general Eqs. (2) and (3). Statement 1, in its essence, is a new theorem of the Information Theory. It establishes fundamental connections between the derivative of the entropy of the environment influence, $\frac{\partial}{\partial x_1}H(x_1)$, and the derivative of the entropy of choice, $\frac{\partial}{\partial X}H(X)$, and the internal variable S.

5. A MODEL OF MULTI-ALTERNATIVE BIPOLAR CHOICE

A connection of function $\Phi_{x_1}^{-1}(X)$ with the derivative of the entropy of choice allows us to extend the two-alternative model on the choice between any finite number of alternatives $n\geq 2$.

Let the self-reflexive system face a choice between n alternatives numbered $i=1,2,...,n$. Distribution of the probabilities of the environment influences is designated as $q_1,q_2,...q_n$ and called *normalized attractiveness*; distribution of the probabilities of choices is $p_1,p_2,...p_n$. Let

$$H_1 = -\sum_{i=1}^{n} q_i \ln q_i \quad \text{and} \quad H_2 = -\sum_{i=1}^{n} p_i \ln p_i \ ,$$

$0<q_i<1$ and $0<p_i<1$.

Let alternatives $1,2,...,n$-1 be correlated with the positive pole, and alternative n with the negative one. We will call alternatives $1,2,...,n$-1 *positive*, and alternative n *negative*. We assume that variables $q_1,q_2, ...,q_{n-1}$ as well as variables $p_1,p_2,...,p_{n-1}$ are locally independent, that is, any small changes of their values can be compensated only by changes of variables q_n and p_n corresponding to the negative alternative. Therefore, in the generalized model, as well as in the two-alternative model, a change of a value belonging to the positive pole for ε leads to a change of a corresponding value belonging to the negative pole for $-\varepsilon$. The following system of equation is a generalization of (13):

$$\frac{1}{c}\left(\frac{\partial H(q_1,q_2,...,q_{n-1})}{\partial q_j}-\frac{\partial H(p_1,p_2,...,p_{n-1})}{\partial p_j}\right)=S_j$$

$S_j \geq 0, j = 1, 2, ..., n-1.$ (14)

Consider each variable p_j where j=1,2,..., n-1, as an implicit function of independent variables $q_1,q_2,...,q_{n-1}$, $S_1,S_2,...,S_{n-1}$. By using (5) we can rewrite (14) as

$$\frac{1}{c}(-\ln\frac{q_j}{q_n}+\ln\frac{p_j}{p_n})=S_j, j=1, 2, ..., n-1 \quad (15)$$

or

$$\frac{p_n}{p_j}=\frac{q_n}{q_j}\exp(-cS_j)\ , \ j=1, 2, ..., n-1, \quad (16)$$

where $q_n = 1 - q_1 - q_2 - ... - q_{n-1}$
and $p_n = 1 - p_1 - p_2 - ... - p_{n-1}$.

By solving system (16) we obtain

$$p_j=\frac{q_j}{\sum_{i=1}^{n-1} q_i\exp(c(S_i-S_j))+q_n\exp(-cS_j)}\ , \quad (17)$$

$j = 1, 2, ..., n-1.$

We will show now that the set of function (17) is compatible with the condition of local independence of variables p_j. Let us fix the values of all q_j. For any set of p_j values, due to (15), there is a set of S_j values which satisfies system (17). Therefore, for any o(p_n) change of p_j it is possible to find a set of $S_1,S_2,...,S_{n-1}$ such that the values of p_i (where $i \neq j$ and $i<n$) do not change. Due to functions (15) continuity, small changes of p_i corresponds to small changes of S_j.

The model can be extended to the arbitrary number of positive and negative alternatives under assumption that the system aggregates a set of negative alternatives into one negative alternative. After such aggregation, the model can predict the probability that an alternative from the set of negative alternatives will be chosen, but cannot predict the probability of choice for every negative alternative separately.

6. THE ENTRO-FIELD

Let us introduce a set of variables $I_j = -cS_j$. Now Eqs. (14) can be rewritten as

$$\frac{\partial H(p_1,p_2,\dots,p_{n-1})}{\partial p_j}= I_j+\frac{\partial H(q_1,q_2,\dots,q_{n-1})}{\partial q_j}, \quad (18)$$

where j = 1, 2, ..., n-1, or in a vector form:

$$grad\ H(p_1,p_2,\dots,p_{n-1}) = I + grad\ H(q_1,q_2,\dots,q_{n-1}). \quad (19)$$

Thus, it follows from (19) that there is an entro-field which is a sum of two entro-fields; the source of the one (internal entro-field, I) is a reflexive system, and the source of the second field is the world influencing the reflexive system (external entro-field, grad $H(q_1, q_2, ..., q_{n-1})$). Partial derivatives of the potential ($H(p_1, p_2, ..., p_{n-1})$) correspond to projections of vector (grad $H(p_1, p_2, ..., p_{n-1})$) to the main axises. Each axis corresponds to one positive alternative. Each projection is interpreted as an entro-force inclining the reflexive system to choose a corresponding positive alternative. It is easy to see that in a situation of "Buridan donkey", i.e., if

$$q_1 = q_2 = \dots = q_n = \frac{1}{n}\ ,$$

then grad $H(q_1, q_2, ..., q_{n-1}) = 0$, and the reflexive system's choice does not depend on the external world's influence at the moment of choice.

7. EMPIRICAL EVIDENCE OF THE EXISTENCE OF ENTRO-FORCES

Herrnstein [13] and Baum [14], [15] and their followers have conducted numerous experiments with pigeons and rats, who were placed in Skinner box with two keys. Reinforcement for a press on a key was given rarely and scarcely. Animals could touch keys with any frequency. This long-term research resulted in the following empirical correlation:

$$\ln\frac{N_2}{N_1}=c+\ln\frac{n_2}{n_1}\ , \quad (20)$$

where N_1 and N_2 are numbers of touching keys, n_1 and n_2 are numbers of reinforcements, and c<0 is a constant parameter characterizing a particular animal. Thus, the animals used only the strategies satisfying Eq. (20) which is analogous to (18) when n = 2.

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