MOISHEZON SPACES IN RIGID GEOMETRY

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Abstract. We prove that all proper rigid-analytic spaces with "enough" algebraically independent meromorphic functions are algebraic (in the sense of proper algebraic spaces). This is a non-archimedean analogue of a result of Artin over $\mathbf{C}$.

1. Introduction

1.1. **Motivation.** We begin with a review some relevant notions in complex-analytic and non-archimedean geometry so as to put the main result in context. Let X be a proper complex-analytic space. It is a classic result of Remmert [R, Thm. 2] that if X is reduced and irreducible then the field of meromorphic functions on X is finitely generated over $\mathbf{C}$ with transcendence degree at most $\dim X$. In general, X is called *Moishezon* if each of its irreducible components X_i (endowed with reduced structure) has meromorphic function field $\mathscr{M}(X_i)$ with transcendence degree over $\mathbf{C}$ equal to $\dim(X_i)$. (We refer to [C1, §2] for the definition and basic properties of irreducible components of rigid-analytic spaces.) In [M1], Moishezon showed that any Moishezon space is related to the analytification $\mathscr{X}^{\rm an}$ of a proper $\mathbf{C}$-scheme $\mathscr{X}$ via a proper birational correspondence, and he gave examples that do not arise in the form $\mathscr{X}^{\rm an}$ for proper $\mathbf{C}$-schemes $\mathscr{X}$.

Moishezon established other stability properties of the category of Moishezon spaces, such as under images and closed subspaces, and in [M2] he proved Chow's Lemma and resolution of singularities for reduced Moishezon spaces by using blow-up along smooth centers. The problem of enlarging the category of schemes to make a good notion of "algebraicity" for Moishezon spaces therefore became the problem of developing an algebro-geometric theory of contractions along closed subvarieties.

Such a contraction operation generally cannot be done within the category of schemes, so in the series of papers [M3] Moishezon independently developed a theory of mini-schemes, contemporaneously with Artin's similar theory of algebraic spaces; his aim was to eventually establish a suitable contraction result for mini-schemes so that he could then prove that Moishezon spaces are algebraic in the sense of proper mini-schemes over $\mathbf{C}$. The required contraction theorem was later proved by Artin [A2, 6.11] for algebraic spaces. The combined work of Moishezon (including his analytic resolution of singularities) and Artin explained the similarities between Moishezon spaces and proper $\mathbf{C}$-schemes: analytification of proper algebraic spaces over $\mathbf{C}$ sets up an equivalence of categories with the category of Moishezon spaces. That is, all Moishezon spaces have a unique and functorial underlying algebraic structure in the sense of GAGA for proper algebraic spaces over $\mathbf{C}$. (In [A2, §7], Artin gave another proof of this result, bypassing Moishezon's work on resolution for Moishezon spaces.)

To make sense of a non-archimedean version of Artin's theorem, it is necessary to construct an analytification functor on proper algebraic spaces over non-archimedean fields k (and to check that the GAGA theorems carry over). This amounts to constructing quotients by certain étale equivalence relations in rigid geometry. Briefly, suppose that $\mathscr{R} \rightrightarrows \mathscr{U}$ is an étale equivalence relation in schemes such that the diagonal $\mathscr{R} \to \mathscr{U} \times \mathscr{U}$ is quasi-compact and the quotient sheaf $\mathscr{U}/\mathscr{R}$ for the big étale site is an algebraic space $\mathscr{X}$ locally of finite type over k. The problem is to determine if the étale equivalence relation $\mathscr{R}^{\rm an} \rightrightarrows \mathscr{U}^{\rm an}$ admits a quotient (suitably defined) in the category of rigid-analytic spaces over k, in which case such a quotient (and its existence) is independent of the choice of étale chart for $\mathscr{X}$ and so may be denoted $\mathscr{X}^{\rm an}$. (See [CT,

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§2] for a more detailed discussion of this definition of analytification, as well as its functorial dependence on $\mathscr{X}$ when it exists.)

In the complex-analytic case this quotient problem is very easy to study, and if $\mathscr{X}$ is an algebraic space locally of finite type over $\operatorname{Spec} \mathbf{C}$ then a necessary and sufficient condition for the analytification of $\mathscr{X}$ to exist is that $\mathscr{X}$ is *locally separated* over $\operatorname{Spec} \mathbf{C}$ (in the sense that its quasi-compact diagonal map $\Delta_{\mathscr{X}/\mathbf{C}}$ is an immersion). In the non-archimedean case, this quotient problem is rather more subtle. An initial indication of the difficulties is the surprising fact that although (as over $\mathbf{C}$) local separatedness remains a necessary condition for analytifiability [CT, 2.2.5], it fails to be sufficient. Counterexamples are given in [CT, 3.1.1] over any non-archimedean field k, including the scalar extension to k of certain smooth algebraic spaces of dimension 2 over the prime field. By using Berkovich spaces (and local techniques for studying them [T1], [T2]), these difficulties were overcome in [CT, 4.2.1], where it was shown that every *separated* algebraic space locally of finite type over k is analytifiable in the sense of rigid geometry. (This is a consequence of a general existence result [CT, 4.2.2] for quotients of étale equivalence relations $R \rightrightarrows U$ in Berkovich's category of k-analytic spaces.) The GAGA formalism carries over [CT, 3.3].

1.2. **Main result.** Since Remmert's theorem on the structure of meromorphic function fields in the proper complex-analytic case remains valid over k (due to Bosch [B2]), it is natural to carry over the definition of a Moishezon space to the case of proper rigid spaces over k. It is straightforward to check (as over $\mathbf{C}$) that if $\mathscr{X}$ is a proper algebraic space over k then its analytification $\mathscr{X}^{\mathrm{an}}$ (in the sense of [CT]) is a Moishezon space over k. Thus, it is reasonable to ask if there is an analogue of Artin's theorem in the non-archimedean case. The affirmative answer is our main result:

Theorem 1.2.1. *The functor $\mathscr{X} \rightsquigarrow \mathscr{X}^{\mathrm{an}}$ from proper algebraic spaces over k to proper rigid spaces over k is an equivalence onto the full subcategory of Moishezon spaces.*

The intervention of inseparability issues when $\operatorname{char}(k) = p > 0$ is the primary new feature. This leads to a larger role for flatness than in Moishezon's work, as well as the use of certain algebraic yet complete ground field extensions with possibly infinite degree (such as $k \to k^{p^{-m}}$ for $m \geq 1$). Also, Berkovich spaces arise in an essential way in our proof of Theorem 1.2.1, even though they do not appear in the statement. Finally, a key ingredient at the end is Artin–Popescu approximation for algebraic equations over excellent rings, applied to the local rings on good k-analytic spaces (which are excellent by a recent result of Ducros [D2]).

We begin by studying meromorphic function fields in §2; this consists largely of known results, which we gather for the convenience of the reader. In §3 we overcome several difficulties in positive characteristic to establish the non-archimedean version of Moishezon's theory. Finally, in §4 we combine the results in §3 with variants on ideas of Artin to prove the algebraicity of Moishezon spaces in rigid geometry. As we mentioned in §1.1, there are two approaches to the algebraicity result over $\mathbf{C}$: (i) Moishezon's strategy via analytic resolution (proved by him) and algebraic contractions (later proved by Artin), (ii) Artin's later direct approach [A2, §7] which avoids resolution of singularities and so is better-suited to adaptation to the non-archimedean case in arbitrary characteristic. This second approach guides our arguments in §4.

1.3. **Erratum.** I would like to take this opportunity to correct a minor error in my paper [C1]. In the final part of [C1, 4.1.1] it should have been assumed that the ground field k is discretely-valued, and so in [C1, 4.2.2, 4.2.3] this assumption should be required. The error was to appeal to an argument with an infinite product of (normalized) units on the open unit disc to "prove" the triviality of line bundles on the open unit disc. Such a product argument makes sense for a discretely-valued ground field, but over a more general non-archimedean ground field there are convergence problems and it is not true that such line bundles are trivial. In fact, over $\mathbf{C}_p$ one can construct an effective divisor in the open unit disc that is not the divisor of zeros of an analytic function (and so the corresponding invertible ideal sheaf is not globally trivial). See [FvP, 2.7.8] for how to construct such examples using that $\mathbf{C}_p$ is not spherically complete.

1.4. **Conventions.** In the rigid-analytic setting, the ground field k is taken to be complete with respect to a fixed nontrivial non-archimedean absolute value (i.e., it is a non-archimedean field). When working with

Berkovich spaces we allow for the possibility that the absolute value on k is trivial. In general, an *analytic* extension field K/k is one that is complete with respect to a fixed absolute value extending the one on k. Algebraic spaces are locally of finite type over k (and maps between them are k-maps) unless otherwise stated, and their diagonals over $\operatorname{Spec} k$ are always assumed to be quasi-compact; in fact, only separated algebraic spaces are relevant in this paper.

Contents

2. Meromorphic function fields

2.1. **Basic properties and definitions.** As in the case of locally ringed spaces [EGA, IV_4, §20], if X is a rigid space then the sheaf of *meromorphic functions* $\mathscr{M}_X$ on X is the localization of $\mathscr{O}_X$ at the multiplicative subsheaf $\mathscr{S}_X \subseteq \mathscr{O}_X$ consisting of regular local sections; i.e., $\mathscr{S}_X(U)$ is the set of $s \in \mathscr{O}_X(U)$ such that multiplication by s on $\mathscr{O}_X|_U$ is injective (or equivalently, for reduced X, s has a nowhere dense zero locus in U). This localization contains $\mathscr{O}_X$ as a subsheaf of algebras, and for any $f \in \mathscr{M}_X(U)$ the associated ideal sheaf of denominators

$$\mathscr{D}_f(V) = \{s \in \mathscr{O}_X(V) \mid sf \in \mathscr{O}_X(V)\} \tag{2.1.1}$$

for $V \subseteq U$ is easily seen to be coherent. The coherent ideal $\mathscr{D}_f$ locally contains a section of $\mathscr{S}_X$, and for any two coherent ideals $\mathscr{I}$ and $\mathscr{J}$ on X that locally contain sections of $\mathscr{S}_X$ the product ideal does too. For any such coherent ideal $\mathscr{I}$ we naturally have $\operatorname{Hom}(\mathscr{I}, \mathscr{O}_X) \subseteq \mathscr{M}_X(X)$, and there is a natural global isomorphism

$$\varinjlim \operatorname{Hom}(\mathscr{I}, \mathscr{O}_X) \simeq \mathscr{M}_X(X) \tag{2.1.2}$$

where $\mathscr{I}$ ranges over the directed system (by reverse inclusion) consisting of coherent ideals on X that locally contain an $\mathscr{O}_X$-regular section. This constraint on $\mathscr{I}$ says exactly that $\mathscr{I}_x$ has positive $\mathscr{O}_{X,x}$-depth for all $x \in X$, and it is equivalent to check this property on completed stalks for all $x \in X$.

The description of $\mathscr{M}_X(X)$ in (2.1.2) has several consequences. First, by rigid GAGA for proper algebraic spaces [CT, 3.3], the analytification functor from proper algebraic spaces over k to proper rigid spaces over k

naturally commutes with the formation of the k-algebra of global rational/meromorphic functions. Second, for each affinoid open $U = \mathrm{Sp}(A)$ in X the A-algebra $\mathscr{M}_X(U)$ is uniquely isomorphic to the total ring of fractions of A, and $\mathscr{M}_{X,x}$ is the total ring of fractions of $\mathscr{O}_{X,x}$ for all $x \in X$. Hence, if $\pi : \widetilde{X} \to X$ is the normalization of a reduced space X then there is a unique isomorphism $\mathscr{M}_X = \pi_*(\mathscr{M}_{\widetilde{X}})$ over the map $\mathscr{O}_X \to \pi_*(\mathscr{O}_{\widetilde{X}})$. By using results in [C1, §2] we can also carry over arguments in the complex-analytic case [CAS, 9.1.2] to show that $\mathscr{M}_X(X)$ is a field when X is reduced and irreducible.

To work with normalizations when studying meromorphic functions, a key ingredient is the Riemann extension theorem for normal rigid spaces. This result was first proved by Lütkebohmert [Lüt, Thm. 1.6] under a hypothesis of geometric normality (in the sense of [C1, 3.3.6]), and then by Bartenwerfer [Bar, §3] in general by an independent method. It is also possible to deduce the general case from the geometrically normal case by using general results concerning ground field extension functors in rigid geometry [C1, §3] and the excellence of affinoid algebras and local rings on rigid spaces ([K3], [C1, §1]). Here is the result.

Theorem 2.1.1 (Bartenwerfer). *If X is a normal rigid space over k and $Z \subseteq X$ is a nowhere dense analytic set then any bounded analytic function h on $U = X - Z$ uniquely extends to X.*

To define a rigid-analytic pullback map for meromorphic functions in the reduced case, the following result is convenient and can be proved exactly as in the complex-analytic case [CAS, 8.4.3] with the help of Theorem 2.1.1; cf. [Ber1, 3.3.18].

Corollary 2.1.2. *Let $f : X \to Y$ be a map between reduced rigid spaces, with respective nowhere dense analytic non-normal loci $N(X)$ and $N(Y)$. If the analytic set $f^{-1}(N(Y))$ in X is nowhere dense then f uniquely lifts to a map $\widetilde{X} \to \widetilde{Y}$ between the normalizations.*

If $\pi : X \to Y$ is a surjective map between connected normal rigid spaces then there is a unique map of sheaves of algebras $\pi^* : \mathscr{M}_Y \to \pi_*(\mathscr{M}_X)$ over $\mathscr{O}_Y \to \pi_*(\mathscr{O}_X)$, and if π is proper and *birational* (in the sense that π restricts to an isomorphism over $Y - Z$ for a nowhere dense analytic set $Z \subseteq Y$ with nowhere dense preimage in X) then Theorem 2.1.1 permits the same argument as in the complex-analytic case [CAS, 8.1.3] to be used to show that the map $\pi^* : \mathscr{M}_Y \to \pi_*(\mathscr{M}_X)$ is an isomorphism. In particular, global meromorphic functions on X are identified with global meromorphic functions on Y when π is proper and birational, even if X and Y are merely reduced (and possibly reducible).

For our work with meromorphic functions, it will be convenient to be able to assign Weil divisors to invertible meromorphic functions. Let us briefly review how this goes on a normal rigid space S. Consider an invertible meromorphic function $g \in \mathscr{M}_S(S)^\times$. By the excellence of affinoid algebras ([K3], [C1, 1.1.1]), the locus $\mathrm{Reg}(S)$ of $s \in S$ such that $\mathscr{O}_{S,s}$ is regular is the complement of an analytic set in S with codimension ≥ 2. Let $Z_\infty(g)$ and $Z_0(g)$ be the closed subspaces of S cut out by the ideal sheaves of "denominators" of g and $1/g$ in the sense of (2.1.1). These closed subspaces meet $\mathrm{Reg}(S)$ in effective Cartier divisors. Define the analytic set $D_0(g) \subseteq Z_0(g)$ (resp. $D_\infty(g) \subseteq Z_\infty(g)$) to be the union of the codimension-1 irreducible components of $Z_0(g)$ (resp. $Z_\infty(g)$) that meet $\mathrm{Reg}(S)$. Since $S - D_\infty(g)$ is a normal space in which $Z_\infty(g) - D_\infty(g)$ has codimension ≥ 2, the analytic function g on $S - Z_\infty(g)$ extends to $S - D_\infty(g)$ by Theorem 2.1.1. Hence, $D_\infty(g) = Z_\infty(g)$ as subsets of S. Working with $1/g$ likewise gives that $D_0(g) = Z_0(g)$ as analytic sets in S, so $D_0(g)$ and $D_\infty(g)$ each admit a natural structure of Weil divisor with positive multiplicities. It is clear that $D_0(g)$ and $D_\infty(g)$ have no irreducible components in common, so the Weil divisor $\mathrm{div}(g) := D_0(g) - D_\infty(g)$ determines $D_0(g)$ and $D_\infty(g)$. We respectively call $D_0(g)$ and $D_\infty(g)$ the *zero locus* and *polar locus* of g on S, and also the *zero part* and *polar part* of $\mathrm{div}(g)$.

Lemma 2.1.3. *Let s be a point in a rigid-analytic space S, and $Z \subseteq S$ an analytic set through $s \in S$. Then s has a base of open neighborhoods $U \subseteq S$ such that the set $\{Z_{i,U}\}$ of irreducible components of $Z \cap U$ is in bijective correspondence with the set of minimal primes of $\mathscr{O}_{Z,s}$ via $Z_{i,U} \mapsto \ker(\mathscr{O}_{Z,s} \twoheadrightarrow \mathscr{O}_{Z_{i,U},s})$.*

In the complex-analytic case, this lemma follows from [CAS, 4.1.3, 9.2.3]. We do not know if U in the lemma can be taken to be affinoid; already in the special case $Z = S$ with $\mathscr{O}_{S,s}$ a domain, the existence of a base of irreducible *affinoid* neighborhoods of s seems to be rather non-trivial; see [P, Cor. 4.7].

Proof. The proof is identical to the complex-analytic case, but in that case the ingredients are somewhat scattered throughout [CAS, 9.1, 9.2], so we give the details here for the convenience of the reader. (A related argument in the setting of Berkovich spaces is given in [D2, 0.11–0.14], combined with the global theory of irreducible components of Berkovich spaces developed in [D2, §4]).

By standard coherence arguments, after shrinking of S around s we may define reduced analytic sets Z_i in Z which cover Z and are cut out by coherent ideals in $\mathscr{O}_Z$ whose stalks at s are the minimal primes of $\mathscr{O}_{Z,s}$. In particular, each $\mathscr{O}_{Z_i,s}$ is a domain, so its normalization is a (local) domain. Hence, by [C1, 2.1.1], the finite surjective normalization $p_i : \widetilde{Z}_i \to Z_i$ has a unique point over s. But the irreducible components of Z_i are (by definition in [C1]) the images of the connected components of $\widetilde{Z}_i$, so there is a unique irreducible component Z_i' of Z_i passing through s.

Since the set of irreducible components of Z_i is locally finite [C1, 2.2.1(3)], the union of any set of irreducible components of Z_i is an analytic set in Z_i and hence in S. Thus, the union Y_i of irreducible components of Z_i distinct from Z_i' is an analytic set in Z_i. Let $Y = \cup Y_i$, so the Zariski-open locus $Z_i - (Z_i \cap Y)$ in Z_i around s has normalization that is the complement in $\widetilde{Z}_i$ of a proper analytic set (by [C1, 2.2.1(2),(3)]). Such a complement in a connected normal rigid space is connected (by Theorem 2.1.1 applied to idempotents), so each $Z_i - (Z_i \cap Y)$ has connected normalization and hence is irreducible (cf. [C1, 2.2.3]). Thus, $U := S - Y$ is an open neighborhood of s such that $Z \cap U$ is covered by the irreducible analytic sets $Z_{i,U} := Z_i - (Z_i \cap Y)$ whose coherent ideals in $\mathscr{O}_{Z,s}$ are the minimal prime ideals. In particular, there are no inclusions among the $Z_{i,U}$, so the $Z_{i,U}$ are the irreducible components of $Z \cap U$ (by [C1, 2.2.8]). ■

Since all local rings on irreducible rigid spaces have the same dimension (see the discussion immediately preceding [C1, 2.2.3]), it follows that under the correspondence in Lemma 2.1.3 the dimension of $Z_{i,U}$ coincides with the dimension of the quotient of $\mathscr{O}_{Z,s}$ by the corresponding minimal prime ideal. Applying this for codimension-1 analytic sets in admissible open subsets of the normal rigid space S, it follows that the formation of $\operatorname{div}(g)$ is local on S (if we allow that irreducible components of a Weil divisor may become reducible under localization) and is compatible with passage to local rings on S. Hence, by normality of S, $\operatorname{div}(g)$ has vanishing polar part if and only if $g \in \mathscr{O}_S(S)$, and $\operatorname{div}(g_1 g_2) = \operatorname{div}(g_1) + \operatorname{div}(g_2)$ for $g_1, g_2 \in \mathscr{M}_S(S)^\times$. In particular, $\operatorname{div}(g)$ determines g up to multiplication by $\mathscr{O}_S(S)^\times$.

This procedure carries over verbatim to associate Weil divisors to trivializations of the invertible $\mathscr{M}_S$-module $\mathscr{M}_S \otimes \mathscr{L}$ for any line bundle $\mathscr{L}$ on S (the preceding being the case $\mathscr{L} = \mathscr{O}_S$). An interesting example is $\mathscr{L} = \Omega^n_{S/k}$ if S is k-smooth with pure dimension n, as we will use in Step 4 of the proof of Theorem 3.3.2.

Example 2.1.4. If X is a reduced rigid space then any morphism $X \to \mathbf{P}^1$ that is not identically equal to ∞ on any irreducible component of X is naturally identified with a meromorphic function on X. For later purposes, we wish to review a technique of removing the indeterminacy locus of an arbitrary meromorphic function on X so as to promote it to a morphism to $\mathbf{P}^1$ (since we will need to generalize the method to the case when X is not reduced; see Remark 4.4.5.) Choose $f \in \mathscr{M}_X(X)$. We seek to construct a proper birational map $\pi : X' \to X$ from a reduced rigid space X' so that when f is viewed in $\mathscr{M}_{X'}(X') = \mathscr{M}_X(X)$ it arises from a morphism $X' \to \mathbf{P}^1$. One approach is to work with the blow-up of X (in the sense of [C2, 4.1.1]) along the coherent ideal sheaf $\mathscr{D}_f$ of denominators of f. The following direct geometric construction seems better-suited to generalization to the non-reduced case, as we will require later (see Remark 4.4.5).

Let $U \subseteq X$ be the dense Zariski-open complement of the analytic set cut out by the coherent ideal sheaf of denominators of f. In $U \times \mathbf{P}^1$ there is a reduced analytic set Γ given by the graph of f, and Γ projects isomorphically onto U. The desired X' will be the unique analytic set in $X \times \mathbf{P}^1$ containing Γ as a dense Zariski-open subspace, but we shall need to work locally to see that such a global analytic set exists.

There is certainly at most one reduced analytic set X' in $X \times \mathbf{P}^1$ that meets the Zariski-open $U \times \mathbf{P}^1$ in Γ as a dense Zariski-open subspace of X', and the same holds if U is replaced with a dense Zariski-open subset $V \subseteq U$ (and Γ is replaced by $\Gamma|_V$). This unique characterization of X' localizes well on X, so for the proof of existence of X' we may work locally on X. Hence, we can assume $f = a/b$ with $a, b \in \mathscr{O}_X(X)$ and b nowhere a zero-divisor on $\mathscr{O}_X$. The Zariski-open subset $\{b \neq 0\}$ in X is dense and contained in U. The closed subspace $Y \subseteq X \times \mathbf{P}^1$ defined by $at_1 = bt_0$ (for homogeneous coordinates t_0, t_1 on $\mathbf{P}^1$) has pure codimension 1 and its restriction over $\{b \neq 0\}$ is the graph of $f = ab^{-1}$ over $\{b \neq 0\}$. Each irreducible component of this graph over $\{b \neq 0\}$ is a dense Zariski-open subspace in a unique irreducible component of Y, so take X' to be the reduced union of these components of Y.

2.2. **Remmert's theorem.** The inspiration for considering Moishezon's theory in rigid geometry is the following analogue of a classical theorem of Remmert over $\mathbf{C}$.

Theorem 2.2.1. *If X is a proper, reduced, and irreducible rigid space over k then the field $\mathscr{M}_X(X)$ of meromorphic functions on X is a finitely generated extension of k with transcendence degree at most* $\dim X$.

See [Ber1, §3.6] for a discussion of this result in the framework of k-analytic spaces, also using a proof modeled on the complex-analytic case. A proof was given by Bosch in [B2], but it is difficult to find a copy of that reference, so for the convenience of the reader we provide a sketch of a proof that is probably the same as Bosch's proof.

Proof. The complex-analytic analogue is proved in [CAS, 10.6.1–10.6.7], and when that method is adapted to the non-archimedean setting (with the help of rigid GAGA, Corollary 2.1.2, and some straightforward modifications to account for both the use of the Tate topology rather than a classical topology and the phenomenon of inseparability when k has positive characteristic) we get a slightly weaker result: $\mathscr{M}_X(X)$ is a purely inseparable algebraic extension of a finitely generated field over k with transcendence degree d at most $\dim X$. (To briefly explain the difficulty with inseparability, the details of which are not essential for what follows, we use the notation in [CAS, 10.6]. The place where inseparability enters in the lemma is [CAS, 10.6.6], where the map $\nu : Y_f \to Y$ between projective normal (rigid-analytic) varieties has geometrically connected fibers and so induces a finite extension on meromorphic function fields that is purely inseparable. In positive characteristic there arises the new difficulty of showing that this map of function fields is actually an isomorphism, as otherwise the first theorem in [CAS, 10.6.7] would have a weaker conclusion, involving not $\iota(\mathscr{M}_Y(Y))$ but rather its its perfect closure in $\mathscr{M}_X(X)$, so the remainder of the argument would collapse.)

To overcome these inseparability problems, we use Stein factorization. (See the second part of the proof of [Ber1, 3.6.8] for an alternative argument.) Choose a transcendence basis $\{t_1, \dots, t_d\}$ for $\mathscr{M}_X(X)$ over k, so $\mathscr{M}_X(X)/k(t_1, \dots, t_d)$ is an algebraic extension. The procedure used in the complex-analytic proof provides (via straightforward adaptation to the non-archimedean case) a pair of normal connected proper rigid spaces X' and Y with Y projective, as well as surjective maps $\mu : X' \to X$ and $\eta : X' \to Y$ such that μ is birational, η is its own Stein factorization, and the subfield $\mathscr{M}_Y(Y) \subseteq \mathscr{M}_{X'}(X') = \mathscr{M}_X(X)$ over k contains $t_1, \dots, t_d$. In particular, $\mathscr{M}_{X'}(X')$ is algebraic over $\mathscr{M}_Y(Y)$. We will show that $\mathscr{M}_Y(Y)$ is algebraically closed in $\mathscr{M}_{X'}(X') = \mathscr{M}_X(X)$, so this algebraic extension is trivial. This would give that $\mathscr{M}_X(X) = \mathscr{M}_Y(Y)$. By GAGA and the projectivity of Y, the field $\mathscr{M}_Y(Y)$ is finitely generated over k with transcendence degree $\dim Y$, so we would therefore be done.

Let K be an intermediate finite extension in the algebraic extension $\mathscr{M}_{X'}(X')/\mathscr{M}_Y(Y)$, and let the finite map $h : Y' \to Y$ be the normalization of the projective Y in K (in either the algebraic or analytic senses, which are compatible [C1, 2.1.3]). Our goal is to prove $K = \mathscr{M}_Y(Y)$, or equivalently that the finite covering map h is an isomorphism. We claim that $\eta : X' \to Y$ uniquely factors through h via a map $X' \to Y'$ inducing the inclusion $K \subseteq \mathscr{M}_{X'}(X')$ on meromorphic function fields. The uniqueness is clear since Y' is projective, and for existence we let $\mathscr{K} = h_*(\mathscr{M}_{Y'})$ on Y. Again using projectiveness, GAGA provides an evident isomorphism of sheaves of algebras $K \otimes_{\mathscr{M}_Y(Y)} \mathscr{M}_Y \simeq \mathscr{K}$ (where we abuse notation by writing K and $\mathscr{M}_Y(Y)$ to denote the associated constant sheaves on Y). Thus, we get an $\mathscr{O}_Y$-algebra map

$$\theta : \mathscr{K} \to \mathscr{M}_{X'}(X') \otimes_{\mathscr{M}_Y(Y)} \mathscr{M}_Y \to \eta_*(\mathscr{M}_{X'})$$

over the map $\mathscr{O}_Y \to \eta_*(\mathscr{O}_{X'})$. Hence, local sections of the subsheaf $h_*(\mathscr{O}_{Y'}) \subseteq \mathscr{K}$ are carried into local sections of $\eta_*(\mathscr{M}_{X'})$ that are integral over $\mathscr{O}_{X'}$. By normality of X', it follows that θ carries $h_*(\mathscr{O}_{Y'})$ into $\eta_*(\mathscr{O}_{X'})$, and by finiteness of h this sheaf map arises from a unique morphism $X' \to Y'$ that does the job.

Our problem now is to show that if a finite surjection $h : Y' \to Y$ between connected normal rigid spaces is intermediate to the map $\eta : X' \to Y$ that is its own Stein factorization (so $X' \to Y'$ is surjective) then h is an isomorphism. Pullback of functions provides inclusions

$$\mathscr{O}_Y \hookrightarrow h_*(\mathscr{O}_{Y'}) \hookrightarrow \eta_*(\mathscr{O}_{X'})$$

whose composite is the canonical map that is an isomorphism since η is its own Stein factorization. Thus, the first inclusion is an equality, and since h is finite this gives $Y' = Y$ as desired. ■

3. Moishezon spaces

3.1. **Behavior under alterations.** By GAGA for proper algebraic spaces over k [CT, 3.3], analytification is a fully faithful functor from proper algebraic spaces over k to proper rigid spaces over k. If $\mathscr{X}$ is a proper algebraic space over k then since $\mathscr{X}$ contains a dense open subscheme it follows (via the compatibility of analytification with respect to the formation of irreducible components with the reduced structure [CT, 2.3.3]) that the upper bound in Theorem 2.2.1 is attained by the irreducible components (endowed with reduced structure) of $\mathscr{X}^{\mathrm{an}}$. In other words, exactly as over $\mathbf{C}$, analytification carries proper algebraic spaces to rigid spaces satisfying the following definition.

Definition 3.1.1. A rigid space X over k is *Moishezon* if it is proper and each irreducible component X_i of X, endowed with its reduced structure, satisfies $\mathrm{trdeg}_k(\mathscr{M}_{X_i}(X_i)) = \dim X_i$.

Our ultimate goal is to prove that the Moishezon property characterizes the essential image of analytification of proper algebraic spaces over k. That is, we will prove that all Moishezon spaces are algebraic in the sense of (proper) algebraic spaces. It is necessary to first record how the Moishezon condition behaves with respect to alterations, which we define as follows.

Definition 3.1.2. A map $f : X \to Y$ between irreducible and reduced proper rigid spaces over k is an *alteration* if $\dim X = \dim Y$ and f is surjective.

The analogy with algebraic geometry and the complex-analytic case suggests that an alteration should be generically finite, and so we first use formal models to check that this is true.

Lemma 3.1.3. *Let $f : X \to Y$ be a proper surjection between non-empty rigid spaces with the same pure dimension $d \geq 0$. There is a nowhere dense analytic set $Z \subseteq Y$ such that $Y - Z$ is the set of $y \in Y$ such that $f^{-1}(y)$ is finite. In particular, f is finite over $Y - Z$.*

Proof. By [K2, 3.7] (or [D1, 3.5]), the locus of $x \in X$ such that x is isolated in $X_{f(x)}$ is a Zariski-open locus $U \subseteq X$, so $Z = Y - f(X - U)$ is the desired analytic set in Y provided that Z is nowhere dense in Y. By working locally on Y and using its global decomposition into irreducible components we see that it suffices to show $Z \neq Y$. Equivalently, we seek a point $x \in X$ that is isolated in $X_{f(x)}$. In view of the dimension hypotheses, this would follow from a generic flatness result for reduced rigid spaces analogous to the well-known version for reduced noetherian schemes. Due to lack a suitable reference for such a result, we give an argument using formal models.

We can assume that Y is quasi-compact and quasi-separated (e.g., affinoid), so there is a formal model $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ for the map f, with $\mathfrak{X}$ and $\mathfrak{Y}$ each R-flat and quasi-compact. In the proof of [C3, Thm. A.2.1] it is shown that the special fibers $\mathfrak{X}_0$ and $\mathfrak{Y}_0$ over $\widetilde{k}$ have pure dimension d. By using rig-points of formal models, we see that the map $\mathfrak{f}_0$ between these special fibers is surjective since f is surjective. Any surjective map between finite-type schemes with the same pure dimension over a field must be finite over a dense open in the target, so we get a dense open $\mathfrak{V}_0 \subseteq \mathfrak{Y}_0$ such that $\mathfrak{f}_0$ is finite over $\mathfrak{V}_0$. Thus, for the corresponding open subset $\mathfrak{V} \subseteq \mathfrak{Y}$ the map $\mathfrak{f}^{-1}(\mathfrak{V}) \to \mathfrak{V}$ is finite. The admissible open $V = \mathfrak{V}^{\mathrm{rig}} \subseteq Y$ is non-empty and $f^{-1}(V) \to V$ is finite. ■

Example 3.1.4. We now show that the Moishezon property is insensitive to alterations. Let $f : X \to Y$ be an alteration between irreducible and reduced proper rigid spaces. We will prove that f induces a finite extension between meromorphic function fields, so X is Moishezon if and only if Y is Moishezon.

Without loss of generality, X and Y are normal (use Corollary 2.1.2 to pass to the induced map between normalizations if necessary). In the birational case we have already observed immediately below Corollary 2.1.2 (via adaption of complex-analytic arguments) that the induced pullback map between meromorphic function fields is an isomorphism. In the general case, consider the Stein factorization $X \to Y' \to Y$ of f. Since Y' is reduced and irreducible, the normality of X forces Y' to be normal.

The finite map $\pi : Y' \to Y$ to the normal Y makes $\pi_*(\mathscr{M}_{Y'})$ into a locally free finitely generated sheaf of modules over $\mathscr{M}_Y$, and its rank is constant since Y is connected. Thus, the sheaf-theoretic characteristic polynomial expresses $\mathscr{M}_{Y'}(Y')$ as an algebraic (hence finite) extension of $\mathscr{M}_Y(Y)$. We can therefore rename Y' as Y so as to assume that f is its own Stein factorization. By Lemma 3.1.3 there is a nowhere dense analytic set $Z \subseteq Y$ such that f is finite over $Y - Z$, so it is an isomorphism over $Y - Z$ due to being its own Stein factorization (and the proper analytic set $f^{-1}(Z) \subseteq X$ is nowhere dense because X is irreducible). Hence, f is birational, so $\mathscr{M}_Y(Y) = \mathscr{M}_X(X)$.

The following useful corollary of the proof of Theorem 2.2.1 and the preceding example is analogous to a result [M1, Ch. I, Thm. 1] that gets Artin's inductive proof of algebraicity of Moishezon spaces off the ground over $\mathbf{C}$. It essentially says that Moishezon spaces are birational to projective schemes; a vast generalization will be given in Corollary 4.1.2.

Corollary 3.1.5. *Let X be an irreducible and reduced Moishezon space. There exists a connected normal proper rigid space X' and a connected normal projective rigid space X'' for which there are proper birational maps $X \leftarrow X' \to X''$.*

Note that X' is necessarily Moishezon by Example 3.1.4. See [Ber1, 3.6.6] for a result similar to Corollary 3.1.5 that applies without a Moishezon hypothesis and gives a proper surjection $f : X' \to X''$ inducing an isomorphism of meromorphic function fields without controlling birationality properties of f. The content in our proof of Corollary 3.1.5 is to check that such birationality must hold in the Moishezon setting.

Proof. We can replace X with its normalization so that X is normal. Take X' as in the proof of Theorem 2.2.1 (so $X' \to X$ is proper and birational), and take $X'' = Y$ in that notation. The proper surjective map $X' \to Y = X''$ was shown to induce an equality on meromorphic function fields in that proof. Since X is Moishezon, the field $\mathscr{M}_X(X) = \mathscr{M}_{X'}(X')$ has transcendence degree over k equal to the common dimension d of X and X'. The projective X'' has the same meromorphic function field, which is then equal to the underlying algebraic rational function field of X'', so its transcendence degree is $\dim X''$. Thus, $\dim X'' = d$ and it remains to prove that $X' \to X''$ is birational. More generally, as we explained in Example 3.1.4 using Lemma 3.1.3, a proper surjection $f : X' \to X''$ between irreducible and reduced rigid spaces is birational if X'' and X' have the same dimension and f is its own Stein factorization. ■

The next corollary (which we shall use later to overcome inseparability problems) is a consequence of Corollary 3.1.5 and relates local geometric properties of certain morphisms to algebraic properties in a global meromorphic function field.

Corollary 3.1.6. *Let X be an irreducible and reduced Moishezon space with dimension $n > 0$, and assume that $\mathscr{M}_X(X)$ is separable over k. Let $f_1, \dots, f_n \in \mathscr{M}_X(X)$ be a separating transcendence basis over k. Let $X' \to X$ be a proper birational map from an irreducible and reduced rigid space X' such that each f_i viewed as a meromorphic function on X' comes from a morphism $f_i : X' \to \mathbf{P}^1$. The map $(f_1, \dots, f_n) : X' \to (\mathbf{P}^1)^n$ is étale on a non-empty Zariski-open locus in X'. In particular, the Zariski-open k-smooth locus $X^{\rm sm}$ in X is non-empty and the meromorphic differential form*

$$\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n \in \Gamma(X, \mathscr{M}_X \otimes \Omega^n_{X/k})$$

restricts over $X^{\rm sm}$ to a global basis of the invertible $\mathscr{M}_{X^{\rm sm}}$-module of meromorphic n-forms.

By Example 2.1.4, an X' as in this corollary always exists.

Proof. Since X' is Moishezon, we may rename X' as X. If $\pi : X_1 \to X$ is a proper and birational map from an irreducible and reduced Moishezon space such that the result is known for X_1, say with the map $X_1 \to (\mathbf{P}^1)^n$ étale away from a proper analytic subset $Z_1 \subseteq X_1$, then the map $X \to (\mathbf{P}^1)^n$ is étale on the Zariski-open overlap of the non-empty $X - \pi(Z_1)$ with any non-empty Zariski-open locus in X over which π is an isomorphism. Hence, by Corollary 3.1.5 we can assume that X is normal and admits a proper and birational map onto a reduced and irreducible rigid space P that is projective. In particular, $\mathscr{M}_P(P) = \mathscr{M}_X(X)$ is separable over k. The algebraic theory of separable function fields provides a non-empty Zariski-open locus W in P on which the f_i's (viewed in $\mathscr{M}_P(P)$) are analytic functions whose product morphism to $(\mathbf{P}^1)^n$ is étale. The desired Zariski-open locus in X may be taken to be the preimage of the overlap of W with a non-empty Zariski-open in P over which $X \to P$ is an isomorphism. ■

3.2. **Ground field extension.** If X is a proper rigid space over k and k'/k is a finite extension then for $X' = k' \otimes_k X$ (viewed as a rigid space over k or k') the natural projection $\pi : X' \to X$ induces an isomorphism $k' \otimes_k \mathscr{M}_X \simeq \pi_*(\mathscr{M}_{X'})$, and hence it induces a k'-algebra isomorphism $k' \otimes_k \mathscr{M}_X(X) \simeq \mathscr{M}_{X'}(X')$. Thus, if k'/k is finite separable (so X' is reduced when X is) then X is Moishezon if and only if X' is (over k or over k'). The same equivalence holds for any finite extension k'/k: the general finite case is immediately reduced to the purely inseparable case, which is part of the next lemma. For later purposes in positive characteristic it is important to also consider a related comparison problem with certain complete and purely inseparable ground field extensions of possibly infinite degree, so we incorporate such generality into this lemma.

Lemma 3.2.1. *Assume* $\mathrm{char}(k) = p > 0$, *and let* k'/k *be an analytic extension with* $k' \subseteq k^{p^{-n}}$ *for some* $n \geq 0$. *A proper rigid space* X *over* k *is Moishezon over* k *if and only if* $X' = k' \widehat{\otimes}_k X$ *is Moishezon over* k'. *If* X *is a reduced Moishezon space then the natural map* $(k' \otimes_k \mathscr{M}_X(X))_{\mathrm{red}} \to \mathscr{M}_{X'_{\mathrm{red}}}(X'_{\mathrm{red}})$ *is an isomorphism.*

A notable instance where Lemma 3.2.1 may be applied is $k' = k^{p^{-n}}$.

Proof. Let $\{X_i\}$ be the set of irreducible components of X, endowed with reduced structure. By [C1, 3.3.4, 3.4.2], the collection $\{(k' \widehat{\otimes}_k X_i)_{\mathrm{red}}\}$ is the set of such components of X', so we can assume that X is irreducible. We may also assume that X is reduced, and so to compare the Moishezon property for X and X' the problem is to prove equality of the transcendence degrees of $\mathscr{M}_{X'_{\mathrm{red}}}(X'_{\mathrm{red}})$ and $\mathscr{M}_X(X)$ over k. The extension $\mathscr{M}_X(X) \to \mathscr{M}_{X'_{\mathrm{red}}}(X'_{\mathrm{red}})$ over k is algebraic because the method of proof of Theorem 2.1.1 shows that every p^{n+m}th-power in $\mathscr{M}_{X'_{\mathrm{red}}}(X'_{\mathrm{red}})$ lies in $\mathscr{M}_X(X)$ for $m \geq 0$ large enough such that the nilradical on X' is killed by the p^m-power map.

For the comparison of meromorphic function fields when X is reduced and Moishezon, it is equivalent to solve the analogous problem for an irreducible and reduced proper rigid space that is related to X by a proper birational map (in either direction). Corollary 3.1.5 thereby reduces the problem to the case when X is projective, so the GAGA-isomorphism for meromorphic function fields reduces us to the trivial analogous algebraic problem. ■

Remark 3.2.2. We emphasize that we do not have a result as in Lemma 3.2.1 when using general non-algebraic complete extensions k'/k (such as a completed perfect or algebraic closure). For this reason, in arguments below with $\mathrm{char}(k) = p > 0$ we will have to restrict ourselves to repeatedly making extensions of the type $k^{p^{-n}}/k$ with $n \geq 0$ rather than making a single extension to the (perfect) completion $\widehat{k_p}$ of the perfect closure k_p of k. Also, if k'/k is an arbitrary analytic extension and X is a reduced Moishezon space over k then the only method we know to prove that the natural injective k'-algebra map $(k' \otimes_k \mathscr{M}_X(X))_{\mathrm{red}} \to \mathscr{M}_{X'_{\mathrm{red}}}(X'_{\mathrm{red}})$ identifies the target with the total ring of fractions of the source is to deduce it from GAGA *after* we have proved that every Moishezon space over k is the analytification of a proper algebraic space over k.

3.3. **Images and subspaces.** Now we turn to proving non-archimedean versions of two further properties of Moishezon spaces that were used in Artin's work over $\mathbf{C}$.

Lemma 3.3.1. *If* $f : X \to Y$ *is a surjective map between irreducible proper rigid spaces over* k *and* X *is Moishezon then so is* Y.

This is the analogue of [M1, Ch. I, Thm. 2], and the only essential new issue is to make the argument work in positive characteristic. We shall use Example 3.1.4 and formal models to handle this.

Proof. Let $\delta = \dim Y$ and $d = \dim X$. We wish to reduce to the case $d = \delta$. We can assume that X and Y are reduced, and by Corollary 3.1.5 we can assume that there is a proper birational map $f' : X \to Y'$ to a target Y' that is projective (so $\dim Y' = d$). By Corollary 2.1.2 we can replace each space with its normalization. The property of being Moishezon or not is unaffected by an analytic scalar extension K/k that is either of finite degree or satisfies $K \subseteq k^{p^{-m}}$ with $m \geq 0$ and $\mathrm{char}(k) = p > 0$ (see Lemma 3.2.1), so by [C1, 3.3.1, 3.3.6, 3.4.2] we can make such a scalar extension (and pass to the normalizations of underlying reduced spaces) to reduce to the case when Y is geometrically irreducible and geometrically normal over k. In particular, Y remains normal and connected after any finite extension on k.

Since Y is geometrically reduced, after an additional finite extension of k there is a flat formal model $\mathfrak{Y}$ for Y such that the $\widetilde{k}$-fiber $\mathfrak{Y}_0$ is reduced [BL, 1.3]. Let $\mathfrak{f}:\mathfrak{X}\to\mathfrak{Y}$ be a formal model of f, so by generic flatness over the reduced target $\mathfrak{Y}_0$ we know that the map $\mathfrak{f}_0$ is flat over a Zariski-dense open $\mathfrak{V}_0\subseteq\mathfrak{Y}_0$. Let $\mathfrak{V}\subseteq\mathfrak{Y}$ be the corresponding Zariski-open formal subscheme. By standard flatness arguments, the map $\mathfrak{f}^{-1}(\mathfrak{V})\to\mathfrak{V}$ is flat modulo π for any nonzero $\pi\in\mathfrak{m}_R$. Hence, $\mathfrak{f}$ is (topologically) flat over $\mathfrak{V}$, so $V=\mathfrak{V}^{\rm rig}$ is a non-empty admissible open in Y such that $f^{-1}(V)\to V$ is flat. In particular, $d\geq\delta$. Assume that $d>\delta$ (so $d>0$). We can choose a proper analytic set $Z'\subseteq Y'$ so that for $Z={f'}^{-1}(Z')\subseteq X$ the map $X-Z\to Y'-Z'$ is an isomorphism. Since X is irreducible, $(X-Z)\cap f^{-1}(V)$ is non-empty. Thus, we can choose $v\in V$ so that $X_v\cap(X-Z)$ is non-empty. By flatness of $f^{-1}(V)$ over V and equidimensionality of X and Y, the fiber X_v has pure dimension $d-\delta>0$. The Cohen-Macaulay locus in any rigid space is a dense Zariski-open set because affinoid algebras are excellent [C1, §1] (here we use that an affinoid algebra A is CM at a maximal ideal $\mathfrak{m}$ if and only if $\mathscr{O}_{{\rm Sp}(A),\mathfrak{m}}$ is CM, since both local rings have the same completion). Thus, we can choose $x\in X_v\cap(X-Z)$ at which X_v is Cohen-Macaulay. Replacing k with a finite extension is compatible with the formation of the CM locus and allows us to assume that $x\in X(k)$, so x maps to a point $y'\in Y'(k)$.

Under the isomorphism $\mathscr{O}_{X,x}\simeq\mathscr{O}_{Y',y'}$ the ideal $\mathfrak{m}_v\mathscr{O}_{X,x}$ goes over to an ideal $J\subseteq\mathscr{O}_{Y',y'}$, so $\mathscr{O}_{Y',y'}/J$ is a CM ring with dimension $d-\delta>0$. In any CM complete local noetherian ring $(A,\mathfrak{m})$ with positive dimension, each of the finitely many associated primes of A has image in $\mathfrak{m}/\mathfrak{m}^2$ that is a proper linear subspace. Thus, when this cotangent space is viewed as an affine space over the residue field it has a dense Zariski-open locus such that any element of $\mathfrak{m}$ reducing to a rational point in this locus is a regular element of A (i.e., is not in any associated prime). There are such rational points when the residue field of A is infinite. Applying this to the completion of $\mathscr{O}_{Y',y'}/J$, since the cotangent space of $\mathscr{O}_{Y',y'}$ is spanned over k by the local equations of projective hyperplanes through y' (with respect to a fixed projective embedding of Y' over k) we can choose a k-rational hyperplane H through y' in the ambient projective space such that $X_v\cap{f'}^{-1}(Y'\cap H)$ is cut out in X_v by a regular element in the maximal ideal $\mathscr{O}_{X_v,x}$. Hence, $X\cap{f'}^{-1}(Y'\cap H)$ is Y-flat at x with relative dimension $(d-\delta)-1$.

Normal rigid spaces are locally irreducible, so some irreducible component X_1 of $X\cap{f'}^{-1}(Y'\cap H)$ through $x\in X-Z$ must have image in the normal connected Y that contains an open around v, and hence X_1 surjects onto Y. Since $f':X\to Y'$ is an isomorphism over $Y'-Z'$, and $Y'\cap H$ has pure codimension 1 in the irreducible projective Y', there is a unique irreducible component Y'_1 of $Y'\cap H'$ such that f' restricts to a proper birational map $f_1:X_1\to Y'_1$ when X_1 and Y'_1 are given their reduced structures. The projective Y'_1 is obviously Moishezon, so X_1 is also Moishezon by Example 3.1.4 . We can therefore replace f' with f_1 and replace $f:X\to Y$ with $X_1\to Y$ to decrease d by 1 without changing Y. Continuing in this way, we eventually get to the case when $d=\delta$. Example 3.1.4 now implies that Y is Moishezon since X is. ■

The following deeper result corresponds to [M1, Ch. I, Thm. 3].

Theorem 3.3.2. *Any closed subspace of a Moishezon space over k is Moishezon.*

Proof. The general strategy of Moishezon's proof over $\mathbf{C}$ carries over, except that we need to modify the argument to address inseparability issues that arise when k has positive characteristic. This requires several technical improvements on the method, so for this reason we shall give the entire proof (which occupies the rest of §3.3). A ubiquitous feature of the proof in positive characteristic p is the use of (possibly infinitely many) scalar extensions of the type $k^{p^{-n}}/k$ with $n\geq 0$. Such extensions are needed to ensure geometric reducedness and geometric normality; see the examples above [C1, 3.3.1] for the insufficiency of using finite-degree purely inseparable extensions for this purpose, and see Remark 3.2.2 for why we cannot get by using the single ground field extension $k\to\widehat{k_p}$ or $k\to\widehat{\overline{k}}$.

Step 1: reduction to hypersurfaces. Let $Y\to X$ be a closed immersion into a Moishezon space X over k. To prove that Y is Moishezon, we can assume that X and Y are irreducible and that $Y\neq X$, so the pure dimension d of X is positive. Let $X'={\rm Bl}_Y(X)$ be the blow-up of X along Y (in the sense of

[C2, 4.1.1]), and let $Y' = Y \times_X X'$. Since X is irreducible and $X - Y = X' - Y'$ is not contained in a proper analytic set in X' (due to Y' being Cartier in X'), it follows from the global theory of irreducible components that X' is irreducible (with dimension d). Hence, Y' has pure dimension $d-1$. Note also that X' is Moishezon since pullback from $\mathscr{M}_X(X)$ provides d algebraically independent meromorphic functions. The proper map $X' \to X$ hits all of $X - Y$, so X' surjects onto X. Thus, since Y is irreducible, some irreducible component Y'_0 of Y' maps onto Y. By Lemma 3.3.1 applied to $Y'_0 \to Y$, the Moishezon property for the subspace $Y \subseteq X$ is reduced to that of the hypersurface $Y'_0 \subseteq X'$. Hence, we may assume that $Y \subseteq X$ is reduced and irreducible with codimension 1.

We may make a finite separable extension on k to get to the case when Y and X are geometrically irreducible over k, and then make a scalar extension by some $k^{p^{-n}}$ when $\mathrm{char}(k) = p > 0$ so that Y and X are geometrically reduced over k as well, with X having normalization that is geometrically normal (and geometrically irreducible) over k. By Example 3.1.4 we can replace X with its normalization and Y with an irreducible component of its preimage in this normalization. Running through the same reduction arguments again, we can get to the case where Y is geometrically reduced and geometrically irreducible and X is geometrically normal and geometrically irreducible over k.

Step 2: meromorphic functions regular near a hypersurface. Moishezon's method for constructing enough algebraically independent meromorphic functions on Y is to restrict meromorphic functions on X whose polar divisor does not contain Y in its support. We shall use the same idea, but new complications arise in positive characteristic. As a preliminary step, we review how to restrict suitable meromorphic functions to a hypersurface.

In general, if $j : H \hookrightarrow S$ is a closed immersion into a normal rigid space S over k with H reduced of pure codimension 1 then there is a natural $\mathscr{O}_S$-subalgebra $\mathscr{M}_{S,H} \subseteq \mathscr{M}_S$ whose sections are the meromorphic functions with no generic pole along H. More precisely, $\mathscr{M}_{S,H}$ is the subsheaf of sections of $\mathscr{M}_S$ whose coherent ideal sheaf of denominators has zero locus in S that intersects H in a nowhere dense subset of H. The formation of $\mathscr{M}_{S,H}$ is local on S, and we can define a natural restriction map $\mathscr{M}_{S,H} \to j_*(\mathscr{M}_H)$ of $\mathscr{O}_S$-algebras as follows. For any admissible open $U \subseteq S$ and any $f \in \mathscr{M}_{S,H}(U)$, locally on U we can express f as a ratio of analytic functions with denominator b whose restriction to $U \cap H$ has nowhere dense zero locus (equivalently, since H is reduced, $b|_{U\cap H}$ is invertible as a meromorphic function when viewed on $U \cap H$). It is therefore clear how to define $\mathscr{M}_{S,H} \to j_*(\mathscr{M}_H)$ via restriction of numerators and denominators of fractions.

For later purposes with extension of the ground field it will be convenient to give a global description of the induced map on global sections from the k-algebra $A_{S,H} = \mathscr{M}_{S,H}(S)$ to the meromorphic function algebra $\mathscr{M}_H(H)$. In terms of ideal sheaves of denominators we have

$$A_{S,H} = \varinjlim \mathrm{Hom}(\mathscr{J}, \mathscr{O}_S),$$

with $\mathscr{J}$ ranging through the directed system (by reverse inclusion) of coherent ideal sheaves on S that are nonzero (hence nowhere zero) on each connected component of S and are the unit ideal somewhere along each irreducible component of H. Each such $\mathscr{J}$ has a coherent pullback ideal $\mathscr{J}_H$ on the reduced H such that $\mathscr{J}_H$ locally has an $\mathscr{O}_H$-regular section, so we get a natural k-algebra map

$$A_{S,H} \to \varinjlim \mathrm{Hom}(\mathscr{J}_H, \mathscr{O}_H) \to \mathscr{M}_H(H)$$

that is the map induced by $\mathscr{M}_{S,H} \to j_*(\mathscr{M}_H)$ on global sections. The kernel $\mathfrak{m}_{S,H}$ of this k-algebra map is the ideal of elements $g \in A_{S,H}$ such that $g|_H \in \mathscr{M}_H(H)$ vanishes (which, for $g \in \mathscr{M}_S(S)^\times$, says exactly that H is contained in the zero locus of g). The k-subalgebra $A_{S,H}/\mathfrak{m}_{S,H} \subseteq \mathscr{M}_H(H)$ is denoted $\kappa_S(H)$, and if H is irreducible then this is clearly a subfield. It is analogous to the residue field at the generic point of an irreducible hypersurface in a normal algebraic k-scheme.

Observe that if S and H are both irreducible and S is Moishezon then $\mathfrak{m}_{S,H}$ must be nonzero, for otherwise we get a k-embedding of fields $\mathscr{M}_S(S) = A_{S,H} \hookrightarrow \mathscr{M}_H(H)$ whose source has transcendence degree $\dim S$ that exceeds the transcendence degree of its target. In particular, consideration of Weil divisors shows that if S and H are irreducible and S is Moishezon then the formation of the order of vanishing along H is a nontrivial discrete valuation ord_H on $\mathscr{M}_S(S)$ (perhaps with proper image in $\mathbf{Z}$) having valuation ring $A_{S,H}$

and maximal ideal $\mathfrak{m}_{S,H}$. Explicitly, if $g \in \mathscr{M}_S(S)^\times$ then $\mathrm{ord}_H(g)$ is the coefficient of H in the Weil divisor $\mathrm{div}(g)$ on the normal rigid space S. The residue field of ord_H is $\kappa_S(H) \subseteq \mathscr{M}_H(H)$.

Lemma 3.3.3. *Let S be a geometrically normal and proper rigid space, $H \subseteq S$ a geometrically reduced closed subspace with pure codimension 1, and k'/k an extension that is finite or satisfies $k' \subseteq k^{p^{-m}}$ for some $m \geq 1$ with $\mathrm{char}(k) = p > 0$. Let $S' = k'\widehat{\otimes}_k S$ and $H' = k'\widehat{\otimes}_k H$. Consider the natural comparison maps*

$$k' \otimes_k \mathscr{M}_S(S) \simeq \mathscr{M}_{S'}(S'), \;\; k' \otimes_k \mathscr{M}_H(H) \simeq \mathscr{M}_{H'}(H') \tag{3.3.1}$$

that carry $k' \otimes_k A_{S,H}$ into $A_{S',H'}$ compatibly with the restriction maps to $k' \otimes_k \mathscr{M}_H(H)$ and $\mathscr{M}_{H'}(H')$.

The induced injective comparison map $k' \otimes_k A_{S,H} \to A_{S',H'}$ is an isomorphism. In particular, if H is geometrically irreducible then $(k' \otimes_k \kappa_S(H))_{\mathrm{red}} \simeq \kappa_{S'}(H')$ for every such k'/k, so if in addition $\mathrm{char}(k) = p > 0$ and $k' = k^{p^{-m}}$ for sufficiently large m then $\kappa_{S'}(H')$ is separable over k'.

Note that the comparison isomorphisms for meromorphic function algebras in (3.3.1) follow from Lemma 3.2.1 (and the trivial case when k'/k is finite separable). Also, the separability claim at the end of Lemma 3.3.3 follows from the elementary fact that if L/k is a finitely generated extension then for sufficiently large m the field $(k^{p^{-m}} \otimes_k L)_{\mathrm{red}}$ is separable over $k^{p^{-m}}$ (as we see via descent from the perfect closure of k).

Proof. We have to show that if $g' \in k' \otimes_k \mathscr{M}_S(S)$ lies in $A_{S',H'}$ then $g' \in k' \otimes_k A_{S,H}$. It suffices to separately treat the cases when k'/k is finite Galois or $k' = k^{p^{-m}}$ with $\mathrm{char}(k) = p > 0$ and $m \geq 1$. The Galois case is trivial by Galois descent (since $A_{S',H'}$ is a Galois-stable k'-subspace of $\mathscr{M}_{S'}(S') = k' \otimes_k \mathscr{M}_S(S)$), so we can assume $k' = k^{p^{-m}}$ with $m \geq 1$ and $\mathrm{char}(k) = p > 0$. The condition $g' \in A_{S',H'}$ says that the polar locus of g' does not contain H'. Since $g' \in \mathscr{M}_{S'}(S') = k' \otimes_k \mathscr{M}_S(S)$, to show $g' \in k' \otimes_k A_{S,H}$ we thereby easily reduce to the analogous problem when k'/k is a finite purely inseparable extension, and then to the special case $k' = k(c')$ with $c'^p = c \in k - k^p$. Thus, we can uniquely write $g' = \sum_{j=0}^{p-1} c'^j \otimes h_j$ for $h_0, \dots, h_{p-1} \in \mathscr{M}_S(S)$. We want $h_j \in A_{S,H}$ for all j, which is to say that H is not in the polar locus of any h_j. If h_{j_0} lies in $A_{S,H}$ then we can replace g' with $g' - (1 \otimes h_{j_0})$ without loss of generality. In this way we can assume that those h_j that are nonzero have H in their polar locus. We seek to show that all h_j vanish (so $g' = 0$).

Since H is geometrically reduced over k, it has a dense Zariski-open smooth locus. The non-smooth locus of the geometrically normal S has codimension ≥ 2, so since H has pure codimension 1 we can choose $s \in H$ at which both H and S are smooth. Let $t \in \mathscr{O}_{S,s}$ be a generator of the height-1 prime $\mathfrak{p} = \mathscr{I}_{H,s}$. Let $b_j := (h_j)_s \in K := \mathrm{Frac}(\mathscr{O}_{S,s})$ for all j, so those b_j that are nonzero have negative order along $\mathfrak{p}$. We may assume that there are some nonzero b_j's, so $\nu = \max_j(-\mathrm{ord}_{\mathfrak{p}}(b_j)) > 0$. Let $h' = t^{\nu-1} g'_s = \sum_{j=0}^{p-1} c'^j \otimes b'_j$ in $k' \otimes_k K$ with $b'_j = t^{\nu-1} b_j \in K$. All b'_j not in $\mathscr{O}_{S,s}$ have a simple pole along $\mathfrak{p}$ and some such b'_j exists. Let s' be the unique point on $S' = k' \otimes_k S$ over s. The ring $\mathscr{O}_{S',s'} = k' \otimes_k \mathscr{O}_{S,s}$ is a regular domain since S' is k'-smooth at s'. Its fraction field is $K' = k' \otimes_k K$, and $\sum c'^j \otimes b'_j = h' = t^{\nu-1} g'_s \in K'$ has non-negative order at the unique height-1 prime $\mathfrak{p}'$ over $\mathfrak{p}$ in $\mathscr{O}_{S',s'}$ since $g' \in k' \otimes A_{S,H}$. Thus, $\sum c'^j \otimes t b'_j = t(\sum c'^j \otimes b'_j) \in \mathfrak{p}'$. But $k' \otimes_k (\mathscr{O}_{S,s}/\mathfrak{p}) = \mathscr{O}_{H',s'}$ is a domain because H' is k'-smooth at s', so $\mathfrak{p}' = k' \otimes_k \mathfrak{p}$. Hence, the element $t b'_j \in \mathscr{O}_{S,s}$ lies in $\mathfrak{p} = t\mathscr{O}_{S,s}$ for all j, so $b'_j \in \mathscr{O}_{S,s}$ for all j. This is a contradiction. ■

Step 3: infinite tower of subfields. Returning to our situation of interest at the end of Step 1 with the geometrically normal and geometrically irreducible Moishezon space X of dimension $d > 0$ and the geometrically irreducible and geometrically reduced analytic set $i : Y \hookrightarrow X$ with codimension 1, we have a k-subalgebra $A_{X,Y} \subseteq \mathscr{M}_X(X)$ that is a discrete valuation ring whose residue field $\kappa_X(Y) = A_{X,Y}/\mathfrak{m}_{X,Y}$ is naturally a subfield of $\mathscr{M}_Y(Y)$ over k obtained by restriction of suitable meromorphic functions on X. The subfield $\kappa_X(Y)$ is therefore *finitely generated* over k since $\mathscr{M}_Y(Y)$ is finitely generated over k. It suffices to show that the transcendence degree of this subfield over k is equal to $d - 1 = \dim(Y)$. By Lemma 3.2.1 and Lemma 3.3.3, if $\mathrm{char}(k) = p > 0$ then we may first make a preliminary ground field extension of the type $k \to k^{p^{-m}}$ to get to the case when $\kappa_X(Y)$ is k-separable.

The rest of the argument goes similarly to the complex-analytic case, but we need special arguments in positive characteristic (to handle several complications caused by inseparability). For the convenience of the reader, we therefore now give the rest of the argument in our situation rather than referring the reader to Moishezon's paper [M1].

The key idea (which will only work after suitable extension of the ground field) is to exploit finite generation of meromorphic function fields over k to get to the situation (after changing X and Y but not $\mathscr{M}_Y(Y)$) in which the discrete valuation ring $A_{X,Y} \subseteq \mathscr{M}_X(X)$ contains a subfield F over k mapping isomorphically onto the residue field $\kappa_X(Y) \subseteq \mathscr{M}_Y(Y)$ (so F is separable over k since we have arranged that $\kappa_X(Y)$ is k-separable). An argument with separability of completions will show that $\mathscr{M}_X(X)$ must be separable over F, and together with k-separability of F it will then follow from a calculation with meromorphic top-degree differential forms on X (via Corollary 3.1.6) that F has transcendence degree $d-1$ over k, as desired.

To carry out this idea, we are going to try to construct a commutative square

$$\begin{array}{ccc} Y_1 & \xrightarrow{i_1} & X_1 \\ \downarrow & & \downarrow \\ Y & \xrightarrow[i]{} & X \end{array}$$

such that the following properties are satisfied:

- the vertical maps are proper surjections, the horizontal maps are closed immersions, and the left vertical map is birational (in the sense that it is an isomorphism over a dense Zariski-open locus in Y whose preimage in Y_1 is dense),
- X_1 is geometrically normal and geometrically irreducible with dimension d (so X_1 is Moishezon, via pullback of meromorphic functions on X),
- Y_1 is geometrically reduced and geometrically irreducible with dimension $d-1$,
- A_{X_1,Y_1} contains a subfield over k whose image in $\kappa_{X_1}(Y_1) \subseteq \mathscr{M}_{Y_1}(Y_1) \simeq \mathscr{M}_Y(Y)$ is $\kappa_X(Y)$.

This procedure can then be repeated infinitely many times, and the rising chain of subfields $\kappa_{X_r}(Y_r) \subseteq \mathscr{M}_{Y_r}(Y_r) = \mathscr{M}_Y(Y)$ over k must eventually stabilize (since $\mathscr{M}_Y(Y)$ is finitely generated over k). However, a caveat is that if $\mathrm{char}(k) = p > 0$ then the preceding description is not quite what we will do: to guarantee geometric normality and geometric reducedness properties for X_1 and Y_1 (as is required to repeat the process) it will be necessary to replace k with $k^{p^{-\mu_1}}$ for some $\mu_1 \geq 0$, so the infinite repetition of the process requires some care.

Now we come to the actual argument. Since $\kappa_X(Y)$ is separable and finitely generated over k, we can choose a separating transcendence basis $h_1, \dots, h_s$ for $\kappa_X(Y)$ over k and a primitive element $h_{s+1} \in \kappa_X(Y)$ over $k(h_1, \dots, h_s)$, say with ϕ its separable minimal polynomial of degree $\delta \geq 1$. Let $H_j \in A_{X,Y} \subseteq \mathscr{M}_X(X)$ represent h_j, so $H_1, \dots, H_s$ are algebraically independent over k. Let $\Phi \in k(H_1, \dots, H_s)[t_0, t_1] \subseteq A_{X,Y}[t_0, t_1]$ be the homogeneous polynomial of degree δ that is obtained from ϕ by lifting h_j to H_j for $1 \leq j \leq s$ (so $\Phi(T, 1)$ lifts $\phi(T)$). We can view Φ as a trivializing meromorphic global section of the line bundle $\mathscr{O}(\delta)$ over the normal connected rigid space $X \times \mathbf{P}^1$ with dimension $d+1$. As such, Φ has zero locus X' with pure dimension d. Let $U \subseteq X$ be the Zariski-open complement of the union of the polar loci of the coefficients of Φ and the non-smooth loci of Y and X. Clearly $X' \cap (U \times \mathbf{P}^1)$ is the effective Cartier divisor in $U \times \mathbf{P}^1$ defined by the vanishing of the nonzero homogeneous polynomial $\Phi|_U$ in $\mathscr{O}(U)[t_0, t_1]$.

Let $Y' = (Y \times \mathbf{P}^1) \cap X' \subseteq Y \times \mathbf{P}^1$. The Zariski-open preimage $Y'_U \subseteq Y'$ of $U \cap Y \subseteq Y$ is the zero locus in the smooth irreducible d-dimensional $(U \cap Y) \times \mathbf{P}^1$ of the nonzero $\Phi|_{U \cap Y} \in \mathscr{O}(U \cap Y)[t_0, t_1]$, and the part of Y'_U lying over $(U \cap Y) \cap (Y - D_\infty(h_{s+1}))$ has an irreducible component Y'_η given by the graph of h_{s+1} over this domain. The component Y'_η arises from a unique global irreducible component Y'_1 of Y' which must be geometrically irreducible (since Y is). We give Y'_1 the reduced structure, so the proper projection $Y'_1 \to Y$ is birational (it is an isomorphism over $(U \cap Y) \cap (Y - D_\infty(h_{s+1}))$). In particular, Y'_1 is geometrically irreducible and is smooth away from a proper analytic subset. Let $\Delta \in k(H_1, \dots, H_s)^\times$ be the discriminant of the separable $\Phi(T, 1) \in k(H_1, \dots, H_s)[T]$. The proper projection $X' \to X$ is visibly étale at points of Y'_1 that lie over the (necessarily non-empty) overlap of $U \cap Y$ and the non-empty Zariski-open $V \subseteq Y$ complementary to the polar loci the coefficients of Φ and the zero locus of Δ. But X and Y are smooth at points of $U \cap Y$, the map $Y'_1 \to Y$ is birational, and Y'_1 is irreducible and reduced, so there is a unique irreducible component

X_1' of X' containing Y_1'. By uniqueness of X_1' and geometric irreducibility of Y_1', it follows via Galois descent (and [C1, 3.4.2]) that X_1' is geometrically irreducible. If we give X_1' its reduced structure then $X_1' \to X$ is étale on a non-empty Zariski-open locus in X_1' that meets Y_1'. (This étale property is a special case of the general fact that a map $T' \to T$ from a reduced rigid space T' with equidimension d onto a normal connected d-dimensional rigid space T is étale on the Zariski-open complement of the support of $\Omega^1_{T'/T}$ because an injective finite map from a normal noetherian domain to another noetherian ring is étale if it is unramified [FK, Ch. I, 1.5].)

Let X_1'' be the normalization of X_1', so there is a unique irreducible analytic set $Y_1'' \subseteq X_1''$ mapping birationally onto Y_1' (and hence onto Y). Both X_1'' and Y_1'' are geometrically irreducible over k. The composite map

$$X_1'' \to X_1' \hookrightarrow X' \hookrightarrow X \times \mathbf{P}^1 \to \mathbf{P}^1$$

restricts to h_{s+1} over $(U \cap Y) \cap (Y - D_\infty(h_{s+1})) \subseteq Y$, so it is not identically ∞, and it defines a meromorphic function $H_{s+1} \in \mathscr{M}_{X_1''}(X_1'')$ that is a root of $\Phi(T,1)$ by definition of X'. In fact, H_{s+1} lies in $A_{X_1'',Y_1''}$ and the resulting restriction $H_{s+1}|_{Y_1''} \in \mathscr{M}_{Y_1''}(Y_1'')$ coincides with the pullback of $h_{s+1} \in \mathscr{M}_Y(Y)$ due to how Y_1' contains a dense Zariski-open locus in the graph of h_{s+1} over $Y - D_\infty(h_{s+1})$. Thus, the inclusion $\mathscr{M}_X(X) \to \mathscr{M}_{X_1''}(X_1'')$ carries $H_1, \dots, H_s \in A_{X,Y}$ to elements of $A_{X_1'',Y_1''}$ that, together with H_{s+1}, generate a subfield $k(H_1, \dots, H_s)[H_{s+1}] \subseteq A_{X_1'',Y_1''}$ whose image in the quotient

$$\kappa_{X_1''}(Y_1'') \subseteq \mathscr{M}_{Y_1''}(Y_1'') \simeq \mathscr{M}_Y(Y)$$

is $k(h_1, \dots, h_s)[h_{s+1}] = \kappa_X(Y)$. If $\mathrm{char}(k) = p > 0$ then X_1'' may not be geometrically normal and Y_1'' may not be geometrically reduced. Letting $k_m = k^{p^{-m}}$ for $m \geq 0$, for some $\mu \geq 0$ the normalization $X_{1,\mu}''$ of $(k_\mu \widehat{\otimes}_k X_1'')_{\mathrm{red}}$ is geometrically normal over k_μ, but the underlying reduced space $Y_{1,\mu}''$ of the pullback of $k_\mu \widehat{\otimes}_k Y_1''$ to this normalization may not be geometrically reduced over k_μ. However, there exists $\mu_1 \geq \mu$ such that in the geometrically normal rigid space $X_1 = k_{\mu_1} \widehat{\otimes}_{k_\mu} X_{1,\mu}''$ over k_{μ_1} the subspace $Y_1 = (k_{\mu_1} \widehat{\otimes}_{k_\mu} Y_{1,\mu}'')_{\mathrm{red}}$ is geometrically reduced over k_{μ_1}. We can also take μ_1 large enough so that $\kappa_{X_1}(Y_1)$ is separable over k_{μ_1}. If $\mathrm{char}(k) = 0$ then we define $k_m = k$ for all $m \geq 0$.

If we iterate this procedure infinitely many times then we get a monotonically increasing sequence of integers $0 = \mu_0 \leq \mu_1 \leq \mu_2 \leq \dots$ and d-dimensional geometrically normal and geometrically irreducible Moishezon spaces X_r over k_{μ_r} equipped with $(d-1)$-dimensional geometrically reduced and geometrically irreducible analytic sets $Y_r \subseteq X_r$ and k_{μ_r}-maps $X_r \to k_{\mu_r} \widehat{\otimes}_{k_{\mu_{r-1}}} X_{r-1}$ inducing birational maps $Y_r \to k_{\mu_r} \widehat{\otimes}_{k_{\mu_{r-1}}} Y_{r-1}$ for all $r \geq 1$ such that $X_0 = X$, $Y_0 = Y$, $\kappa_{X_r}(Y_r)$ is k_{μ_r}-separable, and the compatible identifications $\mathscr{M}_{Y_r}(Y_r) \simeq k_{\mu_r} \otimes_k \mathscr{M}_Y(Y)$ induce containments

$$k_{\mu_r} \otimes_{k_{\mu_{r-1}}} \kappa_{X_{r-1}}(Y_{r-1}) \subseteq \kappa_{X_r}(Y_r)$$

for $r \geq 1$. Moreover, this subfield of $\kappa_{X_r}(Y_r)$ over k_{μ_r} can be isomorphically lifted to a k_{μ_r}-subalgebra $F_r \subseteq A_{X_r,Y_r}$. Observe that Y is Moishezon if and only if some Y_r is Moishzeon.

Step 4: coefficient field and differential forms. Let $k_\infty = \varinjlim k_{\mu_r}$ ($k_\infty = k$ in characteristic 0), so

$$L_r = k_\infty \otimes_{k_{\mu_r}} \kappa_{X_r}(Y_r) \subseteq k_\infty \otimes_k \mathscr{M}_Y(Y)$$

is a rising sequence of subextensions of the finitely generated extension $k_\infty \otimes_k \mathscr{M}_Y(Y)$ of k_∞. Such a sequence must terminate, so there exists an $r \geq 0$ such that $L_{r+1} = L_r$. Hence, the $k_{\mu_{r+1}}$-subalgebra $k_{\mu_{r+1}} \otimes_{k_{\mu_r}} F_r \subseteq A_{X_{r+1},Y_{r+1}}$ maps isomorphically onto $\kappa_{X_{r+1}}(Y_{r+1})$. In particular, upon replacing (X, Y, k) with $(X_{r+1}, Y_{r+1}, k_{\mu_{r+1}})$ (as we may), we reach the situation in which the k-subalgebra $A_{X,Y} \subseteq \mathscr{M}_X(X)$ contains a k-subalgebra F that maps isomorphically onto the quotient $\kappa_X(Y) \subseteq \mathscr{M}_Y(Y)$. Note that this forces F to be a field that is k-separable (since $\kappa_X(Y)$ is k-separable).

In this situation we will prove that F has transcendence degree $d-1$ over k, as desired. The crucial technical observation is that even in case of positive characteristic, $\mathscr{M}_X(X)$ is automatically separable over F. To prove this, consider the nontrivial discrete valuation ord_Y on $\mathscr{M}_X(X)$ given by generic order along Y; this has image $\nu\mathbf{Z} \subseteq \mathbf{Z}$ for a unique $\nu \geq 1$. Choose a uniformizer $t \in \mathfrak{m}_{X,Y}$, so $\mathrm{ord}_Y(t) = \nu$ and the completion of $\mathscr{M}_X(X)$ with respect to this valuation is identified with $F(\!(t)\!)$ as an F-algebra. In particular, since $F(\!(t)\!)$ is separable over F (as for any field), so is $\mathscr{M}_X(X)$. Moreover, $\mathrm{d}t \neq 0$ in $\Omega^1_{\mathscr{M}_X(X)/F}$ since its image in $\Omega^1_{F(\!(t)\!)/F}$ is nonzero. Thus, we can choose a separating transcendence basis of $\mathscr{M}_X(X)$ over F containing t, say $\{G_1, \dots, G_e, t\}$ (with $e \geq 0$), and by replacing each G_j with $G_j/t^{\mathrm{ord}_Y(G_j)/\nu}$ for each j we can ensure that $G_j \in A_{X,Y}$ for all j. Since F/k is separable, we may also choose a separating transcendence basis $H_1, \dots, H_s \in F \subseteq A_{X,Y}$ over k. The collection $\{G_1, \dots, G_e, t, H_1, \dots, H_s\}$ is a separating transcendence basis for $\mathscr{M}_X(X)$ over k, so $1 + e + s = d$. We want to prove that $s = d - 1$, or equivalently $e = 0$.

By Corollary 3.1.6, the top-degree meromorphic differential form

$$\omega_0 = \mathrm{d}G_1 \wedge \cdots \wedge \mathrm{d}G_e \wedge \mathrm{d}t \wedge \mathrm{d}H_1 \wedge \cdots \wedge \mathrm{d}H_s$$

on X is nonzero. Since the irreducible and reduced Y meets the Zariski-open k-smooth locus X^{sm} of X, we can define the generic order $\mathrm{ord}_Y(\omega) \in \mathbf{Z}$ along Y for any nonzero global meromorphic top-degree differential form ω on X. Fix a choice of $n \geq 0$, so for each $G_j \in A_{X,Y} \subseteq F[\![t]\!]$ (if $e > 0$) we have $G_j = \gamma_j + t^{n+1} g_j$ for some $\gamma_j \in F[t]$ and $g_j \in A_{X,Y}$. Thus,

$$\mathrm{d}G_j = \mathrm{d}\gamma_j + (n+1)t^n g_j \mathrm{d}t + t^{n+1}\mathrm{d}g_j \tag{3.3.2}$$

in $\Omega^1_{\mathscr{M}_X(X)/k}$. The same identity holds as meromorphic 1-forms on X by means of the natural $\mathscr{M}_X(X)$-linear map

$$\Omega^1_{\mathscr{M}_X(X)/k} \to \Gamma(X, \mathscr{M}_X \otimes \Omega^1_{X/k}).$$

If $e > 0$ then for each $1 \leq j \leq e$ we have

$$\mathrm{d}\gamma_j \wedge \mathrm{d}t \wedge \mathrm{d}H_1 \wedge \cdots \wedge \mathrm{d}H_s \in \Omega^{s+2}_{F(t)/k},$$

but $\Omega^{s+2}_{F(t)/k} = 0$ because $F(t)/k$ is separable with transcendence degree $s + 1$. Thus, inserting the formulas (3.3.2) into the definition of ω_0 (if $e > 0$) kills the contributions from each $\mathrm{d}\gamma_j$ and $\mathrm{d}t$ in $\mathrm{d}G_j$. Hence,

$$\omega_0 = t^{e(n+1)}\mathrm{d}g_1 \wedge \dots \mathrm{d}g_e \wedge \mathrm{d}t \wedge \mathrm{d}H_1 \wedge \dots \mathrm{d}H_s.$$

Since no g_j or H_i has a generic pole along Y, and nor does t, it follows that the generic order of the nonzero ω_0 along Y is at least $e(n+1)$. But $n \geq 0$ was arbitrary, so $e = 0$ as desired. ■

4. Algebraicity of Moishezon spaces

4.1. **Main result.** We have already seen that analytification identifies the category of proper algebraic spaces over k with a full subcategory of the category of Moishezon spaces over k. Artin's argument in the complex-analytic case will now be adapted to prove that this is an equivalence, as follows.

Theorem 4.1.1. *Every Moishezon space over k is the analytification of a proper algebraic space over k.*

Using Chow's Lemma for algebraic spaces [Kn, IV, 3.1] and the invariance of meromorphic function fields under proper birational maps in the reduced case, we deduce the following generalization of Corollary 3.1.5.

Corollary 4.1.2. *A proper rigid space X over k is Moishezon if and only if there exists a birational map $P^{\mathrm{an}} \to X$ from a projective k-scheme P.*

In [A2, §7], Artin proves the complex-analytic analogue of Theorem 4.1.1 by using input from work of Moishezon for which we have proved non-archimedean analogues in §3. The main tool in Artin's argument is a general theorem on modifications of finite type separated algebraic spaces along closed subspaces, all over a base S that is an algebraic space of finite type over a field or excellent Dedekind domain. For the complex-analytic applications Artin works with $S = \operatorname{Spec} \mathbf{C}$, and we want to use analogues of the same arguments with $S = \operatorname{Spec} k$. There are some mild complications caused by the fact that k may be imperfect.

For example, Artin uses a lifting criterion for adic S-maps from $\mathrm{Spf}(A)$ to formal algebraic spaces over S, with A a complete discrete valuation ring whose residue field κ satisfies the condition that $\operatorname{Spec} \kappa \to S$ is of finite type. For $S = \operatorname{Spec} \mathbf{C}$ only the case $A = \mathbf{C}[\![t]\!]$ arises. However, for $S = \operatorname{Spec} k$ the residue field k' of such an A is a finite extension of k that may be inseparable (if k is imperfect). In such cases A may not be k-isomorphic to $k'[\![t]\!]$. This leads to some minor changes in the use of formal lifting arguments for such A. We shall now go through Artin's argument, focusing almost entirely on where some new technical ingredients are required in the non-archimedean setting. Throughout our discussion, we consider the base space S in Artin's theory to be $\mathrm{Spec}\, k$ for a fixed non-archimedean field k, and all algebraic spaces will be understood to be locally of finite type and *separated* over k (and morphisms between them will be over k as well) unless otherwise specified.

4.2. **Formal modification.** As in Artin's work, a *modification* of rigid spaces over k is a pair consisting of a proper map $f : X' \to X$ between such spaces and a closed immersion $Y \hookrightarrow X$ such that f is an isomorphism over $X - Y$. (It is not required that Y is nowhere dense in X or that f is surjective, so we could take $Y = X$ but that is not useful.) We will be especially interested in the case when X and X' have some common dimension d (though they may be reducible) and Y and $Y' = f^{-1}(Y)$ have dimension $< d$. The notion of modification is defined for algebraic spaces in exactly the same way.

Consider a proper map $f : X' \to X$ between algebraic spaces and let $Y \subseteq X$ be a closed subspace. There is an induced map $\mathfrak{f} : \mathfrak{X}' \to \mathfrak{X}$ between the associated (locally noetherian and separated) formal algebraic spaces over $\mathrm{Spec}\, k$ obtained by completing along Y and $Y' = f^{-1}(Y)$ respectively. In [A2, §1], Artin studies the notion of *formal modification*, a property for $\mathfrak{f}$ that is closely related to $(f, Y \subseteq X)$ being a modification. There is one aspect of this notion that requires some extra care in the non-archimedean case, so we now recall how it is defined in general.

Let $\mathfrak{f} : \mathfrak{X}' \to \mathfrak{X}$ be a proper k-map between locally noetherian and separated formal algebraic spaces (over k) such that the underlying ordinary algebraic spaces (modulo an ideal of definition) are locally of finite type over k.

Definition 4.2.1. The proper map $\mathfrak{f} : \mathfrak{X}' \to \mathfrak{X}$ is a *formal modification* if the following three conditions hold locally over $\mathfrak{X}$:

(1) the coherent Cramer and Jacobian ideals $\mathscr{C}(\mathfrak{f})$ and $\mathscr{J}(\mathfrak{f})$ on $\mathfrak{X}'$ contain ideals of definition of $\mathfrak{X}'$,
(2) the diagonal $\Delta_{\mathfrak{X}'/\mathfrak{X}} : \mathfrak{X}' \to \mathfrak{X}' \times_{\mathfrak{X}} \mathfrak{X}'$ (which is a closed immersion, due to the separatedness of these formal algebraic spaces) has associated coherent defining ideal $\mathscr{I}$ on $\mathfrak{X}' \times_{\mathfrak{X}} \mathfrak{X}'$ that is locally killed by some power of the pullback of an ideal of definition of $\mathfrak{X}$,
(3) for every complete discrete valuation ring $\overline{R}$ over k with residue field k' of finite degree over k, any k-morphism $\mathfrak{g} : \mathrm{Spf}(\overline{R}) \to \mathfrak{X}$ as formal algebraic spaces lifts through $\mathfrak{f}$ to a morphism $\mathfrak{g}' : \mathrm{Spf}(\overline{R}) \to \mathfrak{X}'$.

We will not need to do anything new with condition (1) in this definition, so we omit a discussion of how the Cramer and Jacobian ideals are defined. (See [A2, §1].) The complications with this definition in comparison with its analogue over $\mathbf{C}$ are in condition (3). To explain this, consider a map $f : X' \to X$ between proper complex-analytic or rigid-analytic spaces and let $Y \subseteq X$ be a (possibly non-reduced) closed subspace with pullback $Y' \subseteq X'$. This yields compatible maps $f_n : X'_n \to X_n$ between the respective corresponding infinitesimal neighborhoods of Y' and Y in X' and X for all $n \geq 0$. Assume that the spaces X'_n and X_n are analytifications of proper algebraic spaces for all $n \geq 0$, so we get a map between proper formal algebraic spaces $\mathfrak{f} : \mathfrak{X}' \to \mathfrak{X}$ obtained by algebraization of the maps f_n. In [A2, 7.7], Artin proves over $\mathbf{C}$ that if f is a modification in the complex-analytic sense then $\mathfrak{f}$ is a formal modification (over $\mathrm{Spec}\, \mathbf{C}$). The verification of conditions (1) and (2) in Definition 4.2.1 works verbatim in the non-archimedean case, but for condition (3) some problems arise, as follows.

Artin's proof that the formal morphism $\mathfrak{f}$ associated to a complex-analytic modification f satisfies the lifting property in (3) (with $k = \mathbf{C}$) appeals to some work of Spallek on analytic approximation over $\mathbf{R}$ and $\mathbf{C}$, and uses two additional facts: for $\overline{R}$ as in (3) automatically $\overline{R} \simeq \mathbf{C}[\![t]\!]$ as $\mathbf{C}$-algebras (since $\mathbf{C}$ is algebraically closed), and any countable inverse limit of non-empty affine finite type $\mathbf{C}$-schemes has a $\mathbf{C}$-point (because $\mathbf{C}$ is uncountable and algebraically closed). To make this work in the non-archimedean case, if $\mathrm{char}(k) = 0$ then for the analytic approximation part one can replace Spallek's work with Artin's work on analytic approximation in characteristic 0 [A1, 1.2]. In [B1], Bosch adapts Artin's analytic approximation techniques, and in [B1, §2] it is noted that this yields a proof of [A1, 1.2] in any characteristic, so the analytic approximation part carries over in general. As long as k is algebraically closed, the other parts of Artin's verification of (3) for $\mathfrak{f}$ carry over verbatim. For a general k we have to circumvent the possibility that $\overline{R}$ may not even admit a coefficient field over k (if k'/k is inseparable). To do this, we will reduce to the settled case of an algebraically closed ground field as follows. Let K/k be a completed algebraic closure. Artin's method of constructing lifts via jets can be applied after the formal algebraic spaces being considered over k are pulled back over $\mathrm{Spec}\, K = \mathrm{Spf}\, K$. We shall exploit this by applying the following lemma.

Lemma 4.2.2. *In order that a map* $\mathfrak{g} : \mathrm{Spf}(\overline{R}) \to \mathfrak{X}$ *as in Definition* 4.2.1(3) *satisfies the lifting property there, it suffices to make a lift after composing* $\mathfrak{g}$ *with* $\mathrm{Spf}(\overline{R}') \to \mathrm{Spf}(\overline{R})$ *for a flat local* (*perhaps not finite*) *extension of complete discrete valuation rings* $\overline{R} \to \overline{R}'$*.*

Before we prove the lemma, let us see how to use it in our situation over k. We take such an $\overline{R}'$ to be a factor ring of the normalization of $(K\widehat{\otimes}_k\overline{R})_{\mathrm{red}}$, where we use the completed tensor product in the sense of commutative algebra, giving K and k the discrete topology. By the known lifting result for the induced K-morphism $\mathrm{Spf}(\overline{R}') \to \mathrm{Spf}(K\widehat{\otimes}_k\mathfrak{X})$ over the algebraically closed non-archimedean ground field K, the criterion in the lemma implies that the lifting problem over k has an affirmative solution. (This lemma is also implicitly used in the proof of [A2, 1.13] with a general complete discrete valuation ring $\overline{R}$ and finite étale local extensions $\overline{R} \to \overline{R}'$.)

Proof. To verify the sufficiency of constructing a lift of $\mathfrak{g} : \mathrm{Spf}(\overline{R}) \to \mathfrak{X}$ through the map $\mathfrak{f} : \mathfrak{X}' \to \mathfrak{X}$ after a flat local extension on $\overline{R}$, consideration of the pullback of $\mathfrak{f}$ along $\mathfrak{g}$ reduces us to settling the following claim: if $\mathfrak{Z}$ is a proper formal algebraic space over $\mathrm{Spf}(R)$ for a complete discrete valuation ring R and if the closed immersion $\Delta_{\mathfrak{Z}/\mathrm{Spf}(R)}$ has defining ideal that is killed by a power of a uniformizer of R then $\mathfrak{Z} \to \mathrm{Spf}(R)$ admits a section if it does so after a flat local extension $R \to R'$ where R' is a complete discrete valuation ring. We can kill the $\mathfrak{m}_R$-torsion in $\mathscr{O}_{\mathfrak{Z}}$ to reduce to the case when $\mathfrak{Z}$ is R-flat (since R' is R-flat), and then its diagonal has vanishing defining ideal. That is, $\mathfrak{Z} \to \mathrm{Spf}(R)$ is a proper flat monomorphism of formal algebraic spaces, so it is a flat closed immersion. That is, either $\mathfrak{Z} = \mathrm{Spf}(R)$ or $\mathfrak{Z}$ is empty. The condition of having a section after a flat local base change on R is then the condition that $\mathfrak{Z} = \mathrm{Spf}(R)$, in which case there is trivially a section. ∎

4.3. **Passage to irreducible case.** We now commence with the proof of algebraizability of Moishezon spaces by inducting on the dimension d, the case $d = 0$ being clear. Thus, let X be a Moishezon space with dimension $d > 0$. We first wish to reduce the algebraicity problem for X to the irreducible case. To this end, assume that X is reducible and let $\{C_i\}$ be the set of irreducible components of X. Rather than endowing each C_i with its reduced structure in general, we seek to construct a structure of closed analytic subspace on the subset C_i such that on the subset $X_i = X - (\cup_{j\neq i}C_j) \subseteq C_i$ that is open in X the open subspace structure induced from that on C_i makes it an open subspace of X. Roughly speaking, when X_i is given the open subspace structure in X, we seek to extend this to a closed subspace structure on C_i in X. This operation is akin to scheme-theoretic closure in algebraic geometry, and it is a non-standard analytic operation. (For example, the potential presence of essential singularities along $C_i - X_i$ has to be ruled out.) Due to lack of a reference, we now record a general construction of this type (as also seems to be implicitly used without explanation in the complex-analytic case in [A2, §7]).

Lemma 4.3.1. *Let X be a complex-analytic or rigid-analytic space, let $V \subseteq X$ be a Zariski-open subspace, and let $Z \hookrightarrow V$ be a closed immersion. Assume that Z_{red} is a dense Zariski-open subspace in a reduced analytic set $Y \hookrightarrow X$. Let $j : Z \to X$ be the canonical map, and let $\mathscr{I}$ be the coherent ideal sheaf of locally nilpotent sections of $\mathscr{O}_Z$.*

Assume that the subsheaf $j_*(\mathscr{I}) \subseteq j_*(\mathscr{O}_Z)$ is locally killed by a finite exponent. The kernel $\mathscr{K} = \ker(\mathscr{O}_X \to j_*(\mathscr{O}_Z))$ is coherent and the closed analytic subspace $Y' \hookrightarrow X$ cut out by $\mathscr{K}$ meets V in Z as analytic spaces. Moreover, $Y'_{\mathrm{red}} = Y$ (*so the Zariski-open immersion $Z = Y' \cap V \hookrightarrow Y'$ is dense*)*.*

We view Y' as serving the role of a Zariski-closure of Z in X, the point of the lemma being that this makes sense analytically and its formation is local on X. The purpose of the hypothesis that Z has topological closure equal to an analytic set Y is to eliminate essential singularities, as otherwise it can happen that $\mathscr{K} = 0$ even if Z is nowhere dense in X. (The curve $w = e^{1/z}$ in $V = \mathbf{C}^{\times} \times \mathbf{C}$ with $X = \mathbf{C}^2$ gives such an example in the complex-analytic case.)

Proof. We may work locally on X, so we can assume that X is Stein/affinoid and that $j_*(\mathscr{I})$ has Nth power equal to 0 within $j_*(\mathscr{O}_Z)$ for some $N \geq 1$. Define the ideal

$$I = \{a \in \mathscr{O}_X(X) \mid a|_Z = 0\} = \{a \in \mathscr{O}_X(X) \mid a|_V \in \mathscr{I}_Z(V)\}$$

in $\mathscr{O}_X(X)$. This generates an ideal sheaf $\mathscr{J}$ over X, and $\mathscr{J}$ is coherent: this is obvious in the rigid-analytic case (since I is finitely generated in that case), and in the complex-analytic case it follows from the fact that any rising chains of coherent subsheaves of a coherent sheaf on a complex-analytic space is locally stationary [CAS, 5.6.1]. Since $\mathscr{O}_X \to \mathscr{O}_X/\mathscr{J}$ induces a surjection on global sections, we may replace X with the zero-space of $\mathscr{J}$ to reduce to the case $I = 0$. In this case we claim that $\mathscr{I}_Y$ consists of locally nilpotent sections, so Y has the same underlying space as X and hence we are done by taking $Y' = X$ in our present situation.

To prove the required nilpotence, since X is Stein/affinoid it suffices to show that each $s \in \Gamma(X, \mathscr{I}_Y)$ is nilpotent. But certainly $s|_{Z_{\rm red}} = 0$, so $s|_Z$ is a locally nilpotent function on Z. Hence, $s^N|_Z = 0$ by the choice of N above, so $s^N \in I = 0$. ■

In the motivating situation with a reducible Moishezon space X of dimension d, we can apply Lemma 4.3.1 with $Z = V = X_i$ (as an open subspace of X) and $Y = C_i$ with its reduced structure. This provides each C_i with a closed subspace structure $C_i' \subseteq X$ extending the open subspace structure on X_i. Thus, for $Y := \cup_{i<j}(C_i' \cap C_j')$ with its natural closed subspace structure in X, we have $\dim Y < d$ and the natural map

$$f : X' := \coprod C_i' \to X$$

is an isomorphism over $X - Y$, with $Y' = f^{-1}(Y)$ also of dimension $< d$. Since X is Moishezon, by Theorem 3.3.2 the analytic spaces X', Y', and Y are Moishezon. Each infinitesimal neighborhood X_n' and X_n of Y' and Y in X' and X respectively is likewise Moishezon, and so X_n' and X_n are algebraic for all $n \geq 0$ by the inductive hypothesis. By GAGA, the family of analytic maps $f_n : X_n' \to X_n$ algebraizes to define a proper map of formal algebraic spaces $\mathfrak{f} : \mathfrak{X}' \to \mathfrak{X}$. By the discussion in §4.2 (especially concerning the analogue of [A2, 7.7]), $\mathfrak{f}$ is a formal modification. Grant the algebraicity of Moishezon spaces in the irreducible case in dimension d, so by combining this with the inductive hypothesis in dimension $< d$ we see that $X' = \coprod C_i'$ is algebraic, say $X' = \mathscr{X}'^{\rm an}$ for a proper algebraic space $\mathscr{X}'$ over k.

By Artin's theorem on algebraization of formal contractions of algebraic spaces [A2, 3.1], up to unique isomorphism there is a modification of separated algebraic spaces $(\varphi : \mathscr{X}' \to \mathscr{X}, \mathscr{Y} \subseteq \mathscr{X})$ of finite type over k such that the associated formal modification $\widehat{\varphi}$ is equal to $\mathfrak{f}$. Since separated algebraic spaces of finite type over k are analytifiable [CT, 4.2.1], the analytification $(\varphi^{\rm an}, \mathscr{Y}^{\rm an} \subseteq \mathscr{X}^{\rm an})$ makes sense and by [CT, 2.3.1] the map $\varphi^{\rm an}$ is proper. Thus, this pair is trivially a modification in the category of rigid spaces. We seek to prove that X is isomorphic to $\mathscr{X}^{\rm an}$, so X is algebraic, and for this we observe that $(f, Y \subseteq X)$ and $(\varphi^{\rm an}, \mathscr{Y}^{\rm an} \subseteq \mathscr{X}^{\rm an})$ are analytic modifications yielding the same (algebraizable) formal modification $\mathfrak{f}$. More specifically, the equality $\widehat{\varphi} = \mathfrak{f}$ of formal modifications provides compatible isomorphisms $X_n \simeq (\mathscr{X}_n)^{\rm an} = (\mathscr{X}_n)^{\rm an}$ carrying f_n to $\varphi_n^{\rm an}$ for all n, so we can invoke the following analogue of a weak form of [A2, 7.9(i)].

Lemma 4.3.2. *Let $f^{(1)} : X' \to X^{(1)}$ and $f^{(2)} : X' \to X^{(2)}$ be modifications of separated rigid spaces with respect to closed subspaces $Y^{(i)} \subseteq X^{(i)}$. Assume that $(f^{(i)})^{-1}(Y^{(i)})$ is a common closed subspace $Y' \subseteq X'$, and that there is given an isomorphism $\mathfrak{h} : \mathfrak{X}^{(1)} \simeq \mathfrak{X}^{(2)}$ of formal analytic completions satisfying $\mathfrak{h} \circ \mathfrak{f}^{(1)} = \mathfrak{f}^{(2)}$, which is to say a compatible family of isomorphisms $h_n : X_n^{(1)} \simeq X_n^{(2)}$ satisfying $h_n \circ f_n^{(1)} = f_n^{(2)}$ for all $n \geq 0$. Finally, assume that some $X^{(i)}$ is the analytification of a separated algebraic space locally of finite type over k, and that $Y^{(i)}$ arises from a closed subspace of the same algebraic model. There is a unique isomorphism $h : X^{(1)} \simeq X^{(2)}$ satisfying $h \circ f^{(1)} = f^{(2)}$ and inducing $\mathfrak{h}$.*

We will prove this lemma in §4.7 (to which the interested reader may now turn). Artin's proof of a stronger result in the complex-analytic case in [A2, 7.9(i)] is too local to work in the rigid-analytic case, and to overcome this problem we shall use Berkovich spaces. These complications are the reason that we assume that each $X^{(i)}$ is separated and some $X^{(i)}$ is algebraic, hypotheses that are not present in the complex-analytic analogue [A2, 7.9(i)].

4.4. **Meromorphic sections.** Now we can assume that the d-dimensional Moishezon space X is irreducible. Consider an analytic modification $(f : X' \to X, Y \subseteq X)$ with an irreducible X' and Y having underlying space not equal to that of X (so $\dim Y < d$). The pullback map between meromorphic function fields of $X_{\rm red}$ and $X'_{\rm red}$ provides d algebraically independent meromorphic functions on $X'_{\rm red}$ over k, so X' is Moishezon. If X' is algebraic then by the uniqueness of analytic contractions in Lemma 4.3.2 and Artin's existence result for algebraization of formal contractions of algebraic spaces [A2, 3.1] as already cited, the algebraicity of X will follow. To construct such an analytic modification of X (with X' algebraic), we first require a generalization of Corollary 3.1.5 in which the reducedness hypothesis is eliminated. The following theorem (analogous to [A2, 7.16]) gives such a generalization, and its proof occupies the rest of §4.4.

Theorem 4.4.1. *Let X be an irreducible Moishezon space with dimension $d > 0$. There exists a nowhere dense closed subspace $Y \subseteq X$ and a modification $(f : X' \to X, Y \subseteq X)$ with irreducible X' such that there is also a modification $(f'' : X' \to X'', Y'' \subseteq X'')$ with X'' irreducible and projective of dimension d and $\dim Y'' < d$.*

Following Artin, to prove this theorem we induct on r such that $\mathscr{N}^{r+1} = 0$, with $\mathscr{N}$ the coherent nilradical in $\mathscr{O}_X$. We will not give the entire argument, but rather will just focus on those aspects that have to be treated differently in the rigid-analytic case (generally due to admissibility problems). The case $r = 0$ is settled by Corollary 3.1.5, and (as in [A2, 7.16]) to carry out the induction we need a notion of meromorphic section of a coherent sheaf on a general rigid space. The definition of meromorphic sections that is used by Artin in the complex-analytic case is too local to be used in rigid geometry, so we now give a different definition that is equivalent to Artin's in the complex-analytic case but avoids locality problems (and also leads to simpler arguments even in the complex-analytic case).

Let X be a complex-analytic or rigid-analytic space and let $Y \hookrightarrow X$ be an analytic set. (We do not assume Y to have empty interior in X, but it does so in applications.) Let $\mathscr{F}$ be a coherent sheaf on X. We define the sheaf $\mathscr{F}_{(X,Y)}$ of *meromorphic sections of $\mathscr{F}$ along* Y to be

$$\mathscr{F}_{(X,Y)} = \varinjlim \mathscr{H}om(\mathscr{I}_Y^n, \mathscr{F})$$

for a coherent ideal sheaf $\mathscr{I}_Y$ that cuts out Y; the global choice of $\mathscr{I}_Y$ clearly does not matter (e.g., we could use the unique radical choice). This definition is not the same as Artin's, but in Remark 4.4.2 we will show that it is equivalent to his definition in the complex-analytic case. Our definition will be used to deduce non-archimedean analogues of some basic facts that are used in Artin's proof over **C**. Beware that if X is non-reduced then the natural map $\mathscr{O}_X \to (\mathscr{O}_X)_{(X,Y)}$ may have a nonzero kernel, even if Y is nowhere dense in X. In particular, if $\mathscr{F} = \mathscr{O}_X$ and Y is nowhere dense in X but X is non-reduced then there may not be a realization of $\mathscr{F}_{(X,Y)}$ as an $\mathscr{O}_X$-subalgebra of the sheaf $\mathscr{M}_X$ of meromorphic functions on X as defined in §2.1. This will not create any problems.

Let $i : U = X - Y \to X$ be the natural open immersion, so we get a natural map

$$\mathscr{F}_{(X,Y)} \to i_*(\mathscr{F}_{(X,Y)}|_U) = i_*(\mathscr{F}|_U).$$

We claim that this map is injective, so $\mathscr{F}_{(X,Y)}$ is functorially identified with a subsheaf of $i_*(\mathscr{F}|_U)$. Indeed, if $s \in \Gamma(V, \mathscr{F}_{(X,Y)})$ for some open $V \subseteq X$ and $s|_{V\cap U} = 0$ then by working over a suitable collection of opens V_j covering V and renaming such a V_j as X we can assume that $\mathscr{I}_Y$ is globally generated and that s arises from a map of coherent sheaves $h : \mathscr{I}_Y^n \to \mathscr{F}$ for some n such that $h|_U = 0$ as a map from $\mathscr{O}_U$ to $\mathscr{F}|_U$. Thus, the image of the map h over X is a coherent sheaf on X that vanishes on U, so by shrinking more on X it is killed by $\mathscr{I}_Y^m$ for some $m \geq 0$. The map h therefore kills the subsheaf $\mathscr{I}_Y^{n+m} \subseteq \mathscr{I}_Y^n$, so in the direct limit sheaf $\mathscr{F}_{(X,Y)}$ the section s vanishes as desired.

Remark 4.4.2. Though it is not logically necessary for our purposes, let us prove the equivalence of our definition of $\mathscr{F}_{(X,Y)}$ with Artin's definition in the complex-analytic case. (The reader may ignore this verification.) It was just shown how to identify our notion of $\mathscr{F}_{(X,Y)}$ with a subsheaf of $i_*(\mathscr{F}|_U)$, and Artin's definition is also as a subsheaf of $i_*(\mathscr{F}|_U)$. Both definitions commute with localization on X, so to identify these subsheaves it suffices to compare their global sections. According to Artin's definition, if we let $\mathscr{I}_Y$ denote the unique radical coherent ideal sheaf on X that cuts out Y (as we may also use in the above definition of $\mathscr{F}_{(X,Y)}$ as a direct limit sheaf) then a global meromorphic section s of $\mathscr{F}$ with respect to Y is an element $s \in \Gamma(X, i_*(\mathscr{F}|_U)) = \Gamma(U, \mathscr{F})$ such that for each $y \in Y$ and local section f of $\mathscr{I}_Y$ on an open V around y in X there is an $n \geq 0$ such that $f^n s \in \Gamma(V \cap U, \mathscr{F})$ extends to $\Gamma(V, \mathscr{F})$. The global sections of the subsheaf $\mathscr{F}_{(X,Y)} \subseteq i_*(\mathscr{F}|_U)$ according to our definition certainly satisfy this requirement. For the reverse inclusion, it therefore suffices to compare stalks at all points of the complex-analytic space X. The situation at any point $x \in U$ is trivial (both subsheaves coincide with $\mathscr{F}|_U$ on U), and for $x \in Y$ we have (under our definition)

$$(\mathscr{F}_{(X,Y)})_x = \varinjlim \mathrm{Hom}(\mathscr{I}^n_{Y,x}, \mathscr{F}_x) = \Gamma(\mathrm{Spec}\, \mathscr{O}_{X,x} - \mathrm{Spec}\, \mathscr{O}_{Y,x}, \widetilde{\mathscr{F}_x}),$$

where $\mathscr{O}_{Y,x} := \mathscr{O}_{X,x}/\mathscr{I}_{Y,x}$, $\widetilde{\mathscr{F}_x}$ denotes the coherent sheaf on $\mathrm{Spec}\, \mathscr{O}_{X,x}$ associated to $\mathscr{F}_x$, and the second equality is a standard formula of Deligne for the module of sections of a quasi-coherent sheaf over an open subscheme of an affine noetherian scheme. If $f_1, \dots, f_r$ are generators of the ideal $\mathscr{I}_{Y,x}$ then $\mathrm{Spec}\, \mathscr{O}_{X,x} - \mathrm{Spec}\, \mathscr{O}_{Y,x}$ is covered by the open affines $\mathrm{Spec}\, \mathscr{O}_{X,x}[1/f_j]$, so

$$(\mathscr{F}_{(X,Y)})_x = \ker(\prod_j \mathscr{F}_x[1/f_j] \to \prod_{i,j} \mathscr{F}_x[1/f_i f_j]).$$

But according to Artin's definition, a meromorphic section s of $\mathscr{F}$ along Y on an open neighborhood in X containing the point $x \in Y$ has the property that (after shrinking around $x \in Y$) there exists $m \geq 1$ for which $f_j^m s$ over U extends to a local section s_j of $\mathscr{F}$ around x in X for each j. The collection of fractions $s_j/f_j^m \in \mathscr{F}_x[1/f_j]$ clearly satisfies the requirements to lie in the above kernel, and so $s \in (\mathscr{F}_{(X,Y)})_x$ inside of $(i_*(\mathscr{F}|_U))_x$. This verifies that our definition of $\mathscr{F}_{(X,Y)}$ and Artin's definition of $\mathscr{F}_{(X,Y)}$ coincide as subsheaves of $i_*(\mathscr{F}|_U)$ in the complex-analytic case.

Now we use our definition of $\mathscr{F}_{(X,Y)}$ to replace two stalkwise arguments of Artin that cannot be applied in the rigid-analytic case. The first argument concerns [A2, 7.18(i)], as follows.

Lemma 4.4.3. *Let $\mathscr{X}$ be an analytifiable (e.g., separated) algebraic space and $i : \mathscr{U} \to \mathscr{X}$ an open immersion with closed complement $\mathscr{Y}$ (given the reduced structure, say). For any coherent sheaf $\mathscr{F}$ on $\mathscr{X}$, the natural map of $\mathscr{O}_{\mathscr{X}^{\mathrm{an}}}$-modules $(i_*(\mathscr{F}|_{\mathscr{U}}))^{\mathrm{an}} \to i_*^{\mathrm{an}}(\mathscr{F}^{\mathrm{an}}|_{\mathscr{U}^{\mathrm{an}}})$ is an isomorphism onto the subsheaf $(\mathscr{F}^{\mathrm{an}})_{(\mathscr{X}^{\mathrm{an}}, \mathscr{Y}^{\mathrm{an}})}$.*

Proof. By Deligne's formula

$$\varinjlim \mathrm{Hom}(J^n, M) \simeq \Gamma(\mathrm{Spec}\, R - \mathrm{Spec}\, R/J, \widetilde{M})$$

for any noetherian ring R and R-module M, we have $i_*(\mathscr{F}|_{\mathscr{U}}) = \varinjlim \mathscr{H}om(\mathscr{I}^n_{\mathscr{Y}}, \mathscr{F})$. The formation of Hom-sheaves between coherent sheaves is trivially compatible with analytification, so it remains to check that analytification (relative to the Tate-étale topology on rigid spaces in the non-archimedean case, as defined in [CT, 2.1]) is compatible with the formation of direct limits. This is a simple exercise in universal mapping properties and the adjointness of pushforward and pullback of sheaves of modules with respect to the analytification morphism of locally ringed topoi $\widetilde{(\mathscr{X}^{\mathrm{an}})_{\text{ét}}} \to \widetilde{\mathscr{X}_{\text{ét}}}$ (as considered in [CT, 3.3] in the non-archimedean case). ■

The second place where the notion of meromorphic sections plays a role is in the verification of the exactness of the sequence [A2, (7.20)] that is defined as follows. Let X be Moishezon and let $\mathscr{N}$ be a coherent ideal sheaf on X such that $\mathscr{N}^2 = 0$ and the zero-space $\overline{X}$ of $\mathscr{N}$ has the form $\overline{X} \simeq \overline{\mathscr{X}}^{\mathrm{an}}$ for a proper algebraic space $\overline{\mathscr{X}}$. Let $\overline{\mathscr{U}} = \mathrm{Spec}\, \overline{A}$ be a dense affine open subscheme of $\overline{\mathscr{X}}$, so $\overline{\mathscr{U}}^{\mathrm{an}}$ is a dense Zariski-open subspace of X. Let $Y := \overline{X} - \overline{\mathscr{U}}^{\mathrm{an}}$ be its complement, say with the reduced structure. There is a naturally associated complex

$$0 \to \mathscr{N}_{(X,Y)} \to \mathscr{O}_{(X,Y)} \to \mathscr{O}_{(\overline{X},Y)} \to 0 \tag{4.4.1}$$

arising by functoriality applied to the exact sequence $0 \to \mathscr{N} \to \mathscr{O}_X \to \mathscr{O}_{\overline{X}} \to 0$.

Lemma 4.4.4. *The sequence* (4.4.1) *is exact.*

We give a proof that replaces the stalkwise argument of Artin with another argument that works in both the complex-analytic and rigid-analytic categories (using Remark 4.4.2 in the complex-analytic case).

Proof. Left-exactness of (4.4.1) is clear from the definition of sheaves of meromorphic sections along an analytic set. Thus, it suffices to prove that $\varinjlim \mathscr{E}xt^1(\mathscr{I}_Y^n, \mathscr{N}) = 0$. This is a direct limit of coherent sheaves, so it is easy to see that it vanishes if and only if its stalks are zero. The stalk at $x \in X$ is $\varinjlim \mathrm{Ext}^1_{\mathscr{O}_{X,x}}(\mathscr{I}_{Y,x}^n, \mathscr{N}_x)$, which obviously vanishes if $x \notin Y$. Hence, we can assume $x = y \in Y$. Let $V = \mathrm{Spec}\,\mathscr{O}_{X,y} - \mathrm{Spec}\,\mathscr{O}_{Y,y}$. The direct limit of Ext^1's at $x = y$ is identified with $\mathrm{H}^1(V, \widetilde{\mathscr{N}_y})$, where $\widetilde{\mathscr{N}_y}$ is the coherent sheaf on $\mathrm{Spec}\,\mathscr{O}_{X,y}$ associated to $\mathscr{N}_y$. But $\mathscr{N}_y$ is naturally an $\mathscr{O}_{\overline{X},y}$-module since $\mathscr{N}$ is a square-zero ideal in $\mathscr{O}_X$ and $\mathscr{O}_{\overline{X}} = \mathscr{O}_X/\mathscr{N}$, so this degree-1 coherent cohomology may be computed on $\overline{V} = \mathrm{Spec}\,\mathscr{O}_{\overline{X},y} - \mathrm{Spec}\,\mathscr{O}_{Y,y}$. Hence, it is enough to prove that $\overline{V}$ is affine. Since the reduced Y is a closed subspace of $\overline{X} = \overline{\mathscr{X}}^{\mathrm{an}}$ with $\overline{\mathscr{X}}$ a proper algebraic space, by GAGA for algebraic spaces it has the form $Y = \overline{\mathscr{Y}}^{\mathrm{an}}$ for a unique reduced closed subspace $\overline{\mathscr{Y}} \subseteq \overline{\mathscr{X}}$. The natural map $\mathrm{Spec}\,\mathscr{O}_{\overline{X},y} \to \mathrm{Spec}\,\mathscr{O}_{\overline{\mathscr{X}},y}$ is an affine morphism under which $\overline{V}$ is the preimage of $\overline{\mathscr{V}} := \mathrm{Spec}\,\mathscr{O}_{\overline{\mathscr{X}},y} - \mathrm{Spec}\,\mathscr{O}_{\overline{\mathscr{Y}},y}$, so it suffices to prove that $\overline{\mathscr{V}}$ is affine. But $\overline{\mathscr{V}} = \mathrm{Spec}\,\mathscr{O}_{\overline{\mathscr{X}},y} \times_{\overline{\mathscr{X}}} \mathrm{Spec}\,\overline{A}$ by definition of Y in terms of $\overline{A}$ above, and the open immersion of algebraic spaces $\mathrm{Spec}\,\overline{A} \to \overline{\mathscr{X}}$ is an affine morphism since $\overline{\mathscr{X}}$ is separated. ∎

The following remark provides a global procedure that replaces a step near the end of Artin's proof of (the complex-analytic analogue of) Theorem 4.4.1, where he uses the preceding results on meromorphic sections to globally remove indeterminacies in a rational morphism defined by a collection of meromorphic functions (with a definition of meromorphic function that is inconsistent with our definition of $\mathscr{M}_X$ in the non-reduced case, though equivalent to ours in the reduced case and better-suited for the present purposes).

Remark 4.4.5. By using the Zariski-closure construction in Lemma 4.3.1, the discussion of globally removing indeterminacies in a rational map in Example 2.1.4 can be carried out without reducedness hypotheses. To be precise, in the setup of that example, we make no reducedness assumptions on X and we let $U \subseteq X$ be a dense Zariski-open subspace on which a morphism $f : U \to \mathbf{P}^1$ is given. We assume that $f_{\mathrm{red}} := f|_{U_{\mathrm{red}}}$ is meromorphic on X_{red} (i.e., f_{red} comes from $(\mathscr{O}_{X_{\mathrm{red}}})_{(X_{\mathrm{red}}, X_{\mathrm{red}} - U_{\mathrm{red}})}(X_{\mathrm{red}}) \subseteq \mathscr{M}_{X_{\mathrm{red}}}(X_{\mathrm{red}})$, but perhaps f does not arise from $\mathscr{M}_X(X)$). Then Example 2.1.4 provides a reduced analytic set $X' \subseteq X \times \mathbf{P}^1$ meeting $U_{\mathrm{red}} \times \mathbf{P}^1 = (U \times \mathbf{P}^1)_{\mathrm{red}}$ in the graph of f_{red}, with $\Gamma_{f_{\mathrm{red}}} \subseteq X'$ a dense Zariski-open subspace. We apply Lemma 4.3.1 to the dense Zariski-open subspace $V = U \times \mathbf{P}^1$ in $X \times \mathbf{P}^1$ with $Z = \Gamma_f$ and $Y = X'$. This provides a closed immersion $\widetilde{X}' \hookrightarrow X \times \mathbf{P}^1$ with underlying reduced space X' such that $\widetilde{X}' \cap (U \times \mathbf{P}^1) = \Gamma_f$ as closed subspaces of $U \times \mathbf{P}^1$. Thus, the second projection $f' : \widetilde{X}' \to \mathbf{P}^1$ is a morphism extending f via the first projection $\widetilde{X}' \to X$ that is an isomorphism over U. By iterating this procedure, we can handle a collection of several such f at once by building a morphism to $(\mathbf{P}^1)^n$. This argument also works in the complex-analytic case.

4.5. **Formal dilatations.** Arguing as at the beginning of §4.4, it follows from Lemma 4.3.2 and Theorem 4.4.1 that to prove the algebraicity of irreducible Moishezon spaces X of dimension d it remains to consider the case when there is a modification $(f : X \to P, Y \subseteq P)$ with P irreducible and projective of dimension d and $\dim Y < d$. We call X a *dilatation* of P (with respect to Y). By GAGA, each infinitesimal neighborhood P_n of Y in P in algebraic. Moreover, the pullback X_n of P_n along f is Moishezon by Theorem 3.3.2, and $\dim X_n < d$ since each X_n is a proper analytic subset in X. Thus, the induction on dimension implies that each X_n is algebraic. The analytic maps $f_n : X_n \to P_n$ therefore algebraize to define a map between proper formal algebraic spaces $\mathfrak{f} : \mathfrak{X} \to \mathfrak{P}$, and by §4.2 this is a formal modification. Let $\mathscr{P}$ be the projective k-scheme that algebraizes the projective rigid space P, and let $\mathscr{Y} \subseteq \mathscr{P}$ be the corresponding algebraization of $Y \subseteq P$. By Artin's existence result on algebraization of formal dilatations [A2, 3.2], there is a unique modification of algebraic spaces $(g : \mathscr{X} \to \mathscr{P}, \mathscr{Y} \subseteq \mathscr{P})$ for which the associated formal modification $\mathfrak{g}$ of $\mathfrak{P}$ is $\mathfrak{f}$. Thus, the analytic modification $(g^{\mathrm{an}} : \mathscr{X}^{\mathrm{an}} \to P, Y \subseteq P)$ also gives rise to $\mathfrak{f}$. To prove $X \simeq \mathscr{X}^{\mathrm{an}}$, thereby completing the proof of Theorem 4.1.1, it remains to prove an analogue for analytic dilatations of the uniqueness result in Lemma 4.3.2 for analytic contractions. Such a result is provided by the following analogue of a weak form of [A2, 7.9(*ii*)].

Lemma 4.5.1. *Consider a pair of analytic modifications $f^{(i)} : X'^{(i)} \to X$ with respect to a common closed subspace $Y \subseteq X$. Let $Y'^{(i)} = (f^{(i)})^{-1}(Y)$, and assume that there is given an isomorphism $h'_0 : Y'^{(1)} \simeq Y'^{(2)}$*

over Y that lifts to a compatible family of isomorphisms between infinitesimal neighborhoods $h'_n : {X'}^{(1)}_n \simeq {X'}^{(2)}_n$ over X_n for all $n \geq 0$.

Assume that each ${X'}^{(i)}$ is separated and that some $f^{(i_0)}$ is the analytification of a proper map between separated algebraic spaces locally of finite type over k, with ${Y'}^{(i_0)} \subseteq {X'}^{(i_0)}$ arising from a closed subspace of the algebraic model for ${X'}^{(i_0)}$. There is a unique X-isomorphism $h' : {X'}^{(1)} \simeq {X'}^{(2)}$ inducing the h'_n's.

The proof of Lemma 4.5.1 is given in §4.8 (building on arguments using in §4.7 to prove Lemma 4.3.2). In the complex-analytic analogue [A2, 7.9(*ii*)] it is not necessary to assume that some $f^{(i_0)}$ algebraizes (nor that the $X^{(i)}$'s are separated). However, the only way we can see to circumvent the locality aspects of that proof is to work with k-analytic spaces in the sense of Berkovich, and the available analytic approximation results of Artin (and Bosch) that are used in Artin's argument in the complex-analytic case are not applicable to local rings on good k-analytic spaces. Hence, rather than approximate formal solutions to analytic equations we prefer to approximate formal solutions to polynomial equations, and that is why we impose the hypothesis (sufficient for our purposes) that some $f^{(i)}$ is algebraic. The ability to make such algebraic approximations will rest on the excellence of local rings on good k-analytic spaces [D2, 2.6].

4.6. **Passage to schemes.** To conclude the proof of algebraicity of Moishezon spaces, it remains to prove Lemma 4.3.2 and Lemma 4.5.1. Before we take up these two proofs, we note that a basic source of complications in the arguments to follow is that we lack a method to characterize when a local k-algebra map $\mathscr{O}_{X,x} \to \mathscr{O}_{X',x'}$ between local rings on good k-analytic spaces (in the sense of Berkovich) arises from a map between germs $(X',x') \to (X,x)$. The only way we know to bypass this problem is to put ourselves in situations where the source local ring (i.e., the target germ) is algebraic. This is why we had to impose algebraicity hypotheses in Lemmas 4.3.2 and 4.5.1. It will also be convenient to replace certain algebraic spaces with schemes, and the following preliminary lemma will be used to carry out this passage from algebraic spaces to schemes in the proofs of Lemmas 4.3.2 and 4.5.1.

Lemma 4.6.1. *Let $\mathscr{X}$ be a separated algebraic space locally of finite type over k, and let $X = \mathscr{X}^{\rm an}$ be its analytification in the sense of k-analytic spaces. For any $x \in X$ there exists an étale map $\mathscr{U} \to \mathscr{X}$ from a (separated) scheme such that the analytified map $U \to X$ admits a point $u \in U$ over x for which the natural map of germs $(U,u) \to (X,x)$ is an isomorphism.*

Moreover, if X' is a good k-analytic space and $x' \in X'$ is a point then a map of k-analytic germs $(X',x') \to (X,x)$ is uniquely determined by the induced local k-algebra map $\mathscr{O}_{X,x} \to \mathscr{O}_{X',x'}$.

We refer the reader to [CT, 4.2.2] for the analytifiability of such algebraic spaces in terms of k-analytic spaces. Note that since X is a Berkovich space (rather than a rigid-analytic space), for $x \in X$ typically $[k(x) : k]$ is not finite and so (due to the construction of X) it is not obvious how to relate x to a point on the algebraic space $\mathscr{X}$ in general; this issue arises in the proof below. We do not address the issue of determining when a given local k-algebra map between local rings on good k-analytic spaces arises from a map between k-analytic germs because this existence problem seems to be nontrivial even when the target k-analytic germ lies on the analytification of an algebraic k-scheme.

Proof. To find such a $\mathscr{U}$, we first explain how to construct a canonical map of sets $|X| \to |\mathscr{X}|$ between underlying sets of points; in the scheme case this will coincide with the map on sets arising from the universal property of k-analytification in the sense of locally ringed spaces over k [Ber2, 2.6.1]. The subtlety in the case of analytification for algebraic spaces is that we have defined the analytification functor on algebraic spaces as merely a (canonical) construction with reasonable functorial properties rather than characterizing it by a universal mapping property. (The topos-theoretic characterization via universal properties in [C] does not seem to yield a simpler approach for our present purposes.)

Make an initial choice of étale chart $\mathscr{R}_0 \rightrightarrows \mathscr{U}_0$ for $\mathscr{X}$, so the k-analytification $R_0 \rightrightarrows U_0$ is an étale equivalence relation with k-analytic quotient X. In particular, $|U_0| \to |X|$ is surjective. We claim that if $\xi \in X$ is a point then any two points $u, u' \in U_0$ over ξ have images in $|\mathscr{U}_0|$ over the same point of $|\mathscr{X}|$. This will give a well-defined map of sets $|X| \to |\mathscr{X}|$ that is easily seen to be independent of the chart (and hence recovers the usual such map when $\mathscr{X}$ is a scheme). Our claim is that the natural map of sets

$|U_0| \times_{|X|} |U_0| \to |\mathscr{U}_0| \times |\mathscr{U}_0|$ lands in the subset $|\mathscr{U}_0| \times_{|\mathscr{X}|} |\mathscr{U}_0|$. But $|U_0 \times_X U_0| \to |U_0| \times_{|X|} |U_0|$ is surjective (as for any fiber product of k-analytic spaces) and $U_0 \times_X U_0 = R_0 = \mathscr{R}_0^{\rm an}$, so by factoring $|R_0| \to |\mathscr{U}_0| \times |\mathscr{U}_0|$ through the canonical (surjective) map

$$|R_0| \twoheadrightarrow |\mathscr{R}_0| = |\mathscr{U}_0 \times_{\mathscr{X}} \mathscr{U}_0| \twoheadrightarrow |\mathscr{U}_0| \times_{|\mathscr{X}|} |\mathscr{U}_0|$$

we deduce the claim. It follows from the construction that the formation of the map of sets $|X| \to |\mathscr{X}|$ is functorial in $\mathscr{X}$ and is compatible with passage to Zariski-open subspaces of $\mathscr{X}$. (More specifically, this map is continuous with respect to Zariski topologies.)

The construction of the map $|X| \to |\mathscr{X}|$ works more generally if $\mathscr{X}$ is assumed to be analytifiable (in the sense of k-analytic spaces) rather than separated. It is an important technical remark that if $\mathscr{X}' \to \mathscr{X}$ is a map between such analytifiable algebraic spaces over k and if $X' \to X$ is the induced map between the analytifications then the natural map of sets $|X'| \to |X| \times_{|\mathscr{X}|} |\mathscr{X}'|$ is surjective. To see this, we first recall that the formation of these analytifications commutes with any analytic extension of the base field [CT, 2.3.5, §4.1], so it suffices to prove surjectivity in the case of k-rational points. By functoriality of analytification this case is trivial.

Let $\mathbf{x} \in |\mathscr{X}|$ be the image of x. By [Kn, II, 6.4] we can choose an étale map $h : \mathscr{U} \to \mathscr{X}$ from a (separated) scheme so that there is a point $\mathbf{u} \in |\mathscr{U}|$ over $\mathbf{x}$ with $k(\mathbf{u}) = k(\mathbf{x})$. Using Zariski-localization on $\mathscr{X}$ and $\mathscr{U}$ around $\mathbf{x}$ and $\mathbf{u}$, we can arrange that $\mathbf{u}$ is the unique point over $\mathbf{x}$. Hence, the étale map of schemes $p_1 : \mathscr{U} \times_{\mathscr{X}} \mathscr{U} \to \mathscr{U}$ pulls back to an isomorphism over $\mathbf{u}$. Since the points $\mathbf{u} \in |\mathscr{U}|$ and $x \in X$ lie over the same point $\mathbf{x} \in |\mathscr{X}|$, we can choose a point $u \in U$ over both $\mathbf{u}$ and x. Since analytification of (analytifiable) algebraic spaces is compatible with the formation of fiber products [CT, 2.2.3, §4.1], the map $p_1^{\rm an} : U \times_X U \to U$ is identified with the first projection. Analytification of algebraic schemes commutes with extension of the ground field, so applying this to p_1 and the field extension $k \to \mathscr{H}(u)$ gives (via a calculation with the canonical $\mathscr{H}(u)$-point over u in the $\mathscr{H}(u)$-analytic space $U \widehat{\otimes}_k \mathscr{H}(u)$) that

$$(p_1^{\rm an})^{-1}(u) = (p_1^{-1}(\mathbf{u}) \otimes_{k(\mathbf{u})} \mathscr{H}(u))^{\rm an} \simeq \mathscr{M}(\mathscr{H}(u))$$

as $\mathscr{H}(u)$-analytic spaces. In other words, $p_1^{\rm an}$ pulls back to an isomorphism over $u \in U$. But $p_1^{\rm an}$ is a base change of the étale map $\pi : U \to X$ by itself, with $u \in U$ over $x \in X$, so it follows that π must also pull back to an isomorphism over x. It then follows from the étaleness of π that $(U, u) \simeq (X, x)$, by [Ber2, 3.4.1].

To prove that a map of good k-analytic germs $(X', x') \to (X, x)$ to an algebraic target $X = \mathscr{X}^{\rm an}$ is uniquely determined by the induced map of analytic local rings, it now suffices to consider the case when $\mathscr{X}$ is a scheme, and even an affine scheme. By standard arguments with coherent ideal sheaves we can assume $\mathscr{X} = \mathbf{A}_k^N = \operatorname{Spec} k[T_1, \ldots, T_N]$ for some $N \geq 1$. In this case we can identify morphisms $X' \to X$ with ordered N-tuples in $\mathscr{O}_{X'}(X')$. Thus, if $f, g : X' \rightrightarrows X$ are two maps satisfying $f(x') = g(x') = x$ and they induce the same pullback map on local rings then the corresponding map $(f, g) : X' \to X \times X$ kills the coherent ideal $(T_1 - T_{N+1}, \ldots, T_N - T_{2N})$ of the diagonal after shrinking X' around x'. This shows that f and g coincide near x', as desired. ■

4.7. **Uniqueness of analytic contractions.** We now prove Lemma 4.3.2. Uniqueness is clear since the map h is determined set-theoretically and on complete local rings. Moreover, if h exists as a morphism then it must be an isomorphism. Indeed, h is automatically separated since each $X^{(i)}$ is separated, and hence h is proper (since $h \circ f^{(1)} = f^{(2)}$ is proper and $f^{(1)}$ is proper with $X^{(1)} = f^{(1)}(X') \cup Y^{(1)}$ and $h : Y^{(1)} \simeq Y^{(2)} \subseteq X^{(2)}$, so [T1, 4.5, 4.6] may be applied). The map h has finite fibers, so it is finite, and thus it is an isomorphism since it is bijective and induces isomorphisms on complete local rings. The problem is therefore merely to construct h as a morphism, so we can switch the roles of $X^{(1)}$ and $X^{(2)}$ if necessary to arrange that $X^{(2)}$ is algebraic.

For existence we wish to work locally and then pass to k-analytic spaces. (We can associate k-analytic spaces to rigid spaces satisfying some finiteness conditions, but not to arbitrary rigid spaces. This is why we first need to localize the problem.) To do this, we first reformulate the assertion to be proved. We suppose there is given a pair of modifications $(f^{(i)} : X'^{(i)} \to X^{(i)}, Y^{(i)} \subseteq X^{(i)})$ and we let $Y'^{(i)} = (f^{(i)})^{-1}(Y^{(i)})$. Assume that there is given a map of rigid spaces $h' : X'^{(1)} \to X'^{(2)}$ satisfying $h'^{-1}(Y'^{(2)}) = Y'^{(1)}$, and a morphism $\mathfrak{h} : \mathfrak{X}^{(1)} \to \mathfrak{X}^{(2)}$ satisfying $\mathfrak{h} \circ \mathfrak{f}^{(1)} = \mathfrak{f}^{(2)} \circ \mathfrak{h}'$. Finally, assume that $X^{(2)} = (\mathscr{X}^{(2)})^{\rm an}$ for a separated algebraic space $\mathscr{X}^{(2)}$, and that $Y^{(2)} = (\mathscr{Y}^{(2)})^{\rm an}$ for a closed subspace $\mathscr{Y}^{(2)} \subseteq \mathscr{X}^{(2)}$. We claim that there is a unique map $h : X^{(1)} \to X^{(2)}$ satisfying $h \circ f^{(1)} = f^{(2)} \circ h'$ and inducing $\mathfrak{h}$. This obviously implies the result that we want to prove, and the uniqueness is once again clear.

We can work locally on $X^{(1)}$ for existence, so we may assume $X^{(1)}$ is affinoid. Let $\{\mathscr{V}_i^{(2)}\}$ be an open cover of $\mathscr{X}^{(2)}$ by quasi-compact opens, so the overlaps $W_i^{(2)} = Y^{(2)} \cap (\mathscr{V}_i^{(2)})^{\rm an}$ are a Zariski-open cover of $Y^{(2)}$. Let $W_i^{(1)} = h_0^{-1}(W_i^{(2)})$, so $\{W_i^{(1)}\}$ is an admissible open cover of the analytic set $Y^{(1)}$ in the affinoid $X^{(1)}$. By the Gerritzen–Grauert theorem there is a finite collection of rational affinoid opens $\{U_j^{(1)}\}$ in $X^{(1)}$ such that $\{U_j^{(1)} \cap Y^{(1)}\}$ refines $\{W_i^{(1)}\}$. The union $U = \cup U_j^{(1)}$ is a quasi-compact admissible open in the affinoid $X^{(1)}$ such that $Y^{(1)} \subseteq U$, so by [Ki, 2.3] there are finitely many affinoid opens U_r in $X^{(1)} - Y^{(1)}$ such that $\{U_r, U_j^{(1)}\}_{j,r}$ is an admissible cover of $X^{(1)}$. It suffices to solve the problem over each of the constituents of this covering. The case $Y^{(1)} = \emptyset$ is trivial, and in general we are reduced to the case when $h_0(Y^{(1)})$ is contained in the analytification of a quasi-compact open subspace of $\mathscr{X}^{(2)}$. We may therefore replace $X^{(2)}$ with such a subspace, so $\mathscr{X}^{(2)}$ is separated and of finite type and $X^{(1)}$ is affinoid.

Under the equivalence of categories in [Ber2, 1.6.1] between the category of paracompact Hausdorff strictly k-analytic spaces and a full subcategory of the category of quasi-separated rigid-analytic spaces over k, let $\mathcal{X}^{(i)}$ be the good separated strictly k-analytic space corresponding to $X^{(i)}$. Let $F^{(i)} : \mathcal{X}'^{(i)} \to \mathcal{X}^{(i)}$ likewise correspond to $f^{(i)}$, $\mathcal{Y}^{(i)} \subseteq \mathcal{X}^{(i)}$ correspond to $Y^{(i)} \subseteq X^{(i)}$, and $H' : \mathcal{X}'^{(1)} \to \mathcal{X}'^{(2)}$ correspond to h'. In particular, $F^{(i)}$ is proper by [T1, 4.5], so each $\mathcal{X}'^{(i)}$ is a good separated strictly k-analytic space. Let $\mathfrak{H}_n : \mathcal{X}_n^{(1)} \to \mathcal{X}_n^{(2)}$ correspond to h_n, so $\mathfrak{H}_n \circ F_n^{(1)} = F_n^{(2)} \circ H'_n$ for all $n \geq 0$. We will construct a map $H : \mathcal{X}^{(1)} \to \mathcal{X}^{(2)}$ inducing each $\mathfrak{H}_n$ and satisfying $H \circ F^{(1)} = F^{(2)} \circ H'$. Such an H corresponds to the desired map $h : X^{(1)} \to X^{(2)}$.

More generally, it suffices to solve this construction problem for H in the category of good separated strictly k-analytic spaces, with $\mathcal{X}^{(2)}$ arising from a separated algebraic space of finite type over k but $\mathcal{X}^{(1)}$ not necessarily k-affinoid. In this setting the desired map H is at least uniquely determined set-theoretically if it exists, so its uniqueness follows as in the rigid case.

It suffices to carry out the construction of H locally on $\mathcal{X}^{(1)}$. The problem is trivial over $\mathcal{X}^{(1)} - \mathcal{Y}^{(1)}$ since $F^{(1)}$ is an isomorphism over this Zariski-open locus. Thus, it suffices to solve the existence problem on an open around y_1 in $\mathcal{X}^{(1)}$ for each $y_1 \in \mathcal{Y}^{(1)}$. To circumvent the problem of constructing maps of germs of good k-analytic spaces inducing a given local k-algebra map between the corresponding local rings, we want to next reduce to the case when $\mathscr{X}^{(2)}$ is a scheme. By Lemma 4.6.1 there is an étale map $\mathscr{U} \to \mathscr{X}^{(2)}$ from a (separated) scheme such that the associated analytic étale map $\pi : \mathcal{U} \to \mathcal{X}^{(2)}$ has a point u over $y_2 = \mathfrak{H}_0(y_1) \in \mathcal{Y}^{(2)}$ for which $(\mathcal{U}, u) \simeq (\mathcal{X}^{(2)}, y_2)$ via π. Let $\mathcal{V} \subseteq \mathcal{X}^{(2)}$ be an open around y_2 over which π restricts to an isomorphism near u. The open pullback $H'^{-1}((F^{(2)})^{-1}(\mathcal{V}))$ in $\mathcal{X}'^{(1)}$ contains $H'^{-1}((F^{(2)})^{-1}(y_2)) \supseteq (F^{(1)})^{-1}(y_1)$, so since $F^{(1)} : \mathcal{X}'^{(1)} \to \mathcal{X}^{(1)}$ is proper we can find an open $\mathcal{W} \subseteq \mathcal{X}^{(1)}$ around y_1 such that $(F^{(1)})^{-1}(\mathcal{W}) \subseteq H'^{-1}((F^{(2)})^{-1}(\mathcal{V}))$. Thus, if we replace $\mathcal{X}^{(1)}$ with $\mathcal{W}$ then we can arrange that H' factors through $(F^{(2)})^{-1}(\mathcal{V})$. Further shrinking lets us assume that $\mathfrak{H}_0$ (and hence every $\mathfrak{H}_n$) factors through $\mathcal{V}$. But we chose $\mathcal{V}$ around y_2 so that $\pi : \mathcal{U} \to \mathcal{X}^{(2)}$ restricts to an isomorphism from an open around u onto $\mathcal{V}$. Hence, we can now replace $\mathscr{X}^{(2)}$ with $\mathscr{U}$. This brings us to the case when $\mathscr{X}^{(2)}$ is a scheme. Beware that $\mathscr{X}'^{(2)}$ is still merely an algebraic space, not necessarily a scheme.

The local construction of the required map $H : \mathcal{X}^{(1)} \to \mathcal{X}^{(2)}$ near $y_1 \in \mathcal{Y}^{(1)}$ (with $y_2 = \mathfrak{H}_0(y_1) \in \mathcal{Y}^{(2)}$) can be carried out by simplifying Artin's pointwise construction in the complex-analytic case, as we shall now explain. (There is also an important technical point whose justification was omitted in Artin's proof, relevant in both the non-archimedean and complex-analytic cases, that is addressed in Lemma 4.7.1 below.) Let $\mathscr{O}^{(i)} = \mathscr{O}_{\mathcal{X}^{(i)}, y_i}$ and define

$$\mathscr{O}'^{(i)} = (F_*^{(i)} \mathscr{O}_{\mathcal{X}'^{(i)}})_{y_i} \simeq \varinjlim \mathrm{H}^0(\mathcal{U}^{(i)}, \mathscr{O}_{\mathcal{X}'^{(i)}}),$$

where the limit is taken over all open subsets $\mathcal{U}^{(i)} \subseteq \mathcal{X}'^{(i)}$ containing the subset $(F^{(i)})^{-1}(y_i)$. This is a finite $\mathscr{O}^{(i)}$-algebra since $F^{(i)}$ is proper. Since H' carries the $F^{(1)}$-fiber over y_1 into the $F^{(2)}$-fiber over y_2, pullback along H' induces a k-algebra map

$$\mathscr{O}'^{(2)} \to \mathscr{O}'^{(1)}. \tag{4.7.1}$$

Finally, define

$$\widetilde{\mathscr{O}'}^{(i)} = \varprojlim (F^{(i)}_*(\mathscr{O}_{\mathfrak{X}'^{(i)}}/\mathscr{I}^n_{\mathfrak{Y}^{(i)}}\mathscr{O}_{\mathfrak{X}'^{(i)}}))_{y_i},$$

so by Kiehl's version of the theorem on formal functions [K1] (which carries over to the complex-analytic case and the k-analytic case) this is naturally identified with the $\mathscr{I}_{\mathfrak{Y}^{(i)},y_i}$-adic completion of $\mathscr{O}'^{(i)}$.

Letting $\overline{\mathscr{O}}^{(i)}$ denote the $\mathscr{I}_{\mathfrak{Y}^{(i)},y_i}$-adic completion of the local noetherian ring $\mathscr{O}^{(i)}$, since $y_i \in \mathfrak{Y}^{(i)}$ we see that the flat map $\mathscr{O}^{(i)} \to \overline{\mathscr{O}}^{(i)}$ is faithfully flat. But the compatibility of completion with finite maps between noetherian rings implies (via Kiehl's version of the theorem on formal functions) that the natural map

$$\overline{\mathscr{O}}^{(i)} \otimes_{\mathscr{O}^{(i)}} \mathscr{O}'^{(i)} \to \widetilde{\mathscr{O}'}^{(i)}$$

is an isomorphism. Hence, since the finitely generated kernel ideal of $\mathscr{O}^{(i)} \to \mathscr{O}'^{(i)}$ is killed by a power of $\mathscr{I}_{\mathfrak{Y}^{(i)},y_i}$ (due to $F^{(i)}$ being an isomorphism over $\mathfrak{X}^{(i)} - \mathfrak{Y}^{(i)}$), so it is unaffected by the faithfully flat scalar extension $\mathscr{O}^{(i)} \to \overline{\mathscr{O}}^{(i)}$, we conclude that the natural k-algebra map

$$\mathscr{O}^{(i)} \to \overline{\mathscr{O}}^{(i)} \times_{\widetilde{\mathscr{O}'}^{(i)}} \mathscr{O}'^{(i)} \tag{4.7.2}$$

is an isomorphism.

The compatible maps $\mathfrak{H}_n : \mathfrak{X}^{(1)}_n \to \mathfrak{X}^{(2)}_n$ induce a k-algebra map $\overline{\mathscr{O}}^{(2)} \to \overline{\mathscr{O}}^{(1)}$, and we claim that this restricts to a local map of local k-subalgebras $\varphi : \mathscr{O}^{(2)} \to \mathscr{O}^{(1)}$. This follows from the description (4.7.2) of each $\mathscr{O}^{(i)}$ as a fiber product ring and the fact that the maps $\mathfrak{H}_n$ covered by the maps H'_n induce (via the theorem on formal functions in Kiehl's form) a map $\widetilde{\mathscr{O}'}^{(2)} \to \widetilde{\mathscr{O}'}^{(1)}$ that is compatible with (4.7.1). Let $\mathbf{y}_2 \in \mathscr{Y}^{(2)}$ be the image of y_2, so composing φ with the canonical map $\theta_{y_2} : \mathscr{O}_{\mathscr{X}^{(2)},\mathbf{y}_2} \to \mathscr{O}^{(2)}$ gives a local k-algebra map $\mathscr{O}_{\mathscr{X}^{(2)},\mathbf{y}_2} \to \mathscr{O}^{(1)}$. This is induced by a k-map of germs of locally ringed spaces $(\mathfrak{X}^{(1)}, y_1) \to (\mathscr{X}^{(2)}, \mathbf{y}_2)$ since $\mathscr{X}^{(2)}$ is a locally finite type k-scheme. Hence, by shrinking $\mathfrak{X}^{(1)}$ we can arrange that there is a k-map $H^{\mathrm{alg}} : \mathfrak{X}^{(1)} \to \mathscr{X}^{(2)}$ carrying y_1 to $\mathbf{y}_2$ and inducing $\varphi \circ \theta_{y_2}$ on local rings. Since $\mathfrak{Y}^{(2)} \subseteq \mathfrak{X}^{(2)}$ is the pullback of $\mathscr{Y}^{(2)} \subseteq \mathscr{X}^{(2)}$ under the analytification morphism $\mathfrak{X}^{(2)} \to \mathscr{X}^{(2)}$, shrinking some more around y_1 allows us to arrange that $H^{-1}(\mathfrak{Y}^{(2)}) = \mathfrak{Y}^{(1)}$, where $H : \mathfrak{X}^{(1)} \to \mathfrak{X}^{(2)}$ is the k-analytic map associated to H^{alg} via the universal property of analytification (in the sense of k-analytic spaces). Beware that we do not yet know that $H(y_1) = y_2$ or that H is related to φ on local rings.

For each $n \geq 0$ the induced map of infinitesimal neighborhoods $H_n : \mathfrak{X}^{(1)}_n \to \mathfrak{X}^{(2)}_n$ must agree with $\mathfrak{H}_n$ on some open subspace $\mathcal{V}_n \subseteq \mathfrak{X}^{(1)}_n$ around y_1 because (again via the universal property of analytification) it suffices to check equality after composition with the natural analytification map $\mathfrak{X}^{(2)}_n \to \mathscr{X}^{(2)}_n$, for which it is enough to verify the equality by computing on local rings (due to the second part of the self-contained Lemma 4.6.1). Such equality on local rings follows from how H^{alg} was constructed in terms of $\varphi \circ \theta_{y_2}$. In particular, $H(y_1) = y_2$ and by replacing $\mathfrak{X}^{(1)}$ with an open subset meeting $\mathfrak{Y}^{(1)}$ in $\mathcal{V}_0$ we can arrange that $H(y) = \mathfrak{H}_0(y)$ for all $y \in \mathfrak{Y}^{(1)}$. Hence, the two maps

$$H \circ F^{(1)}, F^{(2)} \circ H' : \mathfrak{X}'^{(1)} \rightrightarrows \mathfrak{X}^{(2)}$$

coincide set-theoretically everywhere. We claim that these agree as morphisms on $(F^{(1)})^{-1}(\mathfrak{U})$ for some open $\mathfrak{U}$ in $\mathfrak{X}^{(1)}$ around y_1. By properness of $F^{(1)}$ it suffices to check equality on the germ $(\mathfrak{X}'^{(1)}, y'_1)$ for each $y'_1 \in (F^{(1)})^{-1}(y_1)$. By the universal property of analytification, since $H(y_1) = y_2$ it suffices to show that for each such y'_1, composition with the canonical map $\mathfrak{X}^{(2)} \to \mathscr{X}^{(2)}$ yields maps of germs $(\mathfrak{X}'^{(1)}, y'_1) \rightrightarrows (\mathscr{X}^{(2)}, \mathbf{y}_2)$ that coincide. But these latter two maps of germs are given on local rings by $H'^* \circ (F^{(2)})^* \circ \theta_{y_2}$ and $(F^{(1)})^* \circ H^* \circ \theta_{y_2} = (F^{(1)})^* \circ (H^{\mathrm{alg}})^* = (F^{(1)})^* \circ \varphi \circ \theta_{y_2}$. Since $H'^* \circ (F^{(2)})^* = (F^{(1)})^* \circ \varphi$ as maps $\mathscr{O}^{(2)} \rightrightarrows \mathscr{O}_{\mathfrak{X}'^{(1)},y'_1}$ by construction of φ, so we have equality of maps of local rings, we get the required equality of maps of germs by the second part of Lemma 4.6.1.

Under the identification of $\mathcal{Y}^{(1)}$ with the underlying topological space of $\mathfrak{X}_n^{(1)}$ for all $n \geq 0$, the only remaining problem is to find a single open subspace $\mathcal{V} \subseteq \mathcal{Y}^{(1)}$ around y_1 that may be taken to be the underlying topological space of $\mathcal{V}_n$ for all $n \geq 0$. Once such a $\mathcal{V}$ is found, for a choice of open $\mathcal{W} \subseteq \mathfrak{X}^{(1)}$ meeting $\mathcal{Y}^{(1)}$ in $\mathcal{V}$ the map $H|_{\mathcal{W}}$ will be the desired local solution to our problem around the initial arbitrary choice of point $y_1 \in \mathcal{Y}^{(1)}$. The existence of such a $\mathcal{V}$ is not explained in Artin's complex-analytic proof, so we justify it in both analytic categories by applying Lemma 4.7.1 below to the pullback of the diagonal in $\mathfrak{X}'^{(2)} \times \mathfrak{X}'^{(2)}$ under the compatible family of maps $(H_n, \mathfrak{H}_n) : \mathfrak{X}'^{(1)}_n \to \mathfrak{X}'^{(2)} \times \mathfrak{X}'^{(2)}$, completing the proof of Lemma 4.3.2.

Lemma 4.7.1. *Let X be a complex-analytic, rigid-analytic, or good k-analytic space, and let $Y \hookrightarrow X$ be a closed immersion with associated coherent ideal sheaf $\mathscr{I} \subseteq \mathscr{O}_X$. Let $\{X_n\}$ be the associated system of infinitesimal neighborhoods of Y. Let $\mathfrak{Z}$ be a formal closed subspace of the associated formal analytic space $\widehat{X}$, by which we mean a compatible family of coherent ideal sheaves $\mathscr{K}_n \subseteq \mathscr{O}_{X_n}$ for all $n \geq 0$.*

Choose $x \in Y$ and assume $(\mathscr{K}_n)_x = 0$ for each $n \geq 0$. Then there exists an open subset $V \subseteq Y$ around x such that $\mathfrak{Z} = \widehat{X}$ over V, which is to say that $\mathscr{K}_n|_V = 0$ for all $n \geq 0$.

Proof. Let $W \subseteq X$ be a neighborhood of x of the following type: a compact Stein set with a Hausdorff Stein neighborhood in the complex-analytic case, an admissible affinoid open in the rigid-analytic case, and a k-affinoid domain in the good k-analytic case. In particular, the functor $\Gamma(W, \cdot)$ is exact on coherent sheaves on X and $A = \Gamma(W, \mathscr{O}_X)$ is a noetherian ring. Moreover, the exactness of this functor ensures that for the ideal $I = \Gamma(W, \mathscr{I})$ in A, the natural ring map $A/I^{n+1} \to \Gamma(W \cap Y, \mathscr{O}_{X_n})$ is an isomorphism and this ring generates $\mathscr{O}_{X_n}|_{W \cap Y}$. Thus, the compatible family $\{\mathscr{K}_n\}$ corresponds to an ideal J in the I-adic completion B of A.

Let $\mathfrak{p} \subseteq B$ be the prime ideal associated to evaluation at x, so this arises from a unique prime ideal $\mathfrak{q}$ of A containing I. (These are maximal ideals in the complex-analytic and rigid-analytic cases.) The natural map $A_{\mathfrak{q}}^{\wedge} \to \mathscr{O}_{X,x}^{\wedge}$ is faithfully flat (even an isomorphism in the complex-analytic and rigid-analytic cases), so each $B_{\mathfrak{p}}/\mathfrak{p}^{r+1}B_{\mathfrak{p}}$ is a quotient of the local subring $(A/I^{r+1})_{\mathfrak{q}} \subseteq \mathscr{O}_{X_r,x}^{\wedge}$. Thus, the initial vanishing hypothesis on $(\mathscr{K}_n)_x$ for each $n \geq 0$ says that J maps to 0 in each $B_{\mathfrak{p}}/\mathfrak{p}^{r+1}B_{\mathfrak{p}}$, so J has vanishing image in $B_{\mathfrak{p}}$. Hence, there is some $b \in B - \mathfrak{p}$ such that $bJ = 0$. But $b \bmod IB \in A/I = \Gamma(W \cap Y, \mathscr{O}_Y)$ is a unit near x, so there is a smaller choice $W' \subseteq W$ around x such that for A', B', I', and J' defined similarly to A, B, I, and J (using W' in place of W) we have that $b \bmod IB$ has unit image in $B'/I'B' = A'/I' = \Gamma(W' \cap Y, \mathscr{O}_Y)$. Since B' is I'-adically separated and complete, the image of b in B' is a unit in B'. But $J' = JB'$ by the Stein/affinoid properties of W and W', so J' is killed by a unit. Hence, $J' = 0$. Any open around x in $W' \cap Y$ therefore serves as the required V. ■

4.8. **Uniqueness of analytic dilatations.** Finally, we prove Lemma 4.5.1. As in the proof of Lemma 4.3.2 in §4.7, the uniqueness of h' as a morphism is immediate and it is automatically an isomorphism if it exists. Thus, we may swap the $f^{(i)}$'s if necessary so that $f^{(2)}$ is the analytification of a proper map between separated algebraic spaces locally of finite type over k and $Y^{(2)}$ arises from analytifying a closed subspace of the algebraic model for $X^{(2)}$. We then generalize the statement to be proved exactly as we did in §4.7: we are given a pair of analytic modifications of separated rigid spaces $(f^{(i)} : X'^{(i)} \to X^{(i)}, Y^{(i)} \subseteq X^{(i)})$ and a map $h : X^{(1)} \to X^{(2)}$ satisfying $h^{-1}(Y^{(2)}) = Y^{(1)}$, and for $Y'^{(i)} := (f^{(i)})^{-1}(Y^{(i)}) \subseteq X'^{(i)}$ we assume that we are also given a map of formal analytic completions $\mathfrak{h}' : \mathfrak{X}'^{(1)} \to \mathfrak{X}'^{(2)}$ satisfying $\mathfrak{f}^{(2)} \circ \mathfrak{h}' = \mathfrak{h} \circ \mathfrak{f}^{(1)}$, where $\mathfrak{h} : \mathfrak{X}^{(1)} \to \mathfrak{X}^{(2)}$ is induced by h. Assuming that $f^{(2)}$ is the analytification of a proper map $F^{(2)} : \mathscr{X}'^{(2)} \to \mathscr{X}^{(2)}$ between separated algebraic spaces locally of finite type over k and that $Y^{(2)} = (\mathscr{Y}^{(2)})^{\mathrm{an}}$ for some closed subspace $\mathscr{Y}^{(2)} \subseteq \mathscr{X}^{(2)}$ such that $F^{(2)}$ is an isomorphism over $\mathscr{X}^{(2)} - \mathscr{Y}^{(2)}$, we seek to construct a map $h' : X'^{(1)} \to X'^{(2)}$ inducing $\mathfrak{h}'$ such that $f^{(2)} \circ h' = h \circ f^{(1)}$; such an h' is clearly unique if it exists. Since we are given the map h downstairs and are trying to construct h' upstairs, by an even easier argument than in §4.7 we can reduce to the case when $\mathscr{X}'^{(2)}$ and $\mathscr{X}^{(2)}$ are of finite type over k and $X^{(1)}$ is affinoid. We may thereby reduce to the analogous construction problem in the setting of good strictly k-analytic spaces, and uniqueness in this case goes as in §4.7.

Exactly as in the complex-analytic case, by an argument with the Artin–Rees lemma and the second part of Lemma 4.6.1 we see that it suffices to prove that for each $x'_1 \in \mathcal{Y}'^{(1)}$ and $n \geq 0$ there is a map of germs $H' : (\mathcal{X}'^{(1)}, x'_1) \to (\mathcal{X}'^{(2)}, x'_2)$ over h inducing $\mathfrak{H}'_n$, where $x'_2 \in \mathcal{Y}'^{(2)}$ is the image of x'_1 under the common topological map of the $\mathfrak{H}'_n$'s. Thus, we are reduced to the following problem. Let $\mathscr{X}$ be a separated algebraic space of finite type over k and let $\mathscr{Y} \subseteq \mathscr{X}$ be a closed subspace; define $\mathcal{X}$ to be the (good, strictly k-analytic) analytification $\mathscr{X}^{\rm an}$ in the sense of k-analytic spaces, and similarly for $\mathcal{Y} = \mathscr{Y}^{\rm an}$. Let $\mathscr{X}' \to \mathscr{X}$ be a separated map of finite type, $\mathscr{Y}'$ the pullback of $\mathscr{Y}$, and $\mathcal{W}$ a good k-analytic space over $\mathcal{X}$. Let $\mathcal{Z} \subseteq \mathcal{W}$ be the pullback of $\mathcal{Y} \subseteq \mathcal{X}$, and assume that there is given an $\mathfrak{X}$-map $\mathfrak{h} : \mathfrak{W} \to \mathfrak{X}'$ between the corresponding formal completions (i.e., compatible $\mathcal{X}_n$-maps $\mathfrak{h}_n : \mathcal{W}_n \to \mathcal{X}'_n$ between infinitesimal neighborhoods for all $n \geq 0$). For each $n \geq 0$ and $w \in \mathcal{Z}$ with image $x' \in \mathcal{Y}'$ under $\mathfrak{h}_0$ and image $x \in \mathcal{X}$, we claim that there exists an $(\mathcal{X}, x)$-map of k-analytic germs $(\mathcal{W}, w) \to (\mathcal{X}', x')$ lifting $\mathfrak{h}_n$.

This assertion only involves k-analytic germs, so by applying Lemma 4.6.1 twice (first to $(\mathcal{X}, x)$ and then to $(\mathcal{X}', x')$) we easily reduce to the case when $\mathscr{X}$ and $\mathscr{X}'$ are schemes. We may also assume that the schemes $\mathscr{X}$ and $\mathscr{X}'$ are affine. Via the universal property of analytification of algebraic k-schemes [Ber2, 2.6.1], an $\mathscr{X}$-map $\mathcal{W} \to \mathscr{X}'$ whose restriction to $\mathcal{Z}$ factors as $\mathcal{Z} \stackrel{\mathfrak{h}_0}{\to} \mathcal{Y}' \to \mathscr{Y}'$ (the final map being the canonical one) corresponds to an $\mathcal{X}$-map of k-analytic spaces $\mathcal{W} \to \mathcal{X}'$ that carries w to $\mathfrak{h}_0(w) = x'$. Hence, we can replace $\mathfrak{h}$ with the composite map $\mathfrak{g} : \mathfrak{W} \to \widehat{\mathscr{X}'}$ to the completion of $\mathscr{X}'$ along $\mathscr{Y}'$ (and x' with its image $\mathbf{x}'$ in $\mathscr{Y}'$) so as to reduce ourselves to the situation in which the target objects are algebraic schemes and formal schemes rather than k-analytic spaces and formal k-analytic spaces.

Choose a presentation $\mathscr{X}' \simeq \operatorname{Spec} A[T_1, \dots, T_N]/(v_1, \dots, v_m)$ over the coordinate ring A of $\mathscr{X}$. Since the $\mathscr{Y}$-map $\mathfrak{g}_0 : \mathcal{Z} \to \mathscr{Y}'$ carries w to $\mathbf{x}'$, the desired map of germs $(\mathcal{W}, w) \to (\mathscr{X}', \mathbf{x}')$ over $(\mathscr{X}, \mathbf{x})$ amounts to an ordered N-tuple $f_1, \dots, f_N \in \mathscr{O}_{\mathcal{W},w}$ that is a simultaneous zero of the v_i's such the image of the f_j's in $\mathscr{O}_{\mathcal{W}_n,w}$ is as specified by the $\mathscr{X}_n$-map $\mathfrak{g}_n$ for all n. If $J \subseteq A$ is the ideal of $\mathscr{Y}$ in $\mathscr{X} = \operatorname{Spec} A$ then $J \cdot \mathscr{O}_{\mathcal{W},w}$ is the proper ideal corresponding to $(\mathcal{Z}, w)$ in $(\mathcal{W}, w)$, so we are given that the system of polynomial equations $v_1 = \cdots = v_m = 0$ over A has a solution in the J-adic completion of the A-algebra $\mathscr{O}_{\mathcal{W},w}$ and we seek a solution in $\mathscr{O}_{\mathcal{W},w}$ that lifts a specified solution modulo $J \cdot \mathscr{O}_{\mathcal{W},w}$. By a standard argument as in the proof of [A1, 1.3] (expressing the congruence condition modulo $J \cdot \mathscr{O}_{\mathcal{W},w}$ in terms of an auxiliary system of polynomial equations with additional variables corresponding to a choice of A-module generators of J), we can dispense with the congruence condition at the expense of introducing more T_i's and enlarging the collection of v_j's. We still have a solution in the J-adic completion, and hence in the $\mathfrak{m}_w$-adic completion, so by Popescu's generalization of Artin approximation to arbitrary excellent rings [S] it suffices to prove that the henselian local noetherian ring $\mathscr{O}_{\mathcal{W},w}$ is excellent. This excellence is a recent result of Ducros [D2, 2.6].

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