DESCENT FOR NON-ARCHIMEDEAN ANALYTIC SPACES

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1. Introduction

In the theory of schemes, faithfully flat descent is a very powerful tool. One wants a descent theory not only for quasi-coherent sheaves and morphisms of schemes (which is rather elementary), but also for geometric objects and properties of morphisms between them. In rigid-analytic geometry, descent theory for coherent sheaves was worked out by Bosch and Görtz [BG, 3.1] under some quasi-compactness hypotheses by using Raynaud's theory of formal models, and their result can be generalized [C, 4.2.8] to avoid quasi-compactness assumptions (as is necessary to include analytifications of faithfully flat maps arising from algebraic geometry [CT, §2.1]). Similarly, faithfully flat descent for morphisms, admissible open sets, and standard properties of morphisms works out nicely in the rigid-analytic category [C, §4.2].

In Berkovich's theory of k-analytic spaces, one can ask if there are similar results. The theory of flatness in k-analytic geometry is more subtle than in the case of schemes or complex-analytic spaces, ultimately because morphisms of k-affinoid spaces generally have non-empty relative boundary (in the sense of [Ber1, 2.5.7]). In the case of quasi-finite morphisms [Ber2, §3.1], which are maps that are finite locally on the source and target, it is not difficult to set up a satisfactory theory of flatness [Ber2, §3.2]. The appendix to this paper (by Ducros) develops a more general theory of flatness for k-analytic maps with empty relative boundary; this includes flat quasi-finite maps, smooth maps, and (relative) analytifications of flat maps between schemes locally of finite type over a k-affinoid algebra.

Let $f : X \to Y$ be a map of k-analytic spaces, and let $Y' \to Y$ be a surjective flat map. For various properties $\mathbf{P}$ of morphisms that are preserved by base change (proper, finite, separated, closed immersion, etc.) we say that $\mathbf{P}$ is *local for the flat topology* if f satisfies $\mathbf{P}$ precisely when the base change $f' : X' \to Y'$ does. For example, consider the property of a morphism $f : X \to Y$ being *without boundary* in the sense that for any k-affinoid W and morphism $W \to Y$, the base change $X \times_Y W$ is a good k-analytic space (i.e., each point has a k-affinoid neighborhood) and the morphism of good spaces $X \times_Y W \to W$ has empty relative boundary in the sense of [Ber1, p. 49]. This property is preserved by k-analytic base change, but it is not at all obvious from the definitions if it is local for the flat topology. Similarly, if Y is a k-analytic space and it has a flat quasi-finite cover $Y' \to Y$ such that Y' is good then it is natural to expect that Y is good but this does not seem to follow easily from the definitions since the target of a finite surjective morphism with affinoid source can be non-affinoid [Liu]. It is also natural to ask if goodness descends even when $Y' \to Y$ is merely a flat cover.

Finally, one can also ask for analogous descent results with respect to extension of the ground field. That is, if $f : Y' \to Y$ is a map of k-analytic spaces and if K/k is an arbitrary analytic field extension then we ask if f satisfies a property $\mathbf{P}$ precisely when $f_K : Y'_K \to Y_K$ satisfies this same property. Likewise, if Y_K is good then is Y good (the converse being obvious)? This latter question seems to be very non-trivial, and in general the problem of descent through a field extension is much harder than descent through flat surjections. The purpose of this paper is to apply the theory of reduction of germs (as developed in [T2]) to provide affirmative answers to all of the above descent questions.

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Remark 1.1. What we are calling a morphism *without boundary* is called a *closed* morphism in [Ber2, 1.5.3(ii)] and [T2]. It is a non-trivial fact [T2, 5.6] that whether or not $f : X \to Y$ is without boundary can be checked locally for the G-topology on Y. This locality is very useful when checking that an abstractly constructed map is without boundary, and it is the reason that for any ground field extension K/k the map f_K is without boundary when f is without boundary. In contrast, the relative notion of good morphism [Ber2, 1.5.3(i)] is not local for the G-topology on the base (one can construct a counterexample using [T1, Rem. 1.6, Thm. 3.1]), so we do not consider it to be an interesting concept for its own sake (although morphisms that are either proper or without boundary are good by definition).

As the proofs will make clear, the essential cases for both kinds of descent situations (flat covers and extension of the base field) are the property of a k-analytic space being good and the property of a k-analytic morphism $f : Y' \to Y$ being without boundary. We now state our main results only for these hardest cases.

Theorem 1.2. *Let $f : X \to Y$ be a morphism of k-analytic spaces.*

1. *If $Y' \to Y$ is a flat surjection and the base change $X' \to Y'$ of f is without boundary then so is f. Also, if f is a flat surjection and X is good then so is Y.*
2. *If K/k is an analytic extension field and f_K is without boundary then so is f. Also, if Y_K is a good K-analytic space then Y is a good k-analytic space.*

In §2 we work out the instances of faithfully flat descent that are "easier" in the sense of only requiring the results already proved in [T2]. Rather more difficult is such descent for the absolute properties of goodness and strict-analyticity of analytic spaces, which we respectively treat in Corollary 6.2 and (under some separatedness hypotheses) Theorem 7.7. These harder results rest on a descent theorem for graded birational spaces (Theorem 6.1) that we prove as a consequence of a lot of work in "graded commutative algebra" in §4–§5. For example, goodness for strictly k-analytic germs (when $|k^\times| \neq 1$) is closely related to affineness for ungraded birational spaces over the residue field $\widetilde{k}$, so to study descent for goodness one is led to seek an analogue of affineness in classical birational geometry. As a warm-up, in §3 we digress to show that affineness for a birational space over $\widetilde{k}$ is equivalent to a certain auxiliary integral $\widetilde{k}$-scheme of finite type having normalization that is proper over an affine algebraic $\widetilde{k}$-scheme. This latter property is a substitute for affineness in birational geometry, and it is a delicate fact that this property descends through modifications: such birational invariance fails if the normalizations are omitted, as shown by an elegant example of de Jong (Example 3.3). de Jong's counterexample gave us new insight into Q. Liu's surprising examples of non-affinoid rigid spaces with affinoid normalization (see Example 3.4), and it also inspired the intervention of integral closures in both §4 and the proof of our main descent theorem for birational spaces (Theorem 6.1).

Though we expect our results to be of general interest in k-analytic geometry, perhaps of greater interest is the techniques of proof. For example, to descend properties after a field extension K/k the essential difficulty is that the topological fibers of the natural map $X_K \to X$ for a k-analytic space X are of the form $\mathscr{M}(\mathscr{H}(x)\widehat{\otimes}_k K)$ for $x \in X$, and these can have a rather complicated structure since the K-Banach algebra $\mathscr{H}(x)\widehat{\otimes}_k K$ can fail to be K-affinoid. We will not try to describe such fibers in their entirety, nor their Shilov boundary (see [Ber1, p. 36]) which may be infinite, but we will prove instead that the fiber over each $x \in X$ contains a "sufficiently generic" point x_K. Then we will see that such points can test X_K very well: for many local properties $\mathbf{P}$, if $\mathbf{P}$ fails for some $x \in X$ then it fails on X_K at such "generic" points $x_K \in X_K$ over x. Corollary 8.5 gives the existence of such x_K, and the proof of this corollary rests on a theory of "transcendence degree" for graded field extensions that we develop in §4.

The proof of Theorem 1.2(2) (see §9) follows the strategy of using such points x_K. In addition to descent for goodness, another absolute property for which descent through a ground field extension K/k is non-trivial and interesting is the strict analyticity property (under the necessary assumption $\sqrt{|K^\times|} = \sqrt{|k^\times|}$). We treat this in Theorem 9.1 subject to some separatedness hypotheses. For properties of morphisms, our descent results through a ground field extension rest on Theorem 1.2(2), and as an interesting application of the invariance of the closed immersion property with respect to arbitrary analytic ground field extensions we conclude the paper by carrying over to k-analytic geometry the basic results for relatively ample line bundles in the rigid-analytic case [C, §3]. We do not know how to adapt the rigid-analytic arguments of [C, §3] to work in the k-analytic case (especially without goodness hypotheses), so we do not obtain a new approach to relative ampleness in rigid geometry.

In the appendix we review basic properties of flatness of analytic morphisms (whose general theory was developed very recently by A. Ducros in [Duc]), and of a related notion of G-smoothness.

Since our proofs rely extensively on the theory of reduction of germs as developed in [T2], we assume that the reader is familiar with this work and we will use its terminology and notation, including the theory of birational spaces over the $\mathbf{R}^{\times}_{>0}$-graded field $\widetilde{k}$ and the "graded commutative algebra" in [T2, §1]. As usual in the theory of k-analytic spaces, we permit the possibility that the absolute value on k may be trivial.

Terminology and notation. For an abelian group G and a G-graded field $\widetilde{k}$, by a *G-graded birational space over* $\widetilde{k}$ we mean an object $\mathfrak{X}$ of the category $\mathrm{bir}_{\widetilde{k}}$ introduced in [T2]. Such an $\mathfrak{X}$ consists of a G-graded field $\widetilde{K}$ over $\widetilde{k}$ and a local homeomorphism $X \to \mathbf{P}_{\widetilde{K}/\widetilde{k}}$ where X is a connected, quasi-compact, and quasi-separated topological space and $\mathbf{P}_{\widetilde{K}/\widetilde{k}}$ is the naturally topologized set of G-graded valuation rings of $\widetilde{K}$ containing $\widetilde{k}$. Since $\mathbf{P}_{\widetilde{K}/\widetilde{k}}$ is irreducible and its irreducible closed sets have unique generic points, the same necessarily holds for X. In particular, by taking $X = \mathbf{P}_{\widetilde{K}/\widetilde{k}}$ we can view $\mathbf{P}_{\widetilde{K}/\widetilde{k}}$ as a G-graded birational space over $\widetilde{k}$. When the group G is understood from context, we will not mention it explicitly (and will simply speak of "graded" objects).

For any extension $\widetilde{L}/\widetilde{K}$ of graded fields over $\widetilde{k}$, restriction of graded valuation rings from $\widetilde{L}$ to $\widetilde{K}$ induces a continuous map $\psi_{\widetilde{L}/\widetilde{K}/\widetilde{k}} : \mathbf{P}_{\widetilde{L}/\widetilde{k}} \to \mathbf{P}_{\widetilde{K}/\widetilde{k}}$. A *morphism* $\mathfrak{Y} \to \mathfrak{X}$ from the birational space $\mathfrak{Y} = (Y \to \mathbf{P}_{\widetilde{L}/\widetilde{k}})$ to the birational space $\mathfrak{X} = (X \to \mathbf{P}_{\widetilde{K}/\widetilde{k}})$ is a pair consisting of a graded embedding $\widetilde{K} \hookrightarrow \widetilde{L}$ over $\widetilde{k}$ and a continuous map $Y \to X$ compatible with $\psi_{\widetilde{L}/\widetilde{K}/\widetilde{k}}$.

An *analytic extension* K/k is an extension field endowed with an absolute value that extends the given one on k and with respect to which K is complete. If X is a k-analytic space then X_K denotes the K-analytic space $X\widehat{\otimes}_k K$. If x is a point in a k-analytic space X then we write (X, x) to denote the associated germ (denoted X_x in [T2]). A k-analytic space X is *locally separated* if each $x \in X$ admits a neighborhood that is a separated k-analytic domain. (By [T2, 4.8(iii)], it is equivalent for each $\mathbf{R}^{\times}_{>0}$-graded birational space $\widetilde{X}_x$ over the $\mathbf{R}^{\times}_{>0}$-graded reduction $\widetilde{k}$ to be separated over $\mathbf{P}_{\widetilde{k}/\widetilde{k}}$ in the sense of [T2, §2].)

A k-analytic map $f : X \to Y$ is *without boundary* (or *has no boundary*) if, for any k-affinoid Y' and morphism $Y' \to Y$, the pullback $X' = X \times_Y Y'$ is a good space and the morphism of good spaces $X' \to Y'$ has empty relative boundary. This concept is called *closed* in [Ber1], [Ber2], and [T2], but we prefer the change in terminology to avoid confusion with the unrelated topological notion of a closed map and because the open unit disc is without boundary over $\mathscr{M}(k)$ whereas the closed unit disc is not. The *relative interior* $\mathrm{Int}(X/Y)$ is the open locus of points $x \in X$ admitting an open neighborhood $V \subseteq X$ such that $V \to Y$ is without boundary; the complement $\partial(X/Y) = X - \mathrm{Int}(X/Y)$ is the *relative boundary* (so $X \to Y$ is without boundary if and only if $\partial(X/Y)$ is empty).

We refer the reader to Appendix A for a discussion of *flatness* for morphisms without boundary.

2. Faithfully flat base change: properties of morphisms

As an application of the theory of birational spaces, we shall now establish some easier descent results. In particular, results in graded commutative algebra in [T2] cover our needs in this section. We begin with a lemma that was recorded (under a mild restriction) in [T2, 5.7]. Since the proof was omitted there and we now need a more general version, we give the proof in its entirety.

Lemma 2.1. *Let $\psi : X \to Y$ and $\varphi : Y \to Z$ be morphisms of k-analytic spaces. The relative interiors satisfy*

$$\mathrm{Int}(X/Y) \cap \psi^{-1}(\mathrm{Int}(Y/Z)) \subseteq \mathrm{Int}(X/Z),$$

and a point $x \in \mathrm{Int}(X/Z)$ *lies in* $\mathrm{Int}(X/Y) \cap \psi^{-1}(\mathrm{Int}(Y/Z))$ *if either* $x \in \mathrm{Int}(X/Y)$ *or if* φ *is separated on an open neighborhood of* $\psi(x)$*. In particular, the inclusion is an equality whenever* $\mathrm{Int}(X/Y) = X$ *or* φ *is locally separated.*

Proof. The inclusion is [Ber2, 1.5.5(ii)], and for the reverse statement at a point $x \in \mathrm{Int}(X/Z)$ such that φ is separated near $\psi(x)$ we can replace our spaces with suitable open subspaces so that φ is separated. In this case the equality is [T2, 5.7], but since we also want to treat the case $x \in \mathrm{Int}(X/Y)$ without separatedness conditions on φ we give the argument here for the convenience of the reader. We choose $x \in \mathrm{Int}(X/Z)$ and let $y = \psi(x) \in Y$ and $z = \varphi(y) \in Z$. Consider the induced maps $\widetilde{\psi} : \widetilde{X}_x \to \widetilde{Y}_y$ and $\widetilde{\varphi} : \widetilde{Y}_y \to \widetilde{Z}_z$ of reductions of germs in the category $\mathrm{bir}_{\widetilde{k}}$ of birational spaces over the $\mathbf{R}^{\times}_{>0}$-graded field $\widetilde{k}$. By [T2, 5.2], a morphism of k-analytic spaces has empty relative boundary near a point of the source if and only if the induced map of reductions of germs at that point and its image is a proper map in the category $\mathrm{bir}_{\widetilde{k}}$ (in the sense of properness defined in [T2, §2]). The condition $x \in \mathrm{Int}(X/Z)$ therefore says exactly that $\widetilde{\varphi} \circ \widetilde{\psi} = \widetilde{\varphi \circ \psi}$ is a proper map in $\mathrm{bir}_{\widetilde{k}}$, and the condition $x \in \mathrm{Int}(X/Y)$ says exactly that $\widetilde{\varphi}$ is proper. By [T2, 4.8(iii)], separatedness for φ near $\psi(x)$ says exactly that $\widetilde{\varphi}$ is a separated map in the category $\mathrm{bir}_{\widetilde{k}}$ (in the sense defined in [T2, §2]).

Our problem is reduced to checking that if $f : \mathfrak{X} \to \mathfrak{Y}$ and $g : \mathfrak{Y} \to \mathfrak{Z}$ are maps in $\mathrm{bir}_{\widetilde{k}}$ (not maps of k-analytic spaces!) with $g \circ f$ proper and either g is separated or f is proper then both f and g are proper. (Since maps in $\mathrm{bir}_{\widetilde{k}}$ are analogous to dominant maps of varieties, it is not unreasonable that g should be proper when $g \circ f$ and f are proper.) By the definition of a birational space over $\widetilde{k}$, $\mathfrak{Z}$ corresponds to a local homeomorphism $U \to \mathbf{P}_{\widetilde{K}/\widetilde{k}}$ where U is a connected, quasi-compact, and quasi-separated topological space and $\widetilde{K}/\widetilde{k}$ is an extension of $\mathbf{R}^{\times}_{>0}$-graded fields. We similarly have that $\mathfrak{Y}$ and $\mathfrak{X}$ respectively correspond to local homeomorphisms $U' \to \mathbf{P}_{\widetilde{K}'/\widetilde{k}}$ and $U'' \to \mathbf{P}_{\widetilde{K}''/\widetilde{k}}$, and the maps f and g respectively correspond to the left and right squares in a commutative diagram of topological spaces

$$\begin{array}{ccccc} U'' & \longrightarrow & U' & \longrightarrow & U \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{P}_{\widetilde{K}''/\widetilde{k}} & \longrightarrow & \mathbf{P}_{\widetilde{K}'/\widetilde{k}} & \longrightarrow & \mathbf{P}_{\widetilde{K}/\widetilde{k}} \end{array}$$

in which the maps along the bottom row are the natural pullback maps induced by maps of graded $\widetilde{k}$-algebras $\widetilde{K} \to \widetilde{K}'$ and $\widetilde{K}' \to \widetilde{K}''$. Separatedness of g (if it holds) means that the natural map $i : U' \to U \times_{\mathbf{P}_{\widetilde{K}/\widetilde{k}}} \mathbf{P}_{\widetilde{K}'/\widetilde{k}}$ is injective, and the properness of $g \circ f$ says that the natural map $U'' \to U \times_{\mathbf{P}_{\widetilde{K}/\widetilde{k}}} \mathbf{P}_{\widetilde{K}''/\widetilde{k}}$ is bijective. This latter map factors as the composition of natural maps

$$U'' \to U' \times_{\mathbf{P}_{\widetilde{K}'/\widetilde{k}}} \mathbf{P}_{\widetilde{K}''/\widetilde{k}} \to U \times_{\mathbf{P}_{\widetilde{K}/\widetilde{k}}} \mathbf{P}_{\widetilde{K}''/\widetilde{k}} \tag{2.1}$$

in which the second map is the topological (or set-theoretic) base change of the map i by the map $\mathbf{P}_{\widetilde{K}''/\widetilde{k}} \to \mathbf{P}_{\widetilde{K}'/\widetilde{k}}$.

The natural map $\mathbf{P}_{\widetilde{K}''/\widetilde{k}} \to \mathbf{P}_{\widetilde{K}'/\widetilde{k}}$ along which we form the base change of i is surjective, due to the easy consequence of Zorn's Lemma that every graded local ring in a graded field F is dominated by a graded valuation ring in F having graded fraction field F. Thus, the base change of i must be injective when i is injective (i.e., when g is separated), so it is an easy set-theoretic argument to deduce that if g is separated and $g \circ f$ is proper then both steps in (2.1) are bijective. For the first step such bijectivity says that f is proper, and for the second step it says that g is proper (since $\mathbf{P}_{\widetilde{K}''/\widetilde{k}} \to \mathbf{P}_{\widetilde{K}'/\widetilde{k}}$ is surjective). This settles the case when g is separated. If instead f is proper, then in the factorization (2.1) we see that the second step is bijective (since the first step and the composition are so), and this is a base change of i by a surjection. Hence, if f is proper then i is bijective, which is to say g is proper. ■

As an immediate consequence of this lemma we obtain a result on étale equivalence relations that answers a question that naturally arose in [CT].

Theorem 2.2. *Let $p_1, p_2 : R \rightrightarrows U$ be a pair of quasi-finite maps of k-analytic spaces such that $\delta = (p_1, p_2) : R \to U \times U$ is a quasi-compact monomorphism. If U is locally separated in the sense that each $u \in U$ has a separated open neighborhood then δ is a closed immersion.*

The case of most interest is when the p_i are étale. (Recall that étale maps are quasi-finite; cf. [Ber2, 3.1.1, 3.3.4].)

Proof. The map δ is separated since it is a monomorphism, so by the locally Hausdorff property for k-analytic spaces it follows that δ is a compact (i.e., topologically proper) map of topological spaces. We claim that δ is a proper morphism of k-analytic spaces, or equivalently (in view of the topological properness and [Ber2, 1.5.3, 1.5.4]) that the relative interior $\mathrm{Int}(R/U \times U)$ with respect to δ is equal to R (i.e., δ is without boundary). Once this is proved then δ is a proper morphism with discrete fibers, so it is finite [Ber2, 1.5.3(iii)]. But a finite monomorphism of k-analytic spaces $X \to Y$ is a closed immersion. Indeed, by [Ber2, 1.3.7] we can assume $X = \mathscr{M}(\mathscr{A})$ and $Y = \mathscr{M}(\mathscr{B})$ are k-affinoid, and since completed tensor products coincide with ordinary tensor products for finite admissible morphisms of k-affinoid algebras it follows that the corresponding map $\mathrm{Spec}(\mathscr{A}) \to \mathrm{Spec}(\mathscr{B})$ is a finite monomorphism of schemes. Hence, it is a closed immersion of schemes, so the finite admissible map of k-affinoid algebras $\mathscr{B} \to \mathscr{A}$ is surjective, as required.

It remains to prove that δ is without boundary. By definition of quasi-finiteness in k-analytic geometry [Ber2, 3.1.1], the quasi-finite maps p_1 and p_2 are without boundary. The first projection $U \times U \to U$ is locally separated since U is locally separated, so by Lemma 2.1 we have that $\mathrm{Int}(R/U \times U)$ contains $\mathrm{Int}(R/U)$ (taken with respect to $p_1 : R \to U$), and this latter interior is R. ■

We next wish to discuss descent of properties of morphisms with respect to base change along maps $f : Y' \to Y$ with *surjective interior* in the sense that the open subset $\mathrm{Int}(Y'/Y) \subseteq Y'$ maps onto Y. This includes the case of surjective morphisms without boundary, such as quasi-finite or flat surjections. In general one cannot expect to have descent results with respect to such base change because surjective morphisms without boundary can have unpleasant topological properties arising from puncturing a space. We illustrate this with a simple example.

Example 2.3. Let D be the closed unit disk, $D^* = D - \{0\}$, and D' the disjoint union $D^* \coprod \{0\}$. The canonical surjective morphism $f : D' \to D$ is a disjoint union of an open immersion and a closed immersion, so it without boundary. For any map $g : X \to D$, the base change $g' : X' \to D'$ along f is the disjoint union of the restrictions of g over D^* and over the origin. For example, the base change of f along itself is an isomorphism $X' \simeq D'$, yet f is neither proper nor étale. Hence, the properties of being proper, finite, étale, an open immersion, an isomorphism, or a closed immersion do not descend through surjective morphisms without boundary. Likewise, separatedness does not descend, since if X denotes the non-separated gluing of D to itself along the identity on D^* then the canonical map $h : X \to D$ is non-separated but its base change along f is separated because h has separated pullback over D^* and over the origin. Note that these kinds of examples also arise from analytifications of algebraic k-schemes.

The surjective morphism $f : D' \to D$ without boundary that is used in Example 2.3 has the defect that no compact neighborhood of the origin in D is the image of a compact set in D'. We say that a continuous map of topological spaces $f : V' \to V$ is *compactly surjective* if every compact subset of V is the image of a compact subset of V'. For example, any surjective topologically open continuous map between locally compact locally Hausdorff spaces is compactly surjective. A notable example is any surjective flat k-analytic map. It is easy to check that the property of being compactly surjective is preserved under topological base change when using locally compact locally Hausdorff spaces, and so it is also preserved under base change in the k-analytic category. Also, any ground field extension functor $Z \rightsquigarrow Z_K$ carries compactly surjective k-analytic maps to compactly surjective K-analytic maps because the natural map $Z_K \to Z$ is topologically proper for any k-analytic space Z and any K/k.

Theorem 2.4. *Let $f : Y' \to Y$ be a morphism of k-analytic spaces such that the induced map $\mathrm{Int}(Y'/Y) \to Y$ is surjective, and let $g : X \to Y$ be any k-analytic morphism, with $g' : X' \to Y'$ the base change of g along f. The map g is without boundary* (*resp. quasi-finite, resp. locally separated, resp. surjective*) *if and only* *if the map g' is. Furthermore, if* $\mathrm{Int}(Y'/Y) \to Y$ *is compactly surjective then g satisfies property* $\mathbf{P}$ *if and only if g' does, where* $\mathbf{P}$ *is any of the following properties: proper, finite, closed immersion, separated.*

Example 2.3 shows the necessity of the compactly surjective hypothesis. In Corollary 6.2 we will prove an instance of descent for absolute properties in the setting of Theorem 2.4: Y is good if Y' is good.

Proof. In each case the "only if" implication is clear, and we have to prove the converse. The map $Y'' := \mathrm{Int}(Y'/Y) \to Y$ is without boundary by definition and surjective by our assumption, and the morphism $X' \times_{Y'} Y'' \to Y''$ is a base change of g'. Thus, we can replace f with its interior to reduce all descent problems to the case when f is without boundary and surjective. It is obvious that if g' is surjective then so is g. We next descend local separatedness. Choose $x \in X$ and let $y = g(x) \in Y$. Choose $x' \in X'$ over x and let $y' = g'(x')$. Since g' is locally separated at x' and f and f' are without boundary, by [T2, 4.8(iii), 5.2] the commutative diagram

$$\begin{array}{ccc} \widetilde{X'}_{x'} & \longrightarrow & \widetilde{Y'}_{y'} \\ \downarrow & & \downarrow \\ \widetilde{X}_x & \longrightarrow & \widetilde{Y}_y \end{array}$$

in $\mathrm{bir}_{\widetilde{k}}$ has separated top side and proper maps along the left and right sides. It is then an easy set-theoretic argument as in the proof of Lemma 2.1 to deduce that the bottom side is separated, so g is separated near the arbitrary $x \in X$.

If g' is without boundary then the composite morphism $X' \to Y' \to Y$ is without boundary, and since the morphism $X' \to X$ is without boundary and surjective (as it is the base change of f) we deduce that $g : X \to Y$ is without boundary via Lemma 2.1. This descent result is the key case from which everything else will be deduced.

By [Ber2, 3.1.10], quasi-finiteness is equivalent to being without boundary and having discrete fibers, so for the descent of quasi-finiteness it suffices to show that if $g : X \to Y$ is any k-analytic map and $f : Y' \to Y$ is a surjective k-analytic map such that the base change $g' : X' \to Y'$ is quasi-finite then g has discrete fibers. Using base change along $y : \mathscr{M}(\mathscr{H}(y)) \to Y$ for any $y \in Y$ and renaming $\mathscr{H}(y)$ as k reduces us to the case $Y = \mathscr{M}(k)$. If $y' : \mathscr{M}(K) \to Y'$ is a point of Y' then X_K is quasi-finite over K, and we wish to deduce that X is topologically discrete. But $X_K \to X$ is topologically a quotient map and X_K is topologically discrete, so we are done.

For descent of the remaining properties, we may assume that the closed surjective morphism f is compactly surjective. Grant the descent of properness for a moment. Since a finite map is the same thing as a proper quasi-finite map, the descent of finiteness follows. If g' is a closed immersion then g is at least finite. By Nakayama's Lemma, a finite map is a closed immersion if and only if its non-empty fibers are 1-point sets corresponding to a trivial field extension. Thus, any finite k-analytic map that becomes a closed immersion after a surjective k-analytic base change is clearly a closed immersion. Hence, g is a closed immersion when g' is (granting the descent for properness). Stability of the compactly surjective property under k-analytic base change implies that $X' \times_{Y'} X' = (X \times_Y X) \times_Y Y' \to X \times_Y X$ is compactly surjective. Thus, by working with diagonal maps and descent for closed immersions we see that g is separated when g' is separated (granting the descent for properness).

Finally, we have to show that g is proper when g' is proper. Since a k-analytic map is proper if and only if it is topologically proper and without boundary, and we know that g must be without boundary (as we have already proved descent for this property), it suffices to check that if $f : Y' \to Y$ is a compactly surjective k-analytic map and $g : X \to Y$ is any k-analytic map whose base change $g' : X' \to Y'$ is a proper morphism then g is topologically proper (so in particular, g is topologically separated). It is easy to use base change along k-affinoid domains in Y to reduce to the case that Y is k-affinoid. Since f is compactly surjective, we can find a finite collection of k-affinoid domains in Y' whose images under f cover Y, so we may assume that Y' is also k-affinoid. Hence, X' is a compact Hausdorff space. Since $\pi : X' \to X$ is a topologically closed surjection, it suffices to prove that if $\pi : Z' \to Z$ is a surjective continuous closed map between topological spaces and Z' is compact Hausdorff then Z is compact Hausdorff. Obviously Z is compact, and all points in Z are closed since all points in Z' are closed (and π is a closed surjection). Let $z_1, z_2 \in Z$ be distinct points, so the fibers $\pi^{-1}(z_i) \subseteq Z'$ are disjoint closed subsets in the compact Hausdorff space Z'. Thus, there are disjoint opens $V_1', V_2' \subseteq Z'$ around $\pi^{-1}(z_1)$ and $\pi^{-1}(z_2)$. Since $\pi : Z' \to Z$ is topologically closed, $V_i = Z - \pi(Z' - V_i')$ is an open set in Z containing z_i, and $V_1 \cap V_2$ is empty because π is surjective and $\pi^{-1}(V_1 \cap V_2) \subseteq V_1' \cap V_2' = \emptyset$. ■

Simple examples with spectra of finite k-algebras shows that the assumptions of Theorem 2.4 do not suffice to descend the properties of being a monomorphism or isomorphism. (The same examples work also in the category of finite k-schemes.) Thus, to descend a few more properties one has to impose a flatness assumption on f. We refer the reader to the appendix for a discussion of k-analytic flatness.

Theorem 2.5. *Let $f : X \to Y$ be a k-analytic morphism, $Y' \to Y$ be a flat surjective morphism and $f' : X' \to Y'$ be the base change of f. The following properties hold for f if and only if they hold for f': surjective, flat, G-smooth, G-étale. Furthermore, if the induced morphism $\mathrm{Int}(Y'/Y) \to Y$ is compactly surjective, then the following properties hold for f if and only if they hold for f': isomorphism, monomorphism, étale, open immersion.*

Proof. The direct implications are all trivial. To descent the properties of the first group we note that they all are G-local (see the appendix), hence we can assume that X, Y and Y' are good (or even affinoid). For a point $x \in X$ choose a preimage $x' \in X'$ and let $y' \in Y'$ and $y \in Y$ be the images of x'. If f' is flat then the composition $X' \to Y' \to Y$ is flat, and in particular $\mathcal{O}_{X',x'}$ is flat over $\mathcal{O}_{Y,y}$. Since $\mathcal{O}_{X',x'}$ is flat over $\mathcal{O}_{X,x}$ because flatness is preserved under base changes, we obtain that $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,y}$. This proves that f is naively flat at x, and to prove that f is actually flat we must prove the same for any good base change of f, but that can be done precisely in the same way. Descent of G-smoothness is done similarly using the fact that if $A \to B$ is a flat local homomorphism and M is a finitely generated A-module, then M is free over A if and only if $M \otimes_A B$ free over B. The case of G-étaleness now follows. The case of étale maps is deduced from the fact that étaleness is the same as G-étaleness and quasi-finiteness, and descent of the latter was established in Theorem 2.4. The case of isomorphisms follows easily by using the finite case from 2.4 and working over k-affinoid domains in the target. Also, open immersions are the same as étale monomorphisms, and monomorphisms are maps for which the diagonal is an isomorphism. ■

It is more difficult to descend properties with respect to a base field extension and to descend absolute properties (e.g., to determine if, for a surjective k-analytic morphism $Y' \to Y$ without boundary, Y is good or strictly k-analytic if and only if Y' is). To prove these descent statements we need certain facts about graded birational spaces and graded reductions that are not covered by [T2] and will be proved in §5-6, building on some additional graded commutative algebra that we develop in §4. To motivate some of these later considerations, we make a digression in the next section.

3. The ungraded case and some examples

By [T2, 5.1], a k-analytic germ (X, x) is good if and only if the corresponding ($\mathbf{R}^\times_{>0}$-graded) birational space $\widetilde{X}_x$ in $\mathrm{bir}_{\widetilde{k}}$ is affine (i.e., corresponds to an open subset $\mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}\{A\} \subseteq \mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$ for a finitely generated $\mathbf{R}^\times_{>0}$-graded $\widetilde{k}$-subalgebra $A \subseteq \widetilde{\mathscr{H}(x)}$). Thus, descent of goodness is related to descent of affineness of (graded) birational spaces over a graded field. In this section we describe an elementary approach to this descent problem for birational spaces in the special case of the trivial grading group, so all fields, rings, birational spaces, etc., in this section are assumed to be *ungraded* (i.e., we take the grading group G to be trivial). Recall from [T1, §1] that any graded birational space $\mathfrak{X} = (X \to \mathbf{P}_{K/k})$ in the category bir_k for the trivial grading group naturally "arises" from a k-map $\eta : \mathrm{Spec}(K) \to \mathscr{X}$, where $\mathscr{X}$ is an integral scheme of finite type over k and η is generic over $\mathscr{X}$. (Explicitly, if $\mathscr{X}$ is separated then $X \subseteq \mathbf{P}_{K/k}$ is the open subset of valuations rings of K containing k and dominating the local ring of a point on $\mathscr{X}$. The general case proceeds by gluing over separated opens in $\mathscr{X}$.) We call such $\eta : \mathrm{Spec}(K) \to \mathscr{X}$ a (pointed) *integral scheme model* of $\mathfrak{X}$. For any two pointed integral scheme models $\eta_j : \mathrm{Spec}(K) \to \mathscr{X}_j$ of $\mathfrak{X} = (X \to \mathbf{P}_{K/k})$ there exists a third such $\mathrm{Spec}(K) \to \mathscr{X}$ with $\mathscr{X}$ is proper over each $\mathscr{X}_j$. (Actually, the collection of all such pointed $\mathscr{X}$ with a fixed K/k is an inverse system and X is naturally homeomorphic to the inverse limit of all such $\mathscr{X}$.)

Given $\mathfrak{X}$ and $\mathscr{X}$ as above and a pair of field extensions K/k and L/l equipped with a map $K \to L$ over a map $k \to l$, there is a naturally induced morphism $\psi_{L/K,l/k} : \mathbf{P}_{L/l} \to \mathbf{P}_{K/k}$ and one easily checks that a pointed integral l-scheme model $\mathrm{Spec}(L) \to \mathscr{Y}$ of the birational space $\mathfrak{Y} = \mathfrak{X} \times_{\mathbf{P}_{K/k}} \mathbf{P}_{L/l}$ in the category bir_l is given by taking $\mathscr{Y}$ to be the Zariski closure (with reduced structure) of the image of the natural composite map $\eta : \mathrm{Spec}(L) \to \mathrm{Spec}(K \otimes_k l) \to \mathscr{X} \otimes_k l$. (Note that $\mathscr{X} \otimes_k l$ does not have to be either irreducible or reduced, and η does not have to hit a generic point of $\mathscr{X} \otimes_k l$.) If $\mathfrak{X}$ is affine then $\mathfrak{Y}$ is affine, since $X = \mathbf{P}_{K/k}\{f_1, \dots, f_n\}$ implies that $\mathfrak{Y} = \mathbf{P}_{L/l}\{f_1, \dots, f_n\}$. It turns out that the converse is true under the additional assumption that any algebraically independent set over k in $l^\times$ is algebraically independent over K. (We will prove in Theorem 5.1(i) that this condition on L/l and K/k is equivalent to the surjectivity of $\psi_{L/K,l/k}$.)

Theorem 6.1 generalizes this converse statement for an arbitrary grading group G, but in the ungraded case it can be proved much more easily: it is a consequence of the following theorem that gives a criterion for affineness of birational spaces in the ungraded case in terms of pointed integral scheme models. To explain this implication, we first note that by Theorem 3.1 affineness descends in two special cases: (i) $l = k$, (ii) $l = k(T)$ and $L = K(T)$ are purely transcendental with a transcendence basis $T = \{T_i\}_{i \in I}$. Also, the descent is obvious when (iii) $K = L$ and l/k is algebraic, because $\psi_{L/K,l/k}$ is then a bijection. It remains to note that in general one can choose a transcendence basis T of l over k and then $\psi_{L/K,l/k} = \psi_{L/K(T),l/l} \circ \psi_{K(T)/K(T),l/k(T)} \circ \psi_{K(T)/K,k(T)/k}$. Thus, the general descent of affineness (in the ungraded case) reduces to the three particular cases as above. Finally, we note that the following theorem should generalize to graded birational spaces if one uses graded integral schemes in the role of $\mathscr{X}$, but a theory of graded schemes has not been developed.

Theorem 3.1. *Let $\mathfrak{X} = (X \to \mathbf{P}_{K/k})$ be a birational space over a field k, and let $\mathrm{Spec}(K) \to \mathscr{X}$ be a pointed integral scheme model for $\mathfrak{X}$. The birational space $\mathfrak{X}$ is affine if and only if the normalization of $\mathscr{X}$ is proper over an affine k-scheme of finite type.*

Proof. The "if" direction easily follows from the definition of the functor $(\mathrm{Spec}(K) \to \mathscr{X}) \rightsquigarrow (X \to \mathbf{P}_{K/k})$ in [T1, §1]. Indeed, the dominant point $\mathrm{Spec}(K) \to \mathscr{X}$ lifts to the normalization $\widetilde{\mathscr{X}}$ of $\mathscr{X}$, and since $\widetilde{\mathscr{X}}$ is proper over $\mathscr{X}$ the pointed integral schemes $\mathrm{Spec}(K) \to \widetilde{\mathscr{X}}$ and $\mathrm{Spec}(K) \to \mathscr{X}$ correspond to isomorphic objects in bir_k. We are given that $\widetilde{\mathscr{X}}$ is proper over an affine k-scheme $\mathscr{X}'$ of finite type, so $\mathrm{Spec}(K) \to \widetilde{\mathscr{X}}$ and the induced morphism $\mathrm{Spec}(K) \to \mathscr{X}'$ have isomorphic images in bir_k. Thus, if $A \subseteq K$ is the finitely generated coordinate ring of $\mathscr{X}'$ over k then $X = \mathbf{P}_{K/k}\{A\} \subseteq \mathbf{P}_{K/k}$ is affine in bir_k. The converse implication follows from the next lemma and the fact that any two pointed integral scheme models for an object in bir_k admit a common refinement that is proper over both of them. ■

Lemma 3.2. *Let S and S' be irreducible and reduced schemes of finite type over a field k, and let $\pi : S' \to S$ be a proper surjection. The normalization $\widetilde{S}'$ of S' is proper over an affine k-scheme of finite type if and only if the normalization $\widetilde{S}$ of S satisfies the same property.*

Our extensive study of graded integral closure in §4 is inspired by the role of normalizations in this lemma. The results in §4 are the main ingredients in the proof of the key descent theorem for graded birational spaces given in Theorem 6.1.

Proof. We can and do replace S and S' with their normalizations. The condition that a k-scheme X be proper over an affine algebraic k-scheme is equivalent to the simultaneous conditions: (i) the domain $A = \mathscr{O}_X(X)$ is of finite type over k, (ii) the canonical k-map $X \to \mathrm{Spec}(A)$ is proper. The nontrivial direction is descent from S' to S, so assume that S' is proper over an affine algebraic k-scheme. In particular, S' is separated. Since π is a proper surjection it follows that Δ_S has closed image, so S is separated over k. Let $A = \mathscr{O}_S(S)$ and $A' = \mathscr{O}_{S'}(S')$, so A' is a k-algebra of finite type and A is a k-subalgebra of A'. If we can show that A'

is a finite A-module then A must be of finite type over k [AM, 7.8], so in the commutative square

$$\begin{array}{ccc} S' & \longrightarrow & S \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

the bottom and left sides are proper, the top is a proper surjection, and the right side is separated and finite type. Hence, the right side is proper, as required.

To prove that A' is A-finite, observe that the A-algebra A' is identified with the global sections of the coherent sheaf of algebras $\mathscr{A}' = \pi_*(\mathscr{O}_{S'})$ on S. If $\eta \in S$ is the generic point then for any section h of $\mathscr{A}'$ over a non-empty open U in S, the characteristic polynomial of $h_\eta \in \mathscr{A}'_\eta$ over $\mathscr{O}_{S,\eta} = k(S)$ has coefficients in $\mathscr{O}_S(U)$ since the coherent $\mathscr{O}_S$-algebra $\mathscr{A}'$ is torsion-free and we can work over an affine open cover of U in the normal scheme S. Thus, A' is integral over A. But A' is finitely generated as an A-algebra (it is even finite type over k), so A' is finite as an A-module. ∎

Though the proof of descent of affineness for birational spaces with trivial grading as explained above is much shorter than the proof required in the general graded case in Theorem 6.1, even in the ungraded case the argument has hidden dangers. One subtlety is that Lemma 3.2 is false without normalizations! The following interesting counterexample along these lines was suggested to us by de Jong.

Example 3.3. Assume k has characteristic 0 (!), and let S be the integral k-scheme obtained from $S' = \mathbf{P}^1 \times \mathbf{A}^1$ by identifying the lines $\{0\} \times \mathbf{A}^1$ and $\{1\} \times \mathbf{A}^1$ via $t \mapsto t+1$. In other words, replace $\mathbf{A}^1 \times \mathbf{A}^1 = \mathrm{Spec}(k[x,t]) \subseteq S'$ with $\mathrm{Spec}(A)$ where $A \subseteq k[x,t]$ is the k-subalgebra of $f \in k[x,t]$ such that $f(0, t+1) = f(1, t)$. (An easy argument in the category of locally ringed spaces shows that S has the expected universal mapping property in the category of k-schemes.) Since $x^2 - x \in A$ and $t + x \in A$, the extension $A \to k[x,t]$ is integral and hence finite, so [AM, 7.8] ensures that A is finitely generated over k. Hence, S is finite type over k. Obviously $A_{x^2-x} = k[x,t]_{x^2-x}$ (since $(x^2-x)k[x,t] \subseteq A$), so S' is the normalization of S (in particular, S is separated) and S' is proper over the affine k-line, with $\mathscr{O}_{S'}(S') = k[t]$. Thus, the global functions on S are those $h \in k[t]$ such that $h(t) = h(t+1)$, so since $\mathrm{char}(k) = 0$ we get $\mathscr{O}_S(S) = k$. Since S is not k-proper it therefore cannot be proper over an algebraic affine k-scheme, though its normalization S' admits such a description.

We will later apply descent of affineness for birational spaces (in the graded case, Theorem 6.1) to prove that the property of being a good analytic space descends through flat surjections and extension of the ground field. As we mentioned in the Introduction, one should be especially careful when dealing with descent of goodness because of an example (due to Q. Liu) of a 2-dimensional separated non-affinoid rigid space D (over any k with $|k^\times| \neq 1$) such that D has an affinoid normalization. This phenomenon has no analogue for schemes of finite type over fields. While reading Liu's paper [Liu], we discovered a much simpler example of the same nature which is a very close relative of de Jong's example above.

Example 3.4. Let k be a non-archimedean field with a non-trivial valuation and residue characteristic zero. We will work with reductions of k-Banach algebras in the traditional (rather than graded) sense, so now $\widetilde{k}$ denotes the ordinary residue field of k (rather than an $\mathbf{R}^\times_{>0}$-graded field as in §2 and [T2]). The general idea of Q. Liu is to construct a Cartesian diagram of rings

$$\begin{array}{ccc} \mathscr{A}_1 & \overset{\psi}{\hookrightarrow} & \mathscr{A} \\ \downarrow & & \downarrow \\ \mathscr{B}_1 & \hookrightarrow & \mathscr{B} \end{array}$$

(so $\mathscr{A}_1 = \mathscr{A} \times_{\mathscr{B}} \mathscr{B}_1$, a ring-theoretic fiber product) where $\mathscr{B}$, $\mathscr{B}_1$, and $\mathscr{A}$ are k-affinoid, $\mathscr{A}_1$ is k-Banach, the embeddings are isometric and all homomorphisms are finite, but the reduction homomorphism $\widetilde{\psi} : \widetilde{\mathscr{A}_1} \to \widetilde{\mathscr{A}}$ is not finite. Then $\mathscr{A}_1$ is not affinoid, as otherwise the latter homomorphism has to be finite by [BGR, 6.3.5/1].

The desired property of $\widetilde{\psi}$ will be achieved using the fact that surjectivity is not generally preserved by the reduction functor [BGR, 6.4.2] (though finiteness is preserved [BGR, 6.3.5]). Given such a diagram of k-algebras, let $Y = \mathscr{M}(\mathscr{B})$, $Y_1 = \mathscr{M}(\mathscr{B}_1)$, and $X = \mathscr{M}(\mathscr{A})$, and assume in addition that there exists a pushout diagram

$$\begin{array}{ccc} X_1 & \longleftarrow & X \\ \uparrow & & \uparrow \\ Y_1 & \longleftarrow & Y \end{array} \tag{3.1}$$

in the k-analytic category, with X_1 strictly k-analytic. From the construction we will see that $X \to X_1$ is a finite surjective map, so X_1 is automatically separated and quasi-compact. (Hence, (3.1) also corresponds to a pushout in the category of separated rigid spaces over k.) Applying $\mathrm{Hom}(\cdot, \mathbf{A}^1_k)$ to the pushout diagram gives that $\mathscr{O}(X_1) = \mathscr{A} \times_{\mathscr{B}} \mathscr{B}_1 = \mathscr{A}_1$ as k-algebras, and this is not an affinoid k-algebra. It follows that X_1 is a non-affinoid space, yet it has an affinoid finite cover X (corresponding to a normalization in the rigid-analytic category).

To make our example, let $\mathscr{A} = k\{x, y\}$ and $\mathscr{B} = \mathscr{A}/(x^2 - \pi x)$, where $\pi \in k$ is a non-zero element with $r = |\pi| < 1$. Thus, $\mathscr{B} \simeq k\{y_0\} \times k\{y_\pi\}$, where (y_0, y_π) is the image of y. In particular, $X = \mathscr{M}(\mathscr{A})$ is a 2-dimensional closed unit polydisc and Y is a disjoint union of two one-dimensional closed unit subdiscs $D_0 = \{x = 0\}$ and $D_\pi = \{x = \pi\}$. We identify D_0 and D_π via $y_0 = y_\pi + 1$, so identify $\mathscr{B}_1 := k\{z\}$ as a closed k-subalgebra of $\mathscr{B}$ via $z \mapsto (y_0, y_\pi + 1)$. Let $\mathscr{A}_1 = \mathscr{A} \times_{\mathscr{B}} \mathscr{B}_1$ be the closed preimage of $\mathscr{B}_1$ in $\mathscr{A}$ provided with the k-Banach norm induced from the Gauss norm on $\mathscr{A}$. Working on the level of abstract rings, it is immediate that the inclusion homomorphism $\psi : \mathscr{A}_1 \to \mathscr{A}$ is finite and both rings have a common fraction field. (In particular, $\mathscr{A}$ is the integral closure of $\mathscr{A}_1$, since it is well-known that $\mathscr{A}$ is integrally closed.) Thus, we obtain a Cartesian diagram of rings as above, and our next task is to show that $\widetilde{\psi}$ is not finite.

Assume to the contrary that $\widetilde{\psi}$ is finite. Then $\widetilde{\mathscr{B}} \simeq \widetilde{k}[y_0] \times \widetilde{k}[y_\pi]$ is finite over the image of $\widetilde{\mathscr{A}_1}$ in $\widetilde{\mathscr{B}}$. The latter image is contained in the image of $\widetilde{\mathscr{A}} \simeq \widetilde{k}[x, y]$ in $\widetilde{\mathscr{B}}$, which is easily seen to be equal to $\widetilde{k}[(y_0, y_\pi)]$ (this is the key point of the construction: we use that the reduction of the epimorphism $\mathscr{A} \twoheadrightarrow \mathscr{B}$ is not surjective!). On the other hand, the image of $\widetilde{\mathscr{A}_1}$ in $\widetilde{\mathscr{B}}$ is contained in the image of $\widetilde{\mathscr{B}}_1$ in $\widetilde{\mathscr{B}}$, which equals to $\widetilde{k}[(y_0 + 1, y_\pi)]$. Since the intersection of $\widetilde{k}[(y_0, y_\pi)]$ and $\widetilde{k}[(y_0 + 1, y_\pi)]$ in $\widetilde{\mathscr{B}}$ is $\widetilde{k}$ (because $\mathrm{char}(\widetilde{k}) = 0$), we obtain a contradiction to the original assumption that $\widetilde{\psi}$ is finite.

Let $Y_1 = \mathscr{M}(\mathscr{B}_1)$ and $Y = \mathscr{M}(\mathscr{B}) = D_0 \coprod D_\pi$. It remains to show that the pushout X_1 of the diagram $Y_1 \leftarrow Y \to X$ exists as a strictly k-analytic space and that $X_1 \to X$ is a finite surjection. Set $z = \frac{x}{\pi}$ and consider the subpolydisc $Z := X\{z\} \hookrightarrow X$ of polyradius $(r, 1)$. Since we want to construct X_1 by identifying the closed unit subdiscs D_0 and D_π that are also closed subdiscs in Z, it suffices to show that there exists a pushout Z_1 of the diagram $Y_1 \leftarrow Y \to Z$ in the k-analytic category such that Z_1 is strictly k-analytic and three properties hold: $Y_1 \to Z_1$ is a closed immersion, $Y = Y_1 \times_{Z_1} Z$ (so $Z_1 - Y_1 = Z - Y$), and $Z \to Z_1$ is a finite surjection. (Then X_1 can be constructed by gluing Z_1 and $X - Y$ along $Z - Y = Z_1 - Y_1$.) We will show that Z_1 exists as an affinoid strictly k-analytic space.

To construct Z_1, the key is to show that the preimage $\mathscr{C}_1 = \mathscr{C} \times_{\mathscr{B}} \mathscr{B}_1$ (ring-theoretic fiber product) of $\mathscr{B}_1 \hookrightarrow \mathscr{B}$ under the epimorphism $\mathscr{C} := k\{\frac{x}{\pi}, y\} \twoheadrightarrow \mathscr{B}$ is a strictly k-analytic affinoid k-subalgebra of $\mathscr{C}$. Since $\mathscr{C}/(z^2 - z) \simeq \mathscr{B}$, we obtain that $\mathscr{C}_1$ contains the affinoid k-subalgebra $\mathscr{C}_0$ generated by $S = z^2 - z$ and $R = y + z$, which has to be isomorphic to $k\{R, S\}$ because $\widetilde{R}, \widetilde{S} \in \widetilde{\mathscr{C}}$ are algebraically independent over $\widetilde{k}$. The map $\widetilde{\mathscr{C}}_0 \to \widetilde{\mathscr{C}}$ is integral, so the embedding homomorphism $\mathscr{C}_0 \to \mathscr{C}$ is finite by [BGR, 6.3.5/1]. Therefore the intermediate ring $\mathscr{C}_1$ is finite over $\mathscr{C}_0$, but being finite over a strictly k-affinoid ring implies that $\mathscr{C}_1$ itself is strictly k-affinoid.

Consider the k-affinoid space $Z_1 = \mathscr{M}(\mathscr{C}_1)$. By construction we see that $\mathscr{C}_1$ has normalization $\mathscr{C}$ and that $Y = Y_1 \times_{Z_1} Z$. It is obvious that Z_1 is a k-analytic pushout of $Y_1 \leftarrow Y \to Z$ within the k-affinoid category (which is enough to carry out the above analysis with X, by exhausting $\mathbf{A}^1_k$ with closed discs centered at the origin), but to justify the pushout property in the larger k-analytic category requires further argument, as follows. The map $\pi : Z \to Z_1$ is finite, so $(\pi_G)_*$ carries coherent sheaves on Z_G to coherent sheaves on $Z_{1,G}$.

We view $\mathscr{O}_{Y_G}$ as a coherent sheaf on Z_G and $\mathscr{O}_{Y_{1,G}}$ as a coherent sheaf on $Z_{1,G}$. Thus, the natural map

$$\mathscr{O}_{Z_{1,G}} \to (\pi_G)_*(\mathscr{O}_{Z_G}) \times_{(\pi_G)_*(\mathscr{O}_{Y_G})} \mathscr{O}_{Y_{1,G}}$$

of coherent $\mathscr{O}_{Z_{1,G}}$-algebras is an isomorphism because it is an isomorphism on global sections and Z_1 is affinoid. The desired universal property of Z_1 is therefore an immediate consequence of the following lemma (which helps us to work G-locally on a target space for the universal property).

Lemma 3.5. *With notation as above, a subset $V_1 \subseteq Z_1$ is a k-analytic domain whenever $\pi^{-1}(V_1) \subseteq Z$ is a k-analytic domain.*

Proof. If $T \to W$ is a closed immersion of k-analytic spaces and $t \in T$ is a point then $\widetilde{T}_t \to \widetilde{W}_t$ is an isomorphism in $\mathrm{bir}_{\widetilde{k}}$ [T2, 4.8(i)]. Hence, by [T2, 4.5, 4.6], $U \mapsto U \cap T$ sets up a bijective correspondence between the sets of germs of k-analytic domains in W through t and in T through t.

Our problem is to check that V_1 is a k-analytic domain locally at each point $v_1 \in V_1$, and since π is an isomorphism over $Z_1 - Y_1$ it suffices to consider $v_1 \in Y_1$. We have $\pi^{-1}(v_1) = \{v, v'\}$, and $\pi : Y \to Y_1$ induces isomorphisms of germs $(Y, v) \simeq (Y_1, v_1)$ and $(Y, v') \simeq (Y_1, v_1)$, so the formation of preimages under π sets up a bijective correspondence between the sets of germs of k-analytic domains in (Z_1, v_1) and in (Z, v) or (Z, v'). Thus, for $V := \pi^{-1}(V_1)$ there are unique germs of k-analytic domains (U_1, v_1) and (U_1', v_1) in (Z_1, v_1) such that $(\pi^{-1}(U_1), v) = (V, v)$ and $(\pi^{-1}(U_1'), v') = (V, v')$. The involution of $Y = D_0 \coprod D_\pi$ over Y_1 swaps (V, v) and (V, v') while preserving $\pi^{-1}(U_1)$, so we can choose $U_1' = U_1$. Hence, $\pi^{-1}(U_1)$ and V topologically agree near $\{v, v'\} = \pi^{-1}(v_1)$. Since π is topologically proper and surjective, the topological germs (V_1, v_1) and (U_1, v_1) in (Z_1, v_1) therefore coincide, so V_1 is a k-analytic domain near v_1 in Z_1. ■

Example 3.4 is very close to de Jong's example. In both cases a global pushout loses a good property (being proper over an affine or being affinoid), yet it can be constructed by restricting to a subspace ($\mathbf{A}^1 \times \mathbf{A}^1 \hookrightarrow \mathbf{P}^1 \times \mathbf{A}^1$ or $\mathscr{M}(k\{\frac{x}{\pi}, y\}) \hookrightarrow \mathscr{M}(k\{x, y\})$) where it behaves nicely and is described by simple explicit formulas that are the same in both examples.

4. Graded commutative algebra

Throughout §4–§6, G is an arbitrary commutative multiplicative group (that will be $\mathbf{R}^\times_{>0}$ in the applications) and k is a G-graded field. We consider only G-gradings in the sequel, so G will often be omitted from the notation. The G-grading on any graded ring $A = \oplus_{g \in G} A_g$ will be denoted $\rho : \coprod(A_g - \{0\}) \to G$. For any nonzero A, by $A^\times$ we denote the group of invertible homogeneous elements, so there is a homomorphism $\rho : A^\times \to G$ whose image in case $A = K$ is a graded field is the subgroup of G that consists of all $g \in G$ such that $K_g \neq 0$. We are going to prove some results about extensions of graded fields. We will see that the theory of graded fields is similar to the classical ungraded case, and many proofs are just mild variants of the classical proofs. Some results on graded fields were proved in [CT, §5.3], and the notions of a finite extension and its degree were introduced there; recall from [T2, 1.2] that any graded module over a graded field K is necessarily a free module (with a homogeneous basis), and if $K \to L$ is a map of graded fields then its *degree* is defined to be the K-rank of L.

A *graded domain* is a graded ring A such that all nonzero homogeneous elements of A are not zero-divisors in A, and a key example of a graded domain is the ring $K[g_0^{-1}T]$ for a graded field K and any $g_0 \in G$; this is the ring $K[T]$ in which K is endowed with its given grading and T is declared to be homogeneous with grading g_0. For example, if $c \in K^\times$ with $\rho(c) = g \in G$ then evaluation at c defines a graded homomorphism of graded rings $K[g^{-1}T] \to K$.

A graded ring A is *graded noetherian* if every homogeneous ideal in A is finitely generated, and by the classical argument it is equivalent to say that the homogeneous ideals of A satisfy the ascending chain condition, in which case every graded A-submodule of a finitely generated graded A-module is finitely generated. The proof of the Hilbert basis theorem carries over, so $A[g^{-1}X]$ is graded noetherian for any $g \in G$ when A is graded noetherian. In particular, if k is a graded field then any finitely generated graded k-algebra is graded noetherian.

Lemma 4.1. *Let K be a graded field. For any $g \in G$, every nonzero homogeneous ideal $I \subseteq K[g^{-1}T]$ is principal with a unique monic homogeneous generator. Moreover, $K[g^{-1}T]^\times = K^\times$ and $K[g^{-1}T]$ is graded-factorial in the sense that every monic homogeneous element in $K[g^{-1}T]$ with positive degree is uniquely* (*up to rearrangement*) *a product of monic irreducible homogeneous elements in $K[g^{-1}T]$.*

Proof. If f and g are nonzero homogeneous elements of A having respective leading terms aT^m and bT^n with $m \leq n$ then $a, b \in K^\times$ and $g - \frac{a}{b}T^{n-m}f$ is homogeneous and either vanishes or is of degree smaller than n, as in the usual Euclidean algorithm. In particular, it follows that I is generated by a monic homogeneous polynomial, and such a generator is obviously unique. It follows that the maximal graded ideals of $K[g^{-1}T]$ are precisely the ideals (f) for a monic irreducible homogeneous $f \in K[g^{-1}T]$. Hence, $K[g^{-1}T]/(f)$ is a graded field for such f. In particular, if $h \in K[g^{-1}T]$ is homogeneous and a given monic homogeneous f does not divide h then $\alpha f + \beta h = 1$ for some homogeneous $\alpha, \beta \in K[g^{-1}T]$. We can therefore show that $K[g^{-1}T]$ is graded-factorial by copying the classical argument (working just with nonzero homogeneous elements). ■

An important instance of this lemma occurs for an extension L/K of graded fields: if $x \in L^\times$ with $\rho(x) = g \in G$ then the graded K-algebra evaluation map $K[g^{-1}T] \to L$ at x has homogeneous kernel denoted I_x, and $K[g^{-1}T]/I_x \to L$ is an isomorphism onto the graded domain $K[x] \subseteq L$, so I_x is a graded-prime ideal of $K[g^{-1}T]$. If $I_x = 0$ then we say x is *transcendental* over K, and otherwise I_x is a graded-maximal ideal with $K[x]$ therefore a graded field of finite degree over K. The case $I_x \neq 0$ always occurs when L has finite degree over K (due to K-freeness of graded K-modules), and whenever $I_x \neq 0$ the unique monic generator of I_x is denoted $f_x(T)$ and is called the *minimal homogeneous polynomial* of x over K.

Corollary 4.2. *For any graded field K and nonzero homogeneous polynomial $f(T) \in K[g^{-1}T]$, there exists a finite extension L/K such that f splits completely into a product of homogeneous degree-*1 *polynomials in $L[g^{-1}T]$.*

Proof. We may assume f is monic and we induct on $\deg_T(f)$. We may assume $\deg_T(f) > 0$. Factoring $f(T)$ as a product of irreducible monic homogeneous polynomials in $K[g^{-1}T]$, we may assume f is a monic irreducible and $f(0) \in K^\times$. Then $K' = K[g^{-1}T]/(f)$ is a graded field of finite degree over K such that $f(x) = 0$ for a homogeneous $x \in K'^\times$ with $\rho(x) = g$. Hence, $T - x \in K'[g^{-1}T]$ is homogeneous and by the Euclidean algorithm in such graded polynomial rings we have $f = (T - x)h(T)$ in $K'[g^{-1}T]$ for some monic homogeneous $h \in K'[g^{-1}T]$. Since $\deg_T(h) < \deg_T(f)$, we are done. ■

It was noted in [T2, §1] that a graded ring A does not contain non-zero homogeneous divisors of zero (i.e., A is a graded domain) if and only if it can be embedded into a graded field: the unique minimal such graded field $\mathrm{Frac}_G(A)$ is obtained by localizing at the multiplicative set of all non-zero homogeneous elements (and it has the expected universal mapping property). We call $\mathrm{Frac}_G(A)$ the *graded field of fractions* of A. Also, we say that a graded domain R is *integrally closed* if it is integrally closed in the graded sense (using monic polynomial relations satisfied by homogeneous elements) in $\mathrm{Frac}_G(R)$. (If $R \to R'$ is a graded map of graded rings and $r' \in R'_g$ satisfies $f(r') = 0$ for some monic $f \in R[X]$ with degree $n > 0$ then we may replace nonzero coefficients of f with suitable nonzero homogeneous parts (depending on $g \in G$) to find such an f that is homogeneous in $R[g^{-1}X]$.) For any injective graded map $R \to R'$ between graded rings, the *graded integral closure* of R in R' is the graded R-subalgebra $\widetilde{R} \subseteq R'$ consisting of elements $r' \in R'$ whose homogeneous parts are integral over R. If $\widetilde{R} = R$ then we say that R is *integrally closed* (in the graded sense) in R'.

Corollary 4.3. *If L/K is an extension of graded fields, A is a graded subring of K integrally closed in K, and $x \in L^\times$ is integral over A with $\rho(x) = g \in G$, then the minimal homogeneous polynomial f_x of x over K is defined over A; i.e. $f_x(T) \in A[g^{-1}T] \subseteq K[g^{-1}T]$.*

Proof. Since $x \in L^\times$, minimality of $d = \deg_T(f_x)$ forces the homogeneous $f_x(0) \in K$ to be nonzero with $\rho(f_x(0)) = g^d$. Increase L so that there is a homogeneous factorization

$$f_x(T) = (T - x_1) \dots (T - x_d)$$

in $L[g^{-1}T]$. Each x_i is homogeneous in L with $\prod x_i = \pm f_x(0) \in K^\times$, so $x_i \in L^\times$ for all i. By homogeneity of the factorization, $\rho(x_i) = \rho(T) = g$ for all i. Obviously each $x_i \in L$ has minimal homogeneous polynomial

f_x over K, so each graded subring $A[x_i] \subseteq L$ is isomorphic to $A[g^{-1}T]/(f_x) \simeq A[x]$ and hence each $A[x_i]$ is finite as an A-module. Thus, the subring $A[x_1, \dots, x_d] \subseteq L$ is finite over A. Since the coefficients of $f_x(T)$ are contained in this latter ring, they are integral over A. But A is integrally closed in K in the graded sense and the coefficients of f_x are homogeneous (or zero), so $f_x(T) \in A[g^{-1}T]$. ■

Corollary 4.4. *Let L/K be an extension of graded fields and let $\{A_i\}$ be a non-empty collection of graded subrings of K that are integrally closed in K in the graded sense. If B_i denotes the graded integral closure of A_i in L then $\cap B_i$ coincides with the graded integral closure of $\cap A_i$ in L.*

Note that the assumption on integral closedness of A_i's is critical for the corollary to hold.

Proof. Obviously, $\cap B_i$ contains the integral closure of $\cap A_i$. Conversely, let x be a nonzero homogeneous element of $\cap B_i$, with $\rho(x) = g$. Clearly, $K[x]$ has finite K-rank (since x satisfies a monic homogeneous relation over any $A_i \subseteq K$), and the coefficients of $f_x \in K[g^{-1}T]$ lie in each A_i by Corollary 4.3. Hence x is integral over $\cap A_i$, as claimed. ■

Theorem 4.5. *Let K'/K be an integral extension of graded fields and let $\mathcal{O}$ be a graded valuation ring of K. Let $\mathcal{O}'$ denote the integral closure of $\mathcal{O}$ in K'. Each graded prime ideal m' of $\mathcal{O}'$ over $m_{\mathcal{O}}$ is a graded maximal ideal and the graded localization $\mathcal{O}'_{m'}$ is a graded valuation ring, with $\mathcal{O}'_{m'} \cap \mathcal{O}' = m'$. Every graded valuation ring of K' dominating $\mathcal{O}$ arises in this way.*

Proof. The maximality of m' reduces to the fact that a graded domain that is integral over a graded field F must be a graded field, the proof of which goes almost exactly as in the classical ungraded case (by using integrality to reduce to considering a graded domain that is finitely generated over F as a graded F-module, for which there is a homogeneous F-basis). As for the description of the graded valuation rings extending $\mathcal{O}$, in the ungraded case this is [ZS, Thm. 12, §7] and its corollaries. The method of proof there (including the proof of [ZS, §5, Lemma]) adapts nearly verbatim to the graded case, due to the fact [T2, 1.4(i)] (where the ground field k plays no role) that the integral closure of $\mathcal{O}$ in K' is the intersection of all graded valuation rings of K' containing $\mathcal{O}$. ■

The following two finiteness and flatness results should have been recorded in [T2, §1].

Lemma 4.6. *If k is a graded field and A is a finitely generated k-algebra that is a graded domain then the graded integral closure $\overline{A}$ of A in $K = \mathrm{Frac}_G(A)$ is finite over A (and so it is a finitely generated k-algebra).*

Proof. Let $k' = \mathrm{Frac}_G(k[g^{-1}T_g]_{g \in G})$, so k'/k is a graded extension field with $\rho(k'^{\times}) = G$. Clearly, $A' = A \otimes_k k'$ is a finitely generated graded k'-subalgebra of $K \otimes_k k'$. But by inspection $K \otimes_k k'$ is a graded localization of the graded ring $K[g^{-1}T_g]_{g \in G}$ that is a graded domain, so $K \otimes_k k'$ is a graded domain. Thus, A' is a graded domain with $K' := \mathrm{Frac}_G(A')$ equal to $\mathrm{Frac}_G(K \otimes_k k')$. The graded integral closure $\overline{A'}$ of A' in K' contains $\overline{A} \otimes_k k'$. Since k' is a free k-module, $\overline{A}$ is a finite A-module if and only if $\overline{A} \otimes_k k'$ is a finite A'-module. But A' is a graded noetherian ring, so such finiteness holds if $\overline{A'}$ is A'-finite. Hence, we may replace k with k' to reduce to the case $\rho(k^{\times}) = G$. To prove that $\overline{A}$ is A-finite, it suffices (by integrality of $\overline{A}$ over A) to show that there cannot be a strictly increasing sequence of A-finite graded subalgebras in $\overline{A}$. Since $\rho(k^{\times}) = G$, [T2, 1.1] gives that $R_1 \mapsto k \otimes_{k_1} R_1$ is an equivalence of categories between k_1-algebras and graded k-algebras, and by chasing gradings we see that a map of k_1-algebras is integral if and only if the corresponding map of graded k-algebras is integral. It is likewise clear that this equivalence respects finiteness of morphisms in both directions, and it also respects the property of being a graded domain or being finitely generated (over k or k_1) in both directions. Hence, if there is an infinite strictly increasing sequence of A-finite graded domains over A with the same graded fraction field as A then we get an infinite strictly increasing sequence of A_1-finite graded domains over A_1 with the same ordinary fraction field as A_1. But A_1 is a domain finitely generated over k_1, so we have a contradiction (as the A_1-finiteness of the integral closure of A_1 is classical). ■

Lemma 4.7. *Let M be a graded module over a graded valuation ring $\mathcal{O}$, and assume M is torsion-free in the graded sense, which is to say that for each nonzero homogeneous $a \in \mathcal{O}$, the self-map $m \mapsto am$ of the $\mathcal{O}$-module M is injective. Then M is $\mathcal{O}$-flat, and if M is finitely generated it admits a homogeneous basis.*

Proof. By consideration of direct limits we may assume that M is finitely generated. Let $K = \mathrm{Frac}_G(\mathcal{O})$, and let $V = K \otimes_{\mathcal{O}} M$. The natural map $M \to V$ is injective, and V admits a finite homogeneous K-basis. We prove the existence of a homogeneous basis of M by induction on the K-rank of V (which we may assume to be positive). If the K-rank is 1 then by shifting the grading on M we can assume that $V = K$ as graded K-modules and that M is a finitely generated graded ideal in $\mathcal{O} \subseteq K$. Hence, this case is settled since $\mathcal{O}$ is a graded valuation ring. If the K-rank n is larger than 1, let $L \subseteq V$ be the graded K-submodule spanned by a member of a homogeneous K-basis of V, so the image M'' of M in V/L admits a homogeneous $\mathcal{O}$-basis. In particular, M'' splits off as a graded direct summand of M, so $M \cap L$ is identified with a complement and thus is also finitely generated over $\mathcal{O}$. By the settled rank-1 case we are done. ∎

Next, we discuss transcendental extensions of graded fields. For any extension of graded fields K/k, a subset $S \subseteq K^\times$ (with $\rho(s) = g_s \in G$ for each $s \in S$) is said to be *algebraically independent over* k if the graded k-algebra map $k[g_s^{-1}T_s]_{s\in S} \to K$ defined by $T_s \mapsto s$ is injective. The following two conditions on a subset $S \subseteq K^\times$ are equivalent: (i) S is a maximal algebraically independent set over k, (ii) S is minimal for the property that K is integral over the graded k-subfield generated by S (i.e., K is integral over the graded fraction field of the graded k-subalgebra generated by S). In condition (ii) it clearly suffices to check integrality for elements of $K^\times$. A subset $S \subseteq K^\times$ satisfying (i) and (ii) is called a *transcendence basis* for K/k (and such subsets clearly exist, via condition (i)). Analogously to the classical arguments, one proves that all transcendence bases have equal cardinality, which is called the *transcendence degree* of K over k and is denoted $\mathrm{trdeg}_k(K)$. Also, one shows akin to the classical case that transcendence degree is additive in towers of graded fields. An extension K/k of graded fields has transcendence degree 0 if and only if each $x \in K$ is integral over k, in which case we say that K/k is *algebraic*.

As with degree for finite extensions of graded fields (studied in [CT, 5.3.1]), the value of the transcendence degree "splits" into a contribution from the extension of 1-graded parts K_1/k_1 (ordinary fields) and a contribution from the extension of grading groups $\rho(k^\times) \subseteq \rho(K^\times)$ in G. Namely, the following lemma holds.

Lemma 4.8. *Let K/k be an extension of graded fields. We have*

$$\mathrm{trdeg}_k(K) = \mathrm{trdeg}_{k_1}(K_1) + \dim_{\mathbf{Q}}\left((\rho(K^\times)/\rho(k^\times)) \otimes_{\mathbf{Z}} \mathbf{Q}\right)$$

in the sense of cardinalities.

Proof. Let $\{x_i\}$ be a transcendence basis for K_1/k_1 in the usual sense, and choose elements $y_j \in K^\times$ such that the gradings $\rho(y_j)$ form a $\mathbf{Q}$-basis of $(\rho(K^\times)/\rho(k^\times)) \otimes_{\mathbf{Z}} \mathbf{Q}$. It suffices to show that $S = \{x_i, y_j\} \subseteq K^\times$ is a transcendence basis of K/k.

First we check that S is a transcendence set. If there is a nontrivial polynomial relation over k satisfied by these elements then by their homogeneity we may take the relation to have homogeneous coefficients. Any monomial $cy_{j_1}^{e_1}\cdots y_{j_r}^{e_r}$ with $r > 0$, $e_1, \dots, e_r > 0$, and $c \in k^\times$ has grading $\rho(c) \cdot \prod \rho(y_{j_m})^{e_m}$ whose image in $(\rho(K^\times)/\rho(k^\times)) \otimes_{\mathbf{Z}} \mathbf{Q}$ is nonzero and uniquely determines r and the ordered r-tuple $(e_1, \dots, e_r)$. Thus, we can decompose a hypothetical nontrivial homogeneous polynomial relation over k according to the y-monomial gradings to get such a relation in which the y-contribution to each monomial in the x's and y's is a common term $y_{j_1}^{e_1}\cdots y_{j_r}^{e_r}$ with $r \geq 0$ and $e_m > 0$ for $1 \leq m \leq r$. This can then be cancelled, so we get a nontrivial relation $\sum c_I x_I^{e_I} = 0$ with all $c_I \in k^\times$ having a common grading. We can then scale by $k^\times$ to get to the case when all $c_I \in k_1^\times$, contradicting that the x_i's are algebraically independent in K_1 over k_1 in the usual sense.

Finally, we check that K is algebraic over the graded subfield generated over k by S. Any $t \in K^\times$ satisfies $\rho(t^e) = \rho(c) \cdot \prod \rho(y_{j_m})^{e_m}$ for some $c \in k^\times$ and $j_1, \dots, j_r$ with $e_1, \dots, e_r > 0$ (for some $r \geq 0$). Replacing t with $t^e/(c\prod y_{j_m}^{e_m})$ then allows us to assume $t \in K_1$, so we are done by the transcendence basis property for $\{x_i\}$. ∎

We will in the sequel need the following natural-looking lemma, which turns out to be surprisingly difficult.

Lemma 4.9. *Let B be an integrally closed graded domain, $T = \{T_i\}_{i\in I}$ a set of variables and $g = \{g_i\}_{i\in I}$ a set of elements of G (where some g_i's may be equal). The G-graded Laurent polynomial ring $B[g^{-1}T, gT^{-1}]$ is an integrally closed graded domain.*

Proof. The problem easily reduces to the case of a finite non-empty I, so we may assume $I = \{1, \ldots, n\}$. By induction on the number of variables we can furthermore assume $n = 1$. Let K be the graded fraction field of B. The set $\rho(K^\times) \subseteq G$ of gradings of nonzero homogeneous elements of K is a subgroup of G. First assume that g is $\mathbf{Z}$-linearly independent from $\rho(K^\times)$ (i.e., $g \bmod \rho(K^\times)$ has infinite order in $G/\rho(K^\times)$), so $K[g^{-1}T, gT^{-1}]$ is a graded field and hence the graded subring $B[g^{-1}T, gT^{-1}]$ is a graded domain with graded fraction field $K[g^{-1}T, gT^{-1}]$. The nonzero homogeneous elements have the form qT^j with homogeneous $q \in K^\times$ since $g = \rho(T)$ is $\mathbf{Z}$-linearly independent from $\rho(K^\times)$. Such an element qT^j is integral over $B[g^{-1}T, gT^{-1}]$ if and only if q is, in which case there is a relation $\sum_{j=0}^{N} h_j(T)q^j = 0$ with $h_N = 1$, $h_0 \neq 0$, homogeneous $h_j \in B[g^{-1}T, gT^{-1}]$, and all nonzero terms $h_j \cdot q^j$ having the same grading. Thus, for each nonzero h_j we have $h_j = b_j T^{e_j}$ for some $e_j \in \mathbf{Z}$ and some homogeneous nonzero $b_j \in B$ such that $\rho(q)^{N-j} = \rho(b_j)g^{e_j}$ in G. Chasing constant terms ($e_j = 0$) gives a monic relation for q over B, so $q \in B$. This settles the case when g is independent from $\rho(K^\times)$, so we can now assume that there is a dependence relation: $g \bmod \rho(K^\times)$ is a torsion element in $G/\rho(K^\times)$. Thus, there is a minimal $e > 0$ such that $g^e = \rho(c)$ for some $c \in K^\times$.

Step 1. We first reduce to the case $e = 1$. The nonzero homogeneous elements of $K[g^{-1}T, gT^{-1}]$ are the elements of the form $h(T^e/c)T^r$ with $0 \leq r < e$ and $h \in K[X]$ a nonzero polynomial whose nonzero coefficients are homogeneous and have a common grading. Such an expression is unique due to the minimality of e. Since K is a graded field, we can uniquely write $h = q \cdot f$ for $q \in K^\times$ and $f \in K_1[X]$ a monic polynomial over the ordinary field K_1 of elements of K with trivial grading. The element T is a homogeneous unit in $B[g^{-1}T, gT^{-1}]$, so we see that $K[g^{-1}T, gT^{-1}]$ is a graded domain with graded fraction field having underlying ring $(K_1(X) \otimes_{K_1} K)[T]/(T^e - X \otimes c)$. The G-grading is determined by $K_1(X)$ having trivial grading, K having its given G-grading, and T assigned grading g.

To prove that the graded domain $B[g^{-1}T, gT^{-1}]$ is integrally closed in its graded fraction field, by minimality of e we can multiply by a suitable power of the homogeneous unit T to reduce to considering homogeneous elements $\xi \in K_1(X) \otimes_{K_1} K$ that are integral over $B[g^{-1}T, gT^{-1}]$ (with $X = T^e/c$). The monic integral relation for ξ can be taken to have its nonzero homogeneous coefficients with the unique form $q_i \cdot f_i(X)T^{r_i}$ for $0 \leq r_i < e$, $f_i \in K_1[X, X^{-1}]$ having monic least-degree X-monomial, and $q_i \in K^\times$ such that $q_i \cdot f_i(T^e/c) \in B[g^{-e}T^e, g^eT^{-e}]$ for all i and $\rho(q_i) = \rho(\xi)^{N-i}$ when $r_i = 0$. We can decompose the monic integral relation for ξ according to those i for which r_i is equal to a fixed integer r between 0 and $e - 1$, and by monicity in ξ we see that passing to the case with $r = 0$ gives a monic integral relation for ξ such that $r_i = 0$ for all i. Since $\xi \in K_1(T^e/c) \otimes_{K_1} K$, we may therefore replace T with T^e and g with g^e to reduce to the case $e = 1$. Thus, $g = \rho(c)$ for some $c \in K^\times$.

Step 2. Next, we reduce to the case when B is a graded valuation ring, and we simplify the integral closedness to be verified. Since K is the graded fraction field of B, we have that the integrally closed graded subring B is the intersection of all graded valuation rings $\mathcal{O}$ of K that contain B; this is [T2, 1.4(i)] (in which the implicit ground field k is not relevant). For each such $\mathcal{O}$, clearly $\mathcal{O}[g^{-1}T, gT^{-1}]$ has the same graded fraction field as $B[g^{-1}T, gT^{-1}]$. Any homogeneous ξ in this graded fraction field that is integral over $B[g^{-1}T, gT^{-1}]$ is also integral over each $\mathcal{O}[g^{-1}T, gT^{-1}]$, so if we can handle the case of graded valuation rings then

$$\xi \in \bigcap_{\mathcal{O} \supseteq B} \mathcal{O}[g^{-1}T, gT^{-1}] = B[g^{-1}T, gT^{-1}],$$

as desired. Hence, it suffices to consider the case when $B = \mathcal{O}$ is a graded valuation ring with graded fraction field K. In other words, in the graded K-algebra $K[X, X^{-1}]$ with X having trivial grading we want to prove that the graded subring $R = \mathcal{O}[cX, c^{-1}X^{-1}]$ for any $c \in K^\times$ and any graded valuation ring $\mathcal{O}$ of K is integrally closed in the graded fraction field $K \otimes_{K_1} K_1(X)$. This graded field has underlying ring that is the localization of $K[X]$ at the multiplicative set of regular elements $K_1[X] - \{0\}$.

Before we consider integral closedness properties over $\mathcal{O}$, it will simplify matters to carry out some integral closedness considerations over K. First, we prove that the graded domain $K[X, X^{-1}]$ (with $\rho(X) = 1$) is integrally closed (in the graded sense) in $K \otimes_{K_1} K_1(X)$. Choose a nonzero homogeneous element $\xi \in K \otimes_{K_1} K_1(X)$ that is integral over $K[X, X^{-1}]$. We want that $\xi \in K[X, X^{-1}]$. It is harmless to multiply ξ by an integral power of X, so we can arrange that ξ is integral over $K[X]$. It suffices to prove that $\xi \in K[X]$. If not, then multiplying by a sufficiently divisible nonzero element of $K_1[X]$ brings us to the case $\xi = f/f_1$ where $f \in K[X]$ is nonzero with all nonzero coefficients homogeneous of the same degree and $f_1 \in K_1[X]$ is a monic irreducible element that does not divide f in $K[X]$. We can scale f by $K^\times$ without loss of generality, so $f \in K_1[X]$. Hence, in an integrality relation for f/f_1 over $K[X]$ we can pass to the trivial graded part to deduce that $f/f_1 \in K_1(X)$ is integral over $K_1[X]$, forcing $f_1|f$ in $K_1[X]$, a contradiction. This reduces us to proving that $\mathcal{O}[cX, c^{-1}X^{-1}]$ is integrally closed (in the graded sense) in $K[X, X^{-1}]$. By swapping the roles of X and X^{-1} if necessary we can assume $c \in \mathcal{O}$.

Step 3. Choose a nonzero homogeneous element $\xi \in K[X, X^{-1}]$ that is integral over $\mathcal{O}[cX, c^{-1}X^{-1}]$. It is harmless to multiply ξ by an integral power of cX, so we can assume that ξ is integral over $\mathcal{O}[cX] \subseteq K[X]$. A variant on the same $K^\times$-scaling argument used in Step 2 shows that $K[X]$ is integrally closed in $K[X, X^{-1}]$ in the graded sense (ultimately because $K_1[X]$ is integrally closed in $K_1[X, X^{-1}]$ in the usual sense), so $\xi \in K[X]$. We have therefore reduced ourselves to proving that a nonzero homogeneous element $\xi \in K[X]$ that is integral over $\mathcal{O}[cX]$ lies in $\mathcal{O}[cX]$.

Let Y denote the nonzero homogeneous element cX (with grading $\rho(c)$), so the identification $K[X] = K[Y]$ preserves homogeneity of coefficients but may change their grading (if $\rho(c) \neq 1$). Since $\mathcal{O}$ is a graded valuation ring, the homogeneity of the coefficients of ξ ensures that the ideal

$$D_\xi = \{a \in \mathcal{O} \mid a\xi \in \mathcal{O}[Y]\}$$

of denominators of ξ with respect to Y is a nonzero principal homogeneous ideal, say $D_\xi = (\delta)$ for some nonzero homogeneous $\delta \in \mathcal{O}$. Hence, $\delta\xi$ has a (homogeneous) unit coefficient with respect to Y, so for any $n \geq 0$ we have $D_{\xi^n} = (\delta^n)$. The integrality hypothesis on ξ provides a monic relation

$$\xi^N + h_{N-1}(Y)\xi^{N-1} + \cdots + h_0(Y) = 0$$

for some $N > 0$ and $h_j \in \mathcal{O}[Y]$ having homogeneous coefficients, so $\delta^{N-1} \in D_{\xi^N} = (\delta^N)$. Thus, $1 \in (\delta)$, so $\delta \in \mathcal{O}^\times$. This gives $\xi \in \mathcal{O}[cX]$, as desired. ■

Corollary 4.10. *Let L/k be a graded field extension, $A \subseteq L$ an integrally closed graded k-subalgebra, $T = \{T_i\}_{i\in I} \in L$ a set of homogeneous elements algebraically independent over A, and $F = \mathrm{Frac}_G(k[T])$. Then the graded subring $FA \subseteq L$ generated by F and A is integrally closed.*

Proof. Obviously F is the graded fraction field of its graded subring B generated by k and the elements $T_i^{\pm 1}$, so F coincides with the graded localization ring B_R, where R is the set of non-zero homogeneous elements of B. It follows that $FA = C_R$, where C is the graded algebra generated by A and the elements $T_i^{\pm 1}$. Since graded integral closedness is preserved by graded localization (similarly to the ungraded case), it suffices to prove that C is integrally closed. Now, it remains to notice that by our assumption on T_i's, $C \simeq A[g^{-1}S, gS^{-1}]$, where $g = \{\rho(T_i)\}_{i\in I}$ and $S = \{S_i\}_{i\in I}$ is a corresponding set of graded indeterminates. Hence, the corollary follows from Lemma 4.9. ■

We conclude the section with the following simple lemma.

Lemma 4.11. *Let L/k be a graded field extension, $\{A_j\}$ a collection of integrally closed k-subalgebras of L, $T = \{T_i\}_{i\in I} \in L$ a set of homogeneous elements algebraically independent over each A_j, and $F = \mathrm{Frac}_G(k[T])$. Then $\cap(FA_j) = F(\cap A_j)$.*

Proof. Since FA_j is the localization of $A_j[T]$ at the multiplicative set of nonzero homogeneous elements of $F[T]$, the natural graded map $A_j \otimes_k F \to FA_j$ is an isomorphism of rings (and hence is a graded isomorphism). By the same reason, $(\cap A_j) \otimes_k F \simeq F(\cap A_j)$, and since F is a free k-module by [T2, 1.2], the lemma now follows from the following lemma. ■

Lemma 4.12. *Let M and N be modules over a commutative ring R, with M projective over R. Let $\{N_i\}$ be a set of R-submodules of N. The inclusion $(\cap N_i) \otimes_R M \subseteq \cap(N_i \otimes_R M)$ inside of $N \otimes_R M$ is an equality.*

Proof. Since M is a direct summand of a free module, we can assume M is free. We may replace N with $N/(\cap N_i)$, so $\cap N_i = 0$. In this case we want $\cap(N_i \otimes_R M) = 0$ inside of $N \otimes_R M$. We have $M \simeq \oplus_j Re_j$, so using linear projection to the factors reduces us to the trivial case $M = R$. ■

5. The category bir_G

In the sequel, we will have to simultaneously consider several G-graded birational spaces with different G-graded ground fields. More specifically, we will study commutative diagrams of G-graded fields

$$\begin{array}{ccc} l & \longrightarrow & L \\ \uparrow & & \uparrow \\ k & \longrightarrow & K \end{array}$$

For any such diagram, restriction of graded valuation rings induces a continuous map $\psi_{L/K,l/k} : \mathbf{P}_{L/l} \to \mathbf{P}_{K/k}$. Note that $\psi_{L/K,l/k}$ is the composition of the obvious topological embedding $\psi_{L/L,l/k} : \mathbf{P}_{L/l} \hookrightarrow \mathbf{P}_{L/k}$ and the natural map $\psi_{L/K/k} : \mathbf{P}_{L/k} \to \mathbf{P}_{K/k}$ of birational spaces over k. The maps $\psi_{L/K,l/k}$ were used in [T2, 5.3] to establish a connection between reduction of germs of the analytic spaces X over k and $X_K = X\widehat{\otimes}_k K$ over K. A deeper study of these maps in this and the next sections will be used later to prove that certain properties of analytic spaces (e.g. goodness) descend from X_K to X.

Theorem 5.1. *Keep the above notation. The map $\psi_{L/K,l/k}$ satisfies the following properties:*

(*i*) *it is surjective if and only if any algebraically independent set over k in $l^\times$ is algebraically independent over K;*

(*ii*) *it has finite fibers* (*resp. is injective*) *if the extension L/K is finite* (*resp. trivial*)*;*

(*iii*) *it is bijective if $L = K$ and l/k is algebraic.*

Part (iii) can be easily generalized: $\psi_{L/K,l/k}$ is bijective when l/k is algebraic, $L_1 = K_1$, and $\rho(L^\times)/\rho(K^\times)$ is a torsion group (this is a graded analogue of totally ramified extensions that are not necessarily purely inseparable). Also, to check the algebraic independence condition in (i) it suffices to work with a single transcendence basis for l over k, and it is insufficient to assume the weaker condition that $K \cap l$ is algebraic over k. Indeed, it can happen that L/l is a finite extension (so $\mathbf{P}_{L/l}$ is a point) but $K = \mathrm{Frac}_G(k[T])$ (so $\mathbf{P}_{K/k}$ is infinite) and $K \cap l = k$. For example, in the ungraded case there are infinite subgroups $\Gamma \subseteq \mathrm{PGL}_2(k)$ generated by a finite collection of non-trivial elements $\gamma_1, \ldots, \gamma_n$ with finite order (using $n = 2$ if $\mathrm{char}(k) = 0$ and $n = 3$ otherwise), so there exists a pair of finite subgroups $H_1, H_2 \subseteq \mathrm{PGL}_2(k)$ that generate an infinite group. Thus, $L = k(t)$ is a finite Galois extension of rational subfields $K = k(t)^{H_1} = k(x_1)$ and $l = k(t)^{H_2} = k(x_2)$, yet $K \cap l = k$.

Proof. The third claim follows from the observation that if $\mathcal{O}$ is a graded valuation ring in $\mathbf{P}_{L/k}$ and l is algebraic over k, then $\mathcal{O}$ must contain l because $\mathcal{O}$ is integrally closed in L in the graded sense. The second claim follows from the fact that any graded valuation ring $\mathcal{O}$ on K admits at most $[L : K]$ extensions to a graded valuation ring on L. To prove this fact, we adapt the classical ungraded argument. By Theorem 4.5, it is equivalent to check that the graded integral closure $\widetilde{\mathcal{O}}$ of $\mathcal{O}$ in L has at most $[L : K]$ graded prime ideals over the graded maximal ideal of $\mathcal{O}$. For this it suffices to show that each graded $\mathcal{O}$-subalgebra $R \subseteq L$ that is $\mathcal{O}$-finite and spans L over K has at most $[L : K]$ graded prime ideals dominating the graded maximal ideal of $\mathcal{O}$. The $\mathcal{O}$-module R is free by Lemma 4.7 so the graded algebra $R/m_{\mathcal{O}}R$ over the graded residue field $\mathcal{O}/m_{\mathcal{O}}$ is a free module of rank $[L : K]$. We are therefore reduced to checking that if F is a graded field and A is a graded finite F-algebra whose underlying F-module has rank n then A has at most n graded prime ideals. Note that all such graded prime ideals are maximal, since a graded domain of finite rank over F is necessarily a graded field. Thus, if F'/F is any graded extension field and $A' = A \otimes_F F'$ then $\mathrm{Spec}_G(A')$ maps onto $\mathrm{Spec}_G(A)$, so it suffices to treat the pair (A', F'). By choosing F' such that $\rho(F'^\times) = G$ (e.g., the graded fraction field of the graded polynomial ring $F[g^{-1}T_g]_{g\in G}$) we can thereby reduce to the case that each homogeneous nonzero $a \in A$ satisfies $\rho(a) \in \rho(F^\times)$. Consequently, by [T2, 1.1(ii)] the natural map $F \otimes_{F_1} A_1 \to A$ is an isomorphism and the operation $I \mapsto I \cap A_1$ identifies the ideal theory of A with that of A_1 respecting primality. Since A_1 is of rank n over the ordinary field F_1, we are reduced to the obvious classical case.

The first claim requires more effort than the other two. Assume that $\psi_{L/K,l/k}$ is surjective. To deduce that algebraically independent sets in $l^\times$ over k are algebraically independent over K when viewed in $L^\times$, by enlarging to a transcendence basis we must show that if $t = \{t_i\}$ is a transcendence basis for l over k then this collection t is an algebraically independent set in $L^\times$ over K. Let $k(t) = \mathrm{Frac}_G(k[t]) \subseteq l$, so l is algebraic over $k(t)$. Thus, $\mathbf{P}_{L/l} = \mathbf{P}_{L/k(t)}$, and hence $\psi_{L/k(t),K/k}$ is surjective. We may therefore assume $l = k(t)$. By functoriality, surjectivity of $\psi_{L/l,K/k}$ forces surjectivity of $\psi_{lK,l,K/k}$, so we can assume $L = lK = K(t)$ (but we do not know that t is an algebraically independent set over K). Letting $t' = \{t'_j\}$ be a subset of t that is maximal for being algebraically independent over K, we see that t' is a transcendence basis for $K(t)$ over K and our problem is to show that $t' = t$.

Let $u = \{u_r\}$ be a transcendence basis for K over k, so K is algebraic over $k(u)$. Arguing as in the classical case, $u \cup t'$ is a transcendence basis for $K(t)$ over k. Since $K(t)$ is algebraic over $k(u,t)$ and K is algebraic over $k(u)$, the natural map $\mathbf{P}_{k(u,t)/k(t)} \to \mathbf{P}_{k(u)/k}$ is surjective. If t' is a proper subset of t then $u \cup t$ is not an algebraically independent set over k. But t is algebraically independent over k, so there is a transcendence basis of $k(u,t)$ over k of the form $u' \cup t$ with u' a proper subset of u. Choose $u_0 \in u$ with $u_0 \notin u'$, so there is a relation $\sum_{i=0}^n h_i(u',t)u_0^i = 0$ such that $h_i \in k[u',t]$ for each i, $n > 0$, $h_n \neq 0$, $h_0 \neq 0$, and each nonzero h_i is homogeneous with grading $\rho(h_i) = \rho(h_0)\rho(u_0)^{-i}$ in G. Let ${u'}^I$ be a monomial in u' that appear in h_n (possibly multiplied against a monomial in t). By multiplying through the relation by $({u'}^I)^{n-1}$ and replacing u_0 in the transcendence basis u with ${u'}^I u_0$, we reach the situation in which the homogeneous element $h_n(0,t) \in k[t]$ is nonzero. Since u is an algebraically independent set over k, there is a graded valuation ring $\mathcal{O}$ on $k(u)$ containing k with $u' \cup \{1/u_0\} \subseteq m_{\mathcal{O}}$. Surjectivity of $\psi_{k(u,t)/k(t),k(u)/k}$ therefore provides a graded valuation ring $\mathcal{O}'$ of $k(u,t)$ that contains $k(t)$ and satisfies $\mathcal{O}' \cap k(u) = \mathcal{O}$, so $m_{\mathcal{O}'}$ contains $1/u_0$ and all elements of u'. The homogeneous $h_n(u',t) \in \mathcal{O}'$ is a unit in $\mathcal{O}'$ since $u' \subseteq m_{\mathcal{O}'}$ and $h_n(0,t) \in k(t)^\times \subseteq {\mathcal{O}'}^\times$, so the chosen polynomial relation with homogeneous coefficients provides an integral dependence relation on u_0 over $\mathcal{O}'$, forcing $u_0 \in \mathcal{O}'$. But $1/u_0 \in m_{\mathcal{O}'}$, so we have reached a contradiction.

Now we treat the converse implication. Assume that algebraically independent sets in $l^\times$ over k are algebraically independent over K. Choose a transcendence basis $T = \{T_i\}$ of l over k, so $T \subseteq L^\times$ is algebraically independent over K. For $l_0 = \mathrm{Frac}_G(k[T]) \subseteq L$ we have $\psi_{L/K,l/k} = \psi_{L/L,l/l_0} \circ \psi_{L/K,l_0/k}$, and the second step is bijective by (iii). Thus, by replacing l with l_0 we can assume that $l = \mathrm{Frac}_G(k[T])$ with T an algebraically independent set over K. The inclusion of graded subfields $k \subseteq K \cap l$ in K must be an equality, since any $x \in (K \cap l)^\times$ not in k is transcendental over k (from the explicit description of l) yet is also visibly algebraic over K.

We can now lift a graded valuation ring $\mathcal{O} \in \mathbf{P}_{K/k}$ to an element of $\mathbf{P}_{L/l}$, as follows. Let $m_{\mathcal{O}}$ be the maximal ideal of $\mathcal{O}$. We claim that the graded ideal $p = lm_{\mathcal{O}}$ is prime in the graded ring $l\mathcal{O} \subseteq L$ generated by $\mathcal{O}$ and l and has restriction $m_{\mathcal{O}}$ to $\mathcal{O}$. Indeed, the graded subring $\mathcal{O}[T] \subseteq L$ generated by $\mathcal{O}$ and T is isomorphic to a graded polynomial ring over $\mathcal{O}$, so $p' = m_{\mathcal{O}}\mathcal{O}[T]$ is a prime in the graded sense. Furthermore, $l = k[T]_R$ where R is the set of all non-zero homogeneous polynomials in $k[T]$, so $l\mathcal{O} = \mathcal{O}[T]_R$. Then $p = p'_R$, so to prove that $p \cap \mathcal{O} = m_{\mathcal{O}}$ it is enough to show that $R \cap m_{\mathcal{O}} = \emptyset$, but the latter is obvious because the equality $l \cap K = k$ implies $R \cap m_{\mathcal{O}} = k^\times \cap m_{\mathcal{O}} = \emptyset$. Hence, $\mathcal{O}$ is dominated by a local graded subring $A \subseteq L$ that is the graded localization of $l\mathcal{O}$ at p. Clearly, such an A contains l, and we have natural embeddings $l\mathcal{O} \subseteq lK \subseteq L$. Therefore, A embeds into L as a local graded l-subalgebra, and any graded valuation ring from $\mathbf{P}_{L/l}$ that dominates A is the required extension of $\mathcal{O}$. ■

The discussion preceding Theorem 5.1 provides the motivation to extend the category of birational spaces to include the maps $\psi_{L/K,l/k}$, so we now introduce the category $\mathrm{bir} = \mathrm{bir}_G$ of all G-graded birational spaces. On the level of objects, bir is just the disjoint union of all categories bir_l. A morphism $f : \mathcal{Y} \to \mathcal{X}$ of birational spaces corresponding to respective local homeomorphisms $Y \to \mathbf{P}_{L/l}$ and $X \to \mathbf{P}_{K/k}$ is a pair of compatible graded embeddings $k \hookrightarrow l$ and $K \hookrightarrow L$ and a continuous map $Y \to X$ compatible with $\psi_{L/K,l/k}$. We naturally extend the properties of objects and morphisms that were defined in [T2, §2]:

Definition 5.2. In the category bir_G of birational spaces over G-graded fields, $\mathcal{Y} = (Y \to \mathbf{P}_{L/l})$ is *affine* if Y maps bijectively onto an affine subset of $\mathbf{P}_{L/l}$, a morphism $(Y \to \mathbf{P}_{L/l}) \to (X \to \mathbf{P}_{K/k})$ is *separated* if the natural map $\phi : Y \to X \times_{\mathbf{P}_{K/k}} \mathbf{P}_{L/l}$ is injective, and such a morphism is *proper* if ϕ is bijective and $\psi_{L/K,l/k}$ is onto.

Furthermore, we say that $\mathcal{X} = (X \to \mathbf{P}_{K/k})$ is *separated* (resp. *proper*) if $X \to \mathbf{P}_{K/k}$ is injective (resp. bijective), which is to say that the canonical morphism $\mathcal{X} \to \mathbf{P}_{K/k}$ in bir is separated (resp. proper).

Lemma 5.3. *For birational spaces $\mathcal{Z} = (Z \to \mathbf{P}_{M/m})$, $\mathcal{Y} = (Y \to \mathbf{P}_{L/l})$, and $\mathcal{X} = (X \to \mathbf{P}_{K/k})$ and morphisms $h : \mathcal{Z} \to \mathcal{Y}$, $g : \mathcal{Y} \to \mathcal{X}$, and $f = g \circ h$, the following properties hold.*
(*i*) *If g and h are separated, then f is separated.*
(*ii*) *If f is separated then h is separated.*
(*iii*) *If g and h are proper, then f is proper.*
(*iv*) *If f and h are proper, then g is proper.*
(*v*) *If f is proper, g is separated, and $\psi_{M/L,m/l}$ is surjective (e.g., $m = l$) then h is proper.*
(*vi*) *If f is separated and h is proper then g is separated.*

Recall from Theorem 5.1(i) that the surjectivity in part (v) says exactly that a transcendence basis for m over l is algebraically independent over L. Also, by definition of properness, the hypothesis on h in (vi) forces $\psi_{M/L,m/l}$ to be surjective.

Proof. The proof is based on the same set-theoretic argument as we used in the proof of Lemma 2.1. Consider the following diagram

$$Z \xrightarrow{\gamma} Y \times_{\mathbf{P}_{L/l}} \mathbf{P}_{M/m} \xrightarrow{\beta'} X \times_{\mathbf{P}_{K/k}} \mathbf{P}_{M/m}$$

where β' is the base change by $\psi_{M/L,m/l}$ of the natural map $\beta : Y \to X \times_{\mathbf{P}_{K/k}} \mathbf{P}_{L/l}$. Separatedness/properness of f, g, and h are connected to injectivity/bijectivity of the maps $\alpha := \beta' \circ \gamma$, β, and γ respectively. In (i), we are given that β and γ are injective. Hence, the base change β' is injective, and so α is also injective, proving (i). Obviously, injectivity of α implies injectivity of γ, thus proving (ii). In (iii), we are given that the maps $\psi_{M/L,m/l}$ and $\psi_{L/K,l/k}$ are surjective, hence their composition $\psi_{M/K,m/k}$ is also surjective. Also, since β and γ are bijective, so are β' and α in this case. In particular, f is proper. In (iv), we deduce bijectivity of β' from bijectivity of α and γ. But $\psi_{M/L,m/l}$ is surjective by properness of h, so bijectivity of β' implies bijectivity of β. Thus, g is proper. It remains to establish (v) and (vi). In either case the map $\psi_{M/L,m/l}$ is surjective, so g is separated if and only if β' is injective. Thus, (v) is the obvious claim that γ is bijective when α is bijective and β' is injective, and (vi) is the obvious claim that β' is injective when α is injective and γ is bijective. ∎

We conclude this section with a brief discussion of H-strictness of birational spaces. Let $\mathcal{X}$ correspond to $X \to \mathbf{P}_{K/k}$ and let $H \supseteq \rho(k^\times)$ be a subgroup of G. Then $\mathcal{X}$ is called *H-strict* if it admits a proper morphism to a birational space $\mathcal{X}_H = (X_H \to \mathbf{P}_{K_H/k})$, where K_H denotes the G-graded subfield $K_H := \oplus_{g \in H} K_g \subseteq K$ over k. Thus, a separated $\mathcal{X}$ is H-strict if and only if the corresponding open subset of $\mathbf{P}_{K/k}$ is the exact preimage of its image in $\mathbf{P}_{K_H/k}$. It is proved in [T2, 2.6, 2.7] that for any H-strict $\mathcal{X}$, the space $\mathcal{X}_H$ and the proper morphism $\mathcal{X} \to \mathcal{X}_H$ are unique up to unique isomorphism. A given $\mathcal{X}$ is H-strict if and only if it admits an *H-strict structure*, which is an open covering of $\mathcal{X}$ by H-strict separated subspaces whose pairwise intersections are also H-strict. (This corresponds to choosing an open covering of $\mathcal{X}_H$ and forming its preimage in $\mathcal{X}$.) By the uniqueness up to unique isomorphism, any two H-strict structures on $\mathcal{X}$ are equivalent in the sense that the pairwise intersections among their members are H-strict.

6. Descent for birational spaces

We study descent on G-graded birational spaces in this section, and later our results will be applied via the reduction functor to study descent on analytic spaces.

Theorem 6.1. *Let $\mathcal{Y} = (h : Y \to \mathbf{P}_{L/l})$ and $\mathcal{X} = (g : X \to \mathbf{P}_{K/k})$ be two birational spaces equipped with a proper morphism $f : \mathcal{Y} \to \mathcal{X}$ and let $\mathbf{P}$ be any of the following properties of objects in* bir*: separated, affine, separated and H-strict for $\rho(l^\times) \subseteq H \subseteq G$. Then $\mathcal{Y}$ satisfies $\mathbf{P}$ if and only if $\mathcal{X}$ satisfies $\mathbf{P}$.*

It seems probable that H-strictness descends without the separatedness assumption, but it is not clear how to attack this problem. The main difficulty is the lack of a useful general criterion for a birational space $\mathcal{X} = (X \to \mathbf{P}_{K/k})$ to fail to be H-strict. (A separated $\mathcal{X}$ fails to be H-strict precisely when the open subset $X \subseteq \mathbf{P}_{K/k}$ is not a union of fibers over $\mathbf{P}_{K_H/k}$.)

Proof. Only the descent implications (from $\mathcal{Y}$ to $\mathcal{X}$) require a proof. Descent of separatedness is purely set-theoretic: given that $Y = X \times_{\mathbf{P}_{K/k}} \mathbf{P}_{L/l}$, so h is the base change of g under the surjective map $\mathbf{P}_{L/l} \to \mathbf{P}_{K/k}$, clearly g is injective if (and only if) h is.

We switch now to descent of H-strictness in the separated case. Consider the commutative diagram

$$\begin{array}{ccc} \mathbf{P}_{L/l} & \longrightarrow & \mathbf{P}_{K/k} \\ \downarrow & & \downarrow \\ \mathbf{P}_{L_H/l} & \longrightarrow & \mathbf{P}_{K_H/k} \end{array}$$

in which the bottom side is surjective since the top and right sides are surjective. We claim that the natural map $\psi : \mathbf{P}_{L/l} \to \mathbf{P}_{K/k} \times_{\mathbf{P}_{K_H/k}} \mathbf{P}_{L_H/l}$ is surjective. Choose graded valuation rings $\mathcal{O}'$, $\mathcal{O}$, and $\mathcal{O}''$ corresponding to compatible points in $\mathbf{P}_{K/k}$, $\mathbf{P}_{K_H/k}$, and $\mathbf{P}_{L_H/l}$ respectively. The natural graded map $K \otimes_{K_H} L_H \to L$ is clearly injective (by consideration of graded parts). Moreover, the natural map $\mathcal{O}' \otimes_{\mathcal{O}} \mathcal{O}'' \to K \otimes_{K_H} L_H$ is injective because $K \otimes_{K_H} L_H = K \otimes_{\mathcal{O}} L_H$ and $\mathcal{O} \to \mathcal{O}''$ and $\mathcal{O} \to K$ are flat (Lemma 4.7). The corresponding tensor product of graded residue fields is nonzero, so by choosing a graded prime ideal of this latter tensor product we get a graded prime ideal P of $\mathcal{O}' \otimes_{\mathcal{O}} \mathcal{O}''$ that dominates $m_{\mathcal{O}'}$, $m_{\mathcal{O}}$, and $m_{\mathcal{O}''}$. The graded localization $(\mathcal{O}' \otimes_{\mathcal{O}} \mathcal{O}'')_P$ is a graded-local subring of L that contains l, so it is dominated by a graded valuation ring $R \in \mathbf{P}_{L/l}$. Clearly $\psi(R) = (\mathcal{O}', \mathcal{O}'')$, establishing the surjectivity of ψ.

Since $\mathcal{Y}$ is assumed to be separated, $\mathcal{X}$ is separated by descent of separatedness. Thus, we can identify $\mathcal{X}$ with an open subspace X in $\mathbf{P}_{K/k}$. If $\mathcal{X}$ is not H-strict then we can find two points $x \in X$ and $x' \in \mathbf{P}_{K/k} - X$ sitting over a point $x_H \in \mathbf{P}_{K_H/k}$. Choose $y_H \in \mathbf{P}_{L_H/l}$ over x_H, so by surjectivity of ψ we can find points $y, y' \in \mathbf{P}_{L/l}$ sitting over (y_H, x) and (y_H, x'), respectively. Then $y \in Y$ and $y' \notin Y$, but their images in $\mathbf{P}_{L_H/l}$ coincide. Thus, $\mathcal{Y}$ is not H-strict. This establishes descent of H-strictness in the separated case.

The deepest and most useful property is being affine, and dealing with it makes use of much of the preliminary work done in §4. Assume that $\mathcal{Y}$ is affine, so we can identify $\mathcal{Y}$ with an affine subset of $\mathbf{P}_{L/l}$, and we can identify $\mathcal{X}$ with a subset of $\mathbf{P}_{K/k}$ using the established descent of separatedness. The first step is to reduce to the case when the extensions K/k and L/l are finitely generated. Choose a finite open covering of the quasi-compact $\mathcal{X}$ by open affine subsets $\mathcal{X}_i = \mathbf{P}_{K/k}\{A_i\}$ with each A_i a finitely generated graded k-subalgebra of K. Let K_0 be the graded subfield of K generated by the A_i's. Clearly $\mathcal{X}$ is the preimage of an open set $\mathcal{X}_0 \subseteq \mathbf{P}_{K_0/k}$, so $\mathcal{X}$ is affine is $\mathcal{X}_0$ is affine. The natural morphism $\mathcal{X} \to \mathcal{X}_0$ is proper, so it induces a proper morphism $\mathcal{Y} \to \mathcal{X}_0$. We can therefore replace $\mathcal{X}$ with $\mathcal{X}_0$, achieving that K/k is finitely generated. At this stage we may and do rechoose the affine sets $\mathcal{X}_i$'s so that K is the graded fraction field of each A_i. (This is done by choosing a finite set of elements $t_1, \dots, t_n \in K^\times$ such that $K = \mathrm{Frac}_G(k[t_1, \dots, t_n])$ and adjoining to the A_i's various homogeneous elements $t_1^{\epsilon_1}, \dots, t_n^{\epsilon_n}$ with $\epsilon_j = \pm 1$.) Moreover, we can replace each A_i with its graded integral closure $\overline{A}_i$ in K because this procedure does not affect $\mathcal{X}_i$ and $\overline{A}_i$ is finitely generated over k by Lemma 4.6.

Similarly, $\mathcal{Y}$ is the preimage in $\mathbf{P}_{L/l}$ of an affine subset $\mathcal{Y}_0 \subseteq \mathbf{P}_{L_0/l}$ for a finitely generated subextension L_0/l, so by replacing L_0 with the composite L_0K (which is also finitely generated over l) we achieve that the morphism $\mathcal{Y} \to \mathcal{X}$ factors through $\mathcal{Y}_0$, and by Lemma 5.3(iv) the resulting morphism $\mathcal{Y}_0 \to \mathcal{X}$ is necessarily proper. Thus, we can assume that L/l is finitely generated as well, and then by finiteness of graded integral closures (Lemma 4.6) we have $\mathcal{Y} = \mathbf{P}_{L/l}\{B\}$ for an integrally closed finitely generated graded l-subalgebra B of L (but $\mathrm{Frac}_G(B)$ can be smaller than L).

Next, we choose any transcendence basis $\{T_j\}_{j \in J}$ of l over k, and let l_0 be the graded subfield of l generated by k and the T_j's. Since l is algebraic over l_0, the map $\psi_{L/L, l/l_0} : \mathbf{P}_{L/l} \to \mathbf{P}_{L/l_0}$ is a bijection by Theorem 5.1(iii). Moreover, if $B = l[b_1, \dots, b_m]$ with homogeneous $b_1, \dots, b_m \in L^\times$ then the image of $\mathcal{Y}$ in $\mathbf{P}_{L/l_0}$ is the affine set $\mathcal{Y}_0 = \mathbf{P}_{L/l_0}\{b_1, \dots, b_m\}$. We again get a natural proper morphism $\mathcal{Y}_0 \to \mathcal{X}$, so once again we can replace $\mathcal{Y}$ with $\mathcal{Y}_0$, this time achieving that l is purely transcendental over k.

Note that the T_j's are algebraically independent over K because of Theorem 5.1(i), so Corollary 4.10 applies to the A_i's and l, giving that the graded rings lA_i are integrally closed and have a common graded fraction field $\mathrm{Frac}_G(lK)$. If B_i denotes the integral closure of lA_i in L then $\cap B_i$ is the integral closure of $\cap lA_i$ in L by Corollary 4.4. Now we make a few observations: $\mathcal{Y}_i = \mathbf{P}_{L/l}\{B_i\}$ is the preimage of $\mathcal{X}_i$ in $\mathbf{P}_{L/l}$; $B = \cap B_i$ because B is an integrally closed graded l-subalgebra and $\mathcal{Y} = \mathbf{P}_{L/l}(B)$ is equal to $\cup\mathcal{Y}_i = \cup\mathbf{P}_{L/k}(B_i) \subseteq \mathbf{P}_{L/l}(\cap B_i)$ (so $B = \cap_{\mathcal{O}\in\mathcal{Y}}\mathcal{O} \subseteq \cap_{\mathcal{O}\in\mathcal{Y}_i}\mathcal{O} = B_i$ for all i, and the containment $B \subseteq \cap B_i$ is an equality due to integrality of $\cap B_i$ over B that follows from $\mathbf{P}_{L/l}(B)$ lying in $\mathbf{P}_{L/l}(\cap B_i)$); by Lemma 4.11, $\cap lA_i = lA$ for $A := \cap A_i$. Summarizing this, we obtain that B is integral over lA.

We have to be careful when working with $A = \cap A_i$: it could a priori happen (without taking $\mathcal{Y}$ into account) that A is not finitely generated over k (e.g., for ungraded k one can construct such an example using that there exists a k-variety X with $\mathrm{H}^0(X, \mathcal{O}_X)$ not a finitely generated k-algebra). However, since the graded l-subalgebra B in L is finitely generated, we can find a finitely generated graded k-subalgebra $A_0 \subseteq A$ such that the integral closure of A_0 in L contains B, hence coincides with it. Thus, $\mathcal{Y} = \mathbf{P}_{L/l}\{B\}$ is the preimage in $\mathbf{P}_{L/l}$ of $\mathbf{P}_{K/k}\{A_0\}$, so we must have $\mathcal{X} = \mathbf{P}_{K/k}\{A_0\}$ by the surjectivity of the map $\mathbf{P}_{L/l} \to \mathbf{P}_{K/k}$ (in particular A, which is the integral closure of A_0 in K, is finitely generated over k). ■

Corollary 6.2. *Let $f : Y' \to Y$ be a k-analytic morphism such that* $\mathrm{Int}(Y'/Y) \to Y$ *is surjective. The k-analytic space Y is good if Y' is good. The converse is true if f has no boundary.*

This corollary applies to any flat surjection.

Proof. The very definition of a morphism being without boundary includes the requirement that the fiber product $Y' \times_Y Z$ is good whenever Z is good, so in particular if Y is good and f is without boundary then it is a tautology that Y' is good. For the more interesting descent claim, we can assume that Y' is good. To prove that Y admits a k-affinoid neighborhood around an arbitrary $y \in Y$, first choose $y' \in \mathrm{Int}(Y'/Y)$ over y. By [T2, 5.2] the reduction morphism $\widetilde{Y'}_{y'} \to \widetilde{Y}_y$ is proper in $\mathrm{bir}_{\widetilde{k}}$, so by Theorem 6.1 the birational space $\widetilde{Y'}_{y'}$ is affine if and only if $\widetilde{Y}_y$ is so. But goodness for the germ (Y', y') is equivalent to affineness for the birational space $\widetilde{Y'}_{y'}$ by [T2, 5.1], and similarly for (Y, y) and $\widetilde{Y}_y$, so we are done. ■

7. H-strict analytic spaces

This section contains material which should have been given in [T2]. In particular, it is logically independent of §2–§6. From now on and until the end of the paper, we consider only $\mathbf{R}^\times_{>0}$-gradings and H denotes a subgroup of $\mathbf{R}^\times_{>0}$ that contains $|k^\times|$. We say that a k-affinoid algebra $\mathscr{A}$ is *H-strict* if the spectral radius of any its elements either vanishes or belongs to the group $\sqrt{H}$. This is equivalent to either of the following conditions: (i) there exists an admissible epimorphism $k\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \to \mathscr{A}$ with $r_1, \ldots, r_n \in H$; (ii) there exists an admissible epimorphism $k\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \to \mathscr{A}$ with $r_1, \ldots, r_n \in \sqrt{H}$. Obviously (i) implies (ii), and that (ii) implies H-strictness is well-known for strictly k-affinoid algebras (i.e., $H \subseteq \sqrt{|k^\times|}$). The general case is reduced to this one by making a ground field extension K/k so that $|K^\times|$ contains all r_i's. Conversely, H-strictness implies (ii) because any admissible epimorphism $k\{r^{-1}T\} \twoheadrightarrow \mathscr{A}$ with $T_i \mapsto a_i \in \mathscr{A}$ where a_i has spectral radius s_i factors through an admissible epimorphism $k\{s^{-1}T\} \twoheadrightarrow \mathscr{A}$. Finally, to see that (ii) implies (i), given (ii) with $r_i^N = h_i \in H$ and $T_i \mapsto a_i \in \mathscr{A}$ we get a finite admissible map $k\{h^{-1}X\} \to \mathscr{A}$ with $X_i \mapsto a_i^N$. Since proper affinoid maps are finite (admissible) maps, we then easily deduce (i) via integrality and properness considerations.

A k-affinoid space $X = \mathscr{M}(\mathscr{A})$ is called *H-strict* if the k-affinoid algebra is. Then for any point $x \in X$, H-strict affinoid neighborhoods of x form a basis of its neighborhoods provided that $H \neq 1$. Note that H-strictness is inherited by direct products and closed subspaces, so the intersection of finitely many H-strict affinoid domains in any separated k-analytic space is H-strict. Also, the argument from [Ber2, 1.2.2] shows that an affinoid space is H-strict if and only if it admits a covering by H-strict affinoids. Thus, the following definition makes sense: a separated k-analytic space is *H-strict* if it admits a covering (for the G-topology on k-analytic spaces) by H-strict affinoid domains. As in the affinoid case, a finite intersection of H-strict analytic domains in a separated H-strict k-analytic space is H-strict. In general (for possibly non-separated k-analytic spaces), H-strictness may not be preserved by intersections of separated H-strict k-analytic domains, so we are led to the following definition in case $H \neq 1$: by an *H-strict structure* on an arbitrary k-analytic space X we mean a net $\{X_i\}$ of compact separated H-strict k-analytic domains. (The stronger condition that X_i's are k-affinoid leads to an equivalent definition.) We say that two H-strict structures $\{X_i\}_{i\in I}$ and $\{X'_j\}_{j\in J}$ are *equivalent* if their union is an H-strict structure. This condition is equivalent to all intersections $X_i \cap X'_j$ (which are separated but possibly non-compact k-analytic domains in X) being H-strict. This really is an equivalence relation: if $\{X''_l\}_{l\in L}$ is a third H-strict structure on X with each $X'_j \cap X''_l$ also H-strict then for each pair (i,l) the separated k-analytic space $X_i \cap X''_l$ is covered by the H-strict overlaps $X_i \cap X'_j \cap X''_l = (X_i \cap X'_j) \cap (X'_j \cap X''_l)$ in the H-strict spaces X'_j for varying j.

Remark 7.1. Let $H \subseteq \mathbf{R}^\times_{>0}$ be a non-trivial subgroup containing $|k^\times|$.

(i) The notion of H-strictness depends only on the group $\sqrt{H}$.

(ii) If $H \subseteq \sqrt{|k^\times|}$ then H-strictness is the usual k-analytic strictness.

(iii) Berkovich defined in [Ber2, §1.2] a general notion of Φ-analytic space, where Φ is a (suitable) family of k-affinoid spaces. His definition was mainly motivated by the case of strictly k-analytic spaces, but one checks immediately that, more generally, the class Φ_H of all H-strict k-affinoid spaces satisfies the conditions (1)–(5) of loc.cit., and the corresponding Φ_H-analytic spaces are exactly the k-analytic spaces with an H-strict structure.

(iv) We excluded the case $H = 1$ (which can only happen for trivially-valued k) because 1-strict affinoids do not satisfy the density condition from [Ber2, §1], so they do not form a net in the sense of [Ber2, §1.1]. (Briefly, the trivial group $\sqrt{H}$ is too small to provide a sufficiently large collection of positive real numbers in the definition of H-strict k-analytic subdomains.) However, one can weaken our definition by removing the density condition in the definition of a net. The resulting definition of H-strictness then makes sense and becomes the usual notion of strict k-analyticity from [Ber2, 1.2] in the case of a trivially-valued field k.

(v) One can, more generally, define H-strictness for any submonoid $H \subseteq \mathbf{R}^\times_{>0}$ containing $|k^\times|$. We do not study this case because some basic results are proved in [T2] only when H is a group. However, it seems very probable that every result stated for a group H (resp. a non-trivial group H) holds true for submonoids (resp. submonoids with an element $r < 1$).

Theorem 7.2. *Let $H \subseteq \mathbf{R}^\times_{>0}$ be a non-trivial subgroup containing $|k^\times|$. If a k-analytic space X admits an H-strict structure then the intersection of any two separated H-strict k-analytic domains* (*not assumed to be compatible with the structure on X*) *is H-strict. In particular, all H-strict structures on X are equivalent and the maximal such structure consists of all compact separated H-strict k-analytic domains.*

Using Remark 7.1(iv), this theorem is true for $H = 1$; see Remark 7.4.

Proof. Given separated H-strict k-analytic domains U and V in X, we have to prove that $W = U \cap V$ is H-strict. We claim that for any H-strict separated space Y and point $y \in Y$, the graded reduction $\widetilde{Y}_y$ is H-strict. The question reduces to the case when Y is affinoid as follows. Find a finite covering of a neighborhood of y by H-strict affinoid domains Y_i. If the reduction of each Y_i at y is H-strict then they provide an H-strict covering of the separated graded birational space $\widetilde{Y}_y$, which, therefore, is itself H-strict. Next we assume that the H-strict space Y is affinoid, say $Y = \mathscr{M}(\mathscr{A})$, so there exists an admissible epimorphism $k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to \mathscr{A}$ with $r_i \in H$. We have that $\widetilde{Y}_y = \mathbf{P}_{\widetilde{\mathscr{H}(y)}/\widetilde{k}}\{\widetilde{f}_1, \dots, \widetilde{f}_n\}$ by [T2, §4], where $\widetilde{f}_i$ is the image of f_i in $\widetilde{\mathscr{H}(y)}$ in degree r_i, so for each i either $\widetilde{f}_i$ vanishes or $\rho(\widetilde{f}_i) = r_i \in H$. We conclude that $\widetilde{Y}_y$ is H-strict as stated.

Thus, we proved that $\widetilde{U}_x$ and $\widetilde{V}_x$ are H-strict, and, moreover, $\widetilde{X}_x$ is H-strict because the reduction at x of the H-strict structure $\{X_j\}_{j\in J}$ on X induces an H-strict structure $\{\widetilde{(X_j)_x}\}_{j\in J}$ on $\widetilde{X}_x$. In particular, $\widetilde{W}_x = \widetilde{U}_x \cap \widetilde{V}_x$ is H-strict by [T2, 2.7], so the "if" implication in the following lemma concludes the proof. ■

Lemma 7.3. *Let $H \subseteq \mathbf{R}^\times_{>0}$ be a non-trivial subgroup containing $|k^\times|$. A point x in a separated k-analytic space W has an H-strict neighborhood if and only if the reduction $\widetilde{W}_x$ is an H-strict birational space.*

Remark 7.4. The lemma is false if H is trivial, but the following weaker version, which suffices for the proof of Theorem 7.2 for trivial H (using the same argument as above), still holds: if U is separated and H-strict and $x \in U$ is a point, then any H-strict open birational subspace $\widetilde{W} \subseteq \widetilde{U}_x$ can be obtained as the reduction at x of an H-strict k-analytic domain $W \subseteq U$ containing x. This is proved by adapting the method of reduction to affine/affinoid spaces in the argument below, and giving a direct analysis of the case $H = 1$.

Proof. We proved above that if the separated W is H-strict in a neighborhood of x then $\widetilde{W}_x$ is H-strict. Let us now assume that $\widetilde{W}_x$ is H-strict. Since W is separated, it suffices to find a covering of (W,x) by H-strict k-analytic subdomains (W_i,x). Find H-strict affine subspaces $\widetilde{W}_1,\dots,\widetilde{W}_n \subseteq \widetilde{W}_x$ that cover $\widetilde{W}_x$, so by [T2, 4.5, 5.1] there exist good subdomains $(W_i,x) \subseteq (W,x)$ that cover (W,x) and lift the $\widetilde{W}_i$'s. It suffices to prove that each (W_i,x) is H-strict, so we are reduced to the particular case of k-affinoid $W = \mathscr{M}(\mathscr{A})$. Fix an admissible epimorphism $k\{r_1^{-1}T_1,\dots,r_n^{-1}T_n\} \to \mathscr{A}$ satisfying $T_i \mapsto f_i \in \mathscr{A}$, and without loss of generality assume that $r_i \in \sqrt{H}$ if and only if $i > m$ for some m. Then $\widetilde{W}_x = \mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}\{\widetilde{f}_1,\dots,\widetilde{f}_n\}$, where $\widetilde{f}_i \in \widetilde{\mathscr{H}(x)}$ is the image of f_i in degree r_i, so each $\widetilde{f}_i$ either vanishes or satisfies $\rho(\widetilde{f}_i) = r_i \in \sqrt{H}$ by H-strictness of $\widetilde{W}_x$. (The reason some $r_i^n \in H$ with $n > 0$ if $\widetilde{f}_i \neq 0$ is that the H-strictness implies that such an $\widetilde{f}_i$ is integral over the H-graded field $\widetilde{\mathscr{H}(x)}_H$, and if n is the degree of the minimal homogeneous polynomial for $\widetilde{f}_i$ over $\widetilde{\mathscr{H}(x)}_H$ then the constant term of this polynomial is nonzero with grading r_i^n.) It follows that $\widetilde{f}_i = 0$ for $1 \le i \le m$; i.e. $r_i > |f_i(x)|$ for those i. Since $H \neq 1$, $\sqrt{H}$ is dense in $\mathbf{R}^\times_{>0}$. Thus, for each $1 \le i \le m$ there exist $s_i \in \sqrt{H}$ such that $r_i > s_i > |f_i(x)|$. Then $W' = W\{s_1^{-1}f_1,\dots,s_m^{-1}f_m\}$ is an H-strict affinoid neighborhood of x, as required. ■

Theorem 7.2 excludes any ambiguity from the following definition for any (possibly trivial) subgroup $H \subseteq \mathbf{R}^\times_{>0}$ containing $|k^\times|$: a k-analytic space is *H-strict* if it admits an H-strict structure. If $H \neq 1$ then a germ (X,x) is called *H-strict* if x admits an H-strict neighborhood in X. (The latter definition makes no sense for trivial H since in that case the existence of such a neighborhood does not imply the existence of a base of such neighborhoods, so the concept is not intrinsic to the germ (X,x).) Now, since the notion of an H-strict germ is defined, we can generalize the above lemma as follows.

Theorem 7.5. *For a non-trivial subgroup $H \subseteq \mathbf{R}^\times_{>0}$ containing $|k^\times|$, a germ (X,x) is H-strict if and only if its reduction $\widetilde{X}_x$ is H-strict.*

The theorem is of local nature, so it does not make sense for trivial H.

Proof. We saw in the proof of Theorem 7.2 that an H-strict germ has H-strict reduction, so now assume that $\widetilde{X}_x$ is H-strict. Find a finite covering of (X,x) by separated germs (X_i,x) such that their reductions $\widetilde{X}_i \subseteq \widetilde{X}_x$ are H-strict (one finds such a covering of $\widetilde{X}_x$ and then lifts it to the germ). By [T2, 4.8] we can shrink each X_i to make them separated. Set $X_{ij} = X_i \cap X_j$, so each germ (X_{ij},x) has H-strict reduction $\widetilde{X}_i \cap \widetilde{X}_j$, and by Lemma 7.3 we can find an open $X'_{ij} \subseteq X_{ij}$ around x that is H-strict. Again using Lemma 7.3, we can shrink the X_i's once again so that all X_i's are H-strict and $X_i \cap X_j \subseteq X'_{ij}$ for any choice of i,j.

We claim that $X' = \cup X_i$ is an H-strict neighborhood of x. Only H-strictness needs a proof, and by definition it suffices to check that each $X_i \cap X_j$ is H-strict. Notice that by the construction $X_i \cup X'_{ij}$ is separated, so H-strictness is inherited by the intersection $X_i \cap X'_{ij}$. By similar reasoning, $X_i \cap X'_{ij} \cap X_j$ is H-strict, but we have chosen X_i's so that the latter intersection is just $X_i \cap X_j$. ■

Finally, for any (possibly trivial) subgroup $H \subseteq \mathbf{R}^\times_{>0}$ containing $|k^\times|$, one can define a suitable notion of morphism to make a category of H-strict k-analytic spaces similar to the category of strictly k-analytic spaces. One possibility for $H \neq 1$ is to apply Berkovich's definition of morphism of Φ-analytic spaces with $\Phi = \Phi_H$ being the class of all H-strict k-affinoid spaces. We prefer a more ad hoc equivalent definition (which has the merit of "working" for $H = 1$ as well): an *H-strict morphism* $Y \to X$ between H-strict k-analytic spaces is a k-analytic morphism for which the preimage of any H-strict k-analytic subdomain of X is H-strict in Y. Note that in case $H = 1$ this recovers the notion of a strictly k-analytic morphism for a trivially-valued field.

Theorem 7.6. *If $H \subseteq \mathbf{R}^\times_{>0}$ is a subgroup containing $|k^\times|$ then the subcategory of H-strict k-analytic spaces with H-strict analytic morphisms is full in the category of all k-analytic spaces.*

The particular case $H = |k^\times|$ (including the case $H = 1$) was proved in [T2, 4.10]. That proof applies verbatim to the more general situation in Theorem 7.6 as soon as one replaces strict analyticity (i.e., $|k^\times|$-strictness) with H-strictness. The special case $H = 1$ for good k-analytic spaces is part of GAGA over a trivially-valued field [Ber1, 3.5.1(v)].

As an application of Theorem 7.5 we can handle descent of H-strictness through morphisms with surjective interior (allowing that H may be trivial):

Theorem 7.7. *Let $f : X \to Y$ be a k-analytic morphism such that* $\mathrm{Int}(X/Y) \to Y$ *is surjective. Let $H \subseteq \mathbf{R}^\times_{>0}$ be a subgroup containing $|k^\times|$. If X is locally separated and H-strict* (*resp. if X is locally separated and strictly k-analytic*) *then so is Y. The converse is true if f has no boundary.*

This theorem applies to any flat surjection $f : X \to Y$.

Proof. First we assume that H is non-trivial (so Theorem 7.5 may be invoked). If f has no boundary then for each $x \in X$ the map $\widetilde{X}_x \to \widetilde{Y}_{f(x)}$ in $\mathrm{bir}_{\widetilde{k}}$ is proper by [T2, 5.2], so in particular it is separated. Hence, by [T2, 4.8(iii)] the map f is separated near x, so if Y is locally separated then so is X. If in addition Y is H-strict then $\widetilde{Y}_{f(x)}$ is H-strict by Theorem 7.5, yet $\widetilde{X}_x = \widetilde{Y}_{f(x)} \times_{\mathbf{P}_{\widetilde{\mathscr{H}(f(x))}/\widetilde{k}}} \mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$ (by properness) so $\widetilde{X}_x$ is H-strict too. Hence, by Theorem 7.5 we deduce that (X, x) is H-strict. The locally separated X admits a covering by separated open sets U_i, and we have shown that each point of each U_i admits an H-strict neighborhood. It follows that each U_i admits an H-strict structure, so by Theorem 7.2 these agree on overlaps to define an H-strict structure on X. Hence, X is H-strict and locally separated when Y is. The strictly k-analytic case is the special case $H = |k^\times|$ when k is not trivially-valued.

For the converse when $H \neq 1$, we assume that $\mathrm{Int}(X/Y)$ surjects onto Y and that X is locally separated and H-strict, and we wish to deduce the same two properties for Y. By the same gluing and uniqueness arguments with H-strict structures in the separated case, our problem is intrinsic to each germ (Y, y) for $y \in Y$. Pick $x \in \mathrm{Int}(X/Y)$ over such a y. Once again the reduction morphism $\widetilde{X}_x \to \widetilde{Y}_y$ is proper in $\mathrm{bir}_{\widetilde{k}}$ and $\widetilde{X}_x$ is H-strict and separated. The H-strictness and separatedness of the germ (Y, y) is equivalent to H-strictness and separatedness of each birational space $\widetilde{Y}_y$ (again using Theorem 7.5 for the H-strictness), and this pair of properties is inherited from $\widetilde{X}_x$ by Theorem 6.1. Once again, taking $H = |k^\times|$ settles the case of strict k-analyticity when k is not trivially-valued.

To handle the case $H = 1$ (so k is trivially-valued and H-strictness means strict k-analyticity), the above arguments permit us to restrict attention to the case when X and Y are separated and f is without boundary. To move the property of strict k-analyticity between Y and X, we can use the preceding arguments by replacing Theorem 7.5 with Remark 7.4. ■

8. Extension of the ground field: key lemma

We want to compare properties of a k-analytic space X and the K-analytic space $X_K = X \widehat{\otimes}_k K$ obtained from X by extending of the ground field. Usually, if X satisfies a property $\mathbf{P}$ then it is easy to see that X_K does so too, but the converse can be much more difficult. A natural approach is to argue by contradiction: assume that X does not satisfy $\mathbf{P}$, take a non-$\mathbf{P}$ point $x \in X$, and show that it lifts to a non-$\mathbf{P}$ point $x_K \in X_K$. The following example shows that one should be very careful with the choice of x_K.

Example 8.1. Consider the property of having a non-empty boundary relative to the ground field (i.e., $X \to \mathscr{M}(k)$ is not without boundary). Let $X = \mathscr{M}(k\{r^{-1}T\})$ be a closed disc of radius $r > 0$, let x be its maximal point (corresponding to the spectral norm on $k\{r^{-1}T\}$), and let $K = \mathscr{H}(x)$. The relative boundary $\partial(X/\mathscr{M}(k))$ consists of the single point x, and the relative boundary of $X_K = \mathscr{M}(K\{r^{-1}T\})$ over $\mathscr{M}(K)$ consists of a single point x_K lying over x. The fiber Z of X_K over x is isomorphic to $\mathscr{M}(K \widehat{\otimes}_k K)$, so it has many points in general. For example, if $r \notin \sqrt{|k^\times|}$ then $K = k\{r^{-1}T\}$ and Z is isomorphic to a closed disc over K, but if $r = 1$ then Z is large but not K-affinoid: as a subset of the closed unit K-disc it is "not defined over k". In both cases x_K is a point of the fiber over x that is "as generic as possible".

Here is another example of a fiber of the map $X_K \to X$.

Example 8.2. Let $X = \mathscr{M}(k\{r^{-1}T\})$ be a disc. Assume for simplicity that k is algebraically closed and let $x \in X$ a point of type 4 (see [Ber1, §1]) equal to the intersection of closed discs whose radii tend to some s from above. (The arithmetic of the field $\mathscr{H}(x)$ depends on whether or not $s \in \sqrt{|k^\times|}$.) Let $K = \mathscr{H}(x)$ over k. One can show that $\mathscr{H}(x)\widehat{\otimes}_k K \simeq K\{s^{-1}T\}$, so the fiber of X_K over x is a disc over K. Note that the graded residue fields $\widetilde{k}$ and $\widetilde{K} = \widetilde{\mathscr{H}(x)}$ are isomorphic, but (by inspection) the graded field $(\mathscr{H}(x)\widehat{\otimes}_k K)^\sim$ is not algebraic over $\widetilde{k}$.

The key to constructing "sufficiently generic" points in fibers of $X_K \to X$ is the following lemma.

Lemma 8.3. *Let l/k and K/k be two analytic extensions. Assume that $d = \operatorname{trdeg}_{\widetilde{k}}(\widetilde{l})$ is finite. There exists a point $x \in \mathscr{M}(l\widehat{\otimes}_k K)$ such that $\operatorname{trdeg}_{\widetilde{K}}(\widetilde{F}) = d$, where $\widetilde{F} \subseteq \widetilde{\mathscr{H}(x)}$ denotes the graded fraction field of $\widetilde{l}\widetilde{K} \subseteq \widetilde{\mathscr{H}(x)}$.*

The idea of the proof is to find an analytic subfield $m \subseteq l$ over k such that $\widetilde{l}/\widetilde{m}$ is algebraic but $\widetilde{m}$ can be described explicitly over $\widetilde{k}$ inside of $\widetilde{l}$. We will first choose a point $x' \in \mathscr{M}(m\widehat{\otimes}_k K)$. Similarly to Example 8.1, the choice is canonical and can be described explicitly. Then, similarly to Example 8.2, any lifting of x' to $x \in \mathscr{M}(l\widehat{\otimes}_k K)$ does the job.

Proof. Pick a transcendence basis $\widetilde{f}_1, \dots, \widetilde{f}_d \in \widetilde{l}^\times$ over $\widetilde{k}$, and let $f_i \in l^\times$ be a lifting of f_i. Let r be the d-tuple whose entries are the gradings $r_i = \rho(\widetilde{f}_i) > 0$ (i.e., $r_i = |f_i|$), so we get a natural embedding $\widetilde{k}[r^{-1}\widetilde{f}] \hookrightarrow \widetilde{l}$ which lifts to an isometric embedding $\phi : \mathscr{A} := k\{r^{-1}T\} \hookrightarrow l$ (pulling the absolute value on l back to the spectral norm on $\mathscr{A}$). The spectral norm on $\mathscr{A}$ is multiplicative: it corresponds to the maximal point y of the polydisc $Y = \mathscr{M}(\mathscr{A})$. Let $x' = y_K$ be the maximal point of the polydisc $Y_K = Y\widehat{\otimes}_k K \simeq \mathscr{M}(\mathscr{A}_K)$ where $\mathscr{A}_K = K\{r^{-1}T\}$. By [T2, 3.1(i)] we have $\widetilde{K}[r^{-1}T] \simeq \widetilde{\mathscr{A}_K}$, so for the graded fraction field $\widetilde{E}$ of $\widetilde{K}[r^{-1}T]$ the natural map $\widetilde{E} \to \widetilde{\mathscr{H}(y_K)}$ of graded fields is surjective and thus is an isomorphism.

Since ϕ preserves spectral norms, it factors through an isometric embedding $m := \mathscr{H}(y) \hookrightarrow l$. Hence

$$\mathscr{M}(l\widehat{\otimes}_k K) \simeq \mathscr{M}(m\widehat{\otimes}_k K)\widehat{\otimes}_m l,$$

and the map $\mathscr{M}(l\widehat{\otimes}_k K) \to \mathscr{M}(m\widehat{\otimes}_k K)$ is surjective. Note that y_K is a point of Y_K lying over y, so it is contained in $\mathscr{M}(m\widehat{\otimes}_k K)$ and we can therefore lift it to a point $x \in \mathscr{M}(l\widehat{\otimes}_k K)$. We claim that x and the corresponding graded field $\widetilde{F} = \operatorname{Frac}_G(\widetilde{l}\widetilde{K}) \subseteq \widetilde{\mathscr{H}(x)}$ are as required. Clearly $\widetilde{\mathscr{H}(x)}$ contains $\widetilde{\mathscr{H}(y_K)} \simeq \widetilde{E}$ and $\widetilde{E}$ is contained in $\operatorname{Frac}_G(\widetilde{l}\widetilde{K})$, so $\operatorname{trdeg}_{\widetilde{K}}(\widetilde{F}) \geq \operatorname{trdeg}_{\widetilde{K}}(\widetilde{E})$. But obviously $\operatorname{trdeg}_{\widetilde{K}}(\widetilde{E}) = d$ and $\operatorname{trdeg}_{\widetilde{K}}(\widetilde{F}) = \operatorname{trdeg}_{\widetilde{K}}(\widetilde{l}\widetilde{K}) \leq \operatorname{trdeg}_{\widetilde{k}}(\widetilde{l}) = d$, so $\operatorname{trdeg}_{\widetilde{K}}(\widetilde{F}) = d$. ■

Let X be a k-analytic space, $x \in X$ a point, and $x_K \in X_K$ a point over x. Let $l = \mathscr{H}(x)$ and $L = \mathscr{H}(x_K)$ for brevity, and use the induced embeddings $\widetilde{l} \hookrightarrow \widetilde{L}$ and $\widetilde{K} \hookrightarrow \widetilde{L}$ of $\mathbf{R}^\times_{>0}$-graded fields to identify $\widetilde{l}$ and $\widetilde{K}$ with graded subfields of $\widetilde{L}$ over $\widetilde{k}$. To measure how close x_K is to being "generic" in the fiber of X_K over $x \in X$ we will measure dependency between $\widetilde{l}$ and $\widetilde{K}$ in $\widetilde{L}$.

Corollary 8.4. *Let X be a k-analytic space with a point x, K/k an analytic field extension, and $Y = X\widehat{\otimes}_k K$. Then there exists a point $y \in Y$ over x such that any algebraically independent set in $\widetilde{\mathscr{H}(x)}^\times$ over $\widetilde{k}$ is algebraically independent over $\widetilde{K}$ when viewed in $\widetilde{\mathscr{H}(y)}^\times$.*

By Theorem 5.1(i), an equivalent formulation of the property of y is that the natural map $\mathbf{P}_{\widetilde{\mathscr{H}(y)}/\widetilde{K}} \to \mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$ is surjective. This surjectivity captures the idea that y is sufficiently generic over x with respect to the ground field extension K/k.

Proof. Using notation as in the discussion preceding the statement of the corollary, let $\widetilde{F} = \operatorname{Frac}_G(\widetilde{l}\widetilde{K}) \subseteq \widetilde{L}$. Since transcendence degree of extensions of graded fields is additive in towers, and for the property of algebraically independent sets in $\widetilde{\mathscr{H}(x)}^\times$ over $\widetilde{k}$ it suffices to work with a single transcendence basis (in the graded sense), we just need to find $y \in Y$ such that $\mathrm{trdeg}_{\widetilde{k}}(\widetilde{l})$ is equal to $\mathrm{trdeg}_{\widetilde{K}}(\widetilde{F})$. Such a y exists by Lemma 8.3. ■

Corollary 8.5. *Let X be a k-analytic space with a point x, K/k be an analytic extension, and $Y = X\widehat{\otimes}_k K$. Then there exists a point $y \in Y$ over x such that the induced morphism $\widetilde{Y}_y \to \widetilde{X}_x$ is proper in* $\mathrm{bir}_{\mathbf{R}^{\times}_{>0}}$ *(in the sense of Definition* 5.2*).*

Proof. Take y as in the above corollary, so $\psi_{\widetilde{\mathscr{H}(y)}/\widetilde{\mathscr{H}(x)},\widetilde{K}/\widetilde{k}} : \mathbf{P}_{\widetilde{\mathscr{H}(y)}/\widetilde{K}} \to \mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$ is surjective (as required in the definition of properness in Definition 5.2). To prove that the reduction map $\widetilde{Y}_y \to \widetilde{X}_x$ is proper it remains to show that the map $\widetilde{Y}_y \to \mathbf{P}_{\widetilde{\mathscr{H}(y)}/\widetilde{K}}$ is the set-theoretic base change of $\widetilde{X}_x \to \mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$. But the latter was proved in [T2, 5.3] for any $y \in Y = X_K$ over any $x \in X$. ■

9. Descent with respect to extensions of the ground field

Theorem 9.1. *Let K/k be an analytic field extension and $H \subseteq \mathbf{R}^{\times}_{>0}$ any (possibly trivial) subgroup containing $|K^{\times}|$. A k-analytic space X is good (resp. H-strict and locally separated) if and only if the K-analytic space X_K is good (resp. H-strict and locally separated). In particular, if $|K^{\times}|/|k^{\times}|$ is a torsion group then X is strictly k-analytic and locally separated if and only if X_K is strictly K-analytic and locally separated.*

Proof. Let $\mathbf{P}$ be the property of being good (resp. H-strict and locally separated) and $\widetilde{\mathbf{P}}$ be the property of an $\mathbf{R}^{\times}_{>0}$-graded birational space being affine (resp. H-strict and separated). If X satisfies $\mathbf{P}$ then X_K obviously satisfies $\mathbf{P}$ too. Conversely, assume that X does not satisfy $\mathbf{P}$ locally at a point x. By Corollary 8.5, we can find a preimage $y \in Y := X_K$ of x so that $\widetilde{Y}_y \to \widetilde{X}_x$ is proper. By Theorem 6.1, $\widetilde{Y}_y$ satisfies $\widetilde{\mathbf{P}}$ if and only if $\widetilde{X}_x$ does. Also, by [T2, 4.8(iii), 5.1] for local separatedness and goodness and Theorem 7.5 (which requires $H \neq 1$) for H-strictness, X (resp. Y) satisfies $\mathbf{P}$ locally at x (resp. y) if and only if $\widetilde{X}_x$ (resp. $\widetilde{Y}_y$) satisfies $\widetilde{\mathbf{P}}$, at least if we require $H \neq 1$. Hence, assuming $H \neq 1$, X satisfies $\mathbf{P}$ at x if and only if Y satisfies $\mathbf{P}$ at y, and since we assumed that X is non-$\mathbf{P}$ at x we conclude that X_K is non-$\mathbf{P}$ at y, so X_K does not satisfy $\mathbf{P}$. Taking $H = |k^{\times}|$ settles the case of strict analyticity when k is not trivially-valued and $|K^{\times}|/|k^{\times}|$ is a torsion group.

It remains to show that if K is trivially-valued and X_K is strictly K-analytic and locally separated then X is strictly k-analytic and locally separated. The preceding argument with local separatedness shows that X is locally separated, so we can assume X is separated. We may then replace Theorem 7.5 with Remark 7.4 to carry over the above argument in the case of trivially-valued k and K. ■

Theorem 9.2. *Let K/k be an analytic field extension and let $h : X' \to X$ be a map of k-analytic spaces. Let $h_K : X'_K \to X_K$ be the induced K-analytic map. Each of the following properties holds for h if and only if it holds for h_K: without boundary, proper, surjective, finite, closed immersion, separated, locally separated, isomorphism, monomorphism, étale, open immersion, quasi-finite, flat, G-smooth, G-étale.*

Proof. In each case, the nontrivial implication is that the hypothesis on h_K implies the same for h. Descent for surjectivity is obvious. Next, assume that h_K is locally separated. To prove that h is locally separated we choose $x' \in X'$ and (by [T2, 4.8(iii)]) the problem is to prove that $\widetilde{X'}_{x'} \to \widetilde{X}_x$ is separated. For ease of notation, let $Y = X_K$ and $Y' = X'_K$. By Corollary 8.5 there is a point $y' \in Y'$ over x' such that $\widetilde{Y'}_{y'} \to \widetilde{X'}_{x'}$ is proper. Let $y = h_K(y') \in Y$, so by [T2, 5.3], $\widetilde{Y}_y \to \widetilde{X}_x$ is separated. (It is even proper.) The local separatedness of h_K gives that $\widetilde{Y'}_{y'} \to \widetilde{Y}_y$ is separated, so by Lemma 5.3(vi) we conclude that $\widetilde{X'}_{x'} \to \widetilde{X}_x$ is separated. Hence, h is separated near the arbitrary $x' \in X'$. The descent of flatness and G-smoothness is done similarly to Theorem 2.5 and using the fact that if X is good then for each $x_K \in X_K$ sitting over a point $x \in X$ the ring $\mathcal{O}_{X_K,x_K}$ is flat over $\mathcal{O}_{X,x}$ (for example use that if $\mathscr{A}$ is k-affinoid then $\mathscr{A}\widehat{\otimes}_k K$ is $\mathscr{A}$-flat).

Exactly as in the proof of Theorem 2.4, for the descent of the other properties it suffices to treat the property of being without boundary. Thus, assume that h_K is without boundary. To deduce the same for h, we assume to the contrary that there exists a point $x' \in X'$ not in $\mathrm{Int}(X'/X)$, so the reduction morphism

$\widetilde{X}'_{x'} \to \widetilde{X}_x$ is not proper by [T2, 5.2]. By Corollary 8.5, we can find a point $y' \in Y'$ over x' so that $\widetilde{Y}'_{y'} \to \widetilde{X}'_{x'}$ is proper (and moreover $\mathrm{trdeg}_{\widetilde{k}} \widetilde{\mathscr{H}(x')} = \mathrm{trdeg}_{\widetilde{K}}(\widetilde{K}\widetilde{\mathscr{H}(x')})$, where the composite graded field $\widetilde{K}\widetilde{\mathscr{H}(x)}$ is formed within $\widetilde{\mathscr{H}(y')}$). The composite map $\widetilde{Y}'_{y'} \to \widetilde{X}_x$ is therefore not proper, by Lemma 5.3(iv).

Let y be the image of y' in Y. The inequality $\mathrm{trdeg}_{\widetilde{k}} \widetilde{\mathscr{H}(x)} \geq \mathrm{trdeg}_{\widetilde{K}}(\widetilde{K}\widetilde{\mathscr{H}(x)})$ (with composite graded field formed within $\widetilde{\mathscr{H}(y)}$) must be an equality because if it is a strict inequality then additivity of graded transcendence degree and the inequality $\mathrm{trdeg}_{\widetilde{\mathscr{H}(x)}} \widetilde{\mathscr{H}(x')} \geq \mathrm{trdeg}_{\widetilde{K}\widetilde{\mathscr{H}(x)}}(\widetilde{K}\widetilde{\mathscr{H}(x')})$ would give $\mathrm{trdeg}_{\widetilde{k}} \widetilde{\mathscr{H}(x')} > \mathrm{trdeg}_{\widetilde{K}}(\widetilde{K}\widetilde{\mathscr{H}(x')})$, contrary to how y' was chosen. Hence, as in the proof of Corollary 8.5 we deduce via [T2, 5.3] that the map $\widetilde{Y}_y \to \widetilde{X}_x$ is proper. Since the morphism $h_K : Y' \to Y$ is without boundary by our assumption, the reduction morphism $\widetilde{Y}'_{y'} \to \widetilde{Y}_y$ is proper. Therefore, the composition $\widetilde{Y}'_{y'} \to \widetilde{Y}_y \to \widetilde{X}_x$ is proper by Lemma 5.3(iii), yet above we saw that this is not proper. The contradiction shows that h has to be without boundary. ∎

As an application of Theorem 9.2, we can use the rigid-analytic theory of ampleness [C] to set up a parallel theory in the k-analytic case (without imposing goodness requirements). We begin with a definition, in which $\mathbf{P}(V)$ for a finite-dimensional k-vector space V is the k-analytic space associated to the algebraic projective space $\mathrm{Proj}(\mathrm{Sym}(V))$; it represents the functor of invertible sheaves $\mathscr{L}$ for the G-topology (on a varying k-analytic space X) equipped with a surjection $V \otimes_k \mathscr{O}_{X_G} \to \mathscr{L}$.

Definition 9.3. An invertible sheaf $\mathscr{L}$ for the G-topology X_G on a proper k-analytic space X is *ample* if there exists an $n > 0$ such that the map $\Gamma(X_G, \mathscr{L}^{\otimes n}) \otimes_k \mathscr{O}_{X_G} \to \mathscr{L}^{\otimes n}$ of coherent $\mathscr{O}_{X_G}$-modules is surjective and the resulting morphism $X \to \mathbf{P}(\Gamma(X_G, \mathscr{L}^{\otimes n}))$ is a closed immersion.

If $f : X \to S$ is a proper map of k-analytic spaces then an invertible $\mathscr{O}_{X_G}$-module $\mathscr{L}$ is *relatively ample* with respect to f if $\mathscr{L}_s = \mathscr{L}|_{X_s}$ is ample on the fibral $\mathscr{H}(s)$-analytic space X_s for every $s \in S$.

For a k-analytic space S and a coherent $\mathscr{O}_{S_G}$-module $\mathscr{E}$, we will use the S-proper k-analytic space $\mathbf{P}(\mathscr{E})$ that classifies invertible $\mathscr{O}_{X_G}$-modules equipped with a surjection from $\mathscr{E} \otimes_{\mathscr{O}_{S_G}} \mathscr{O}_{X_G}$ (where X is a varying k-analytic space over S), exactly as for schemes. Via the universal property and gluing for the G-topology [Ber2, 1.3.3], to construct $\mathbf{P}(\mathscr{E})$ it suffices to do for k-affinoid S provided that it is compatible with k-affinoid base change. Relative analytification over affinoid algebras in the sense of [Ber2, 2.6.1] provides such a construction over an affinoid base $\mathscr{M}(\mathscr{A})$ by using the corresponding algebraic construction over $\mathrm{Spec}(\mathscr{A})$. Via the universal property and the behavior of relative analytification with respect to closed immersions, the formation of $\mathbf{P}(\mathscr{E})$ commutes with any base change on S and surjections $\mathscr{E}' \to \mathscr{E}$ on S_G induce closed immersions $\mathbf{P}(\mathscr{E}) \hookrightarrow \mathbf{P}(\mathscr{E}')$ over S. In particular, this shows that $\mathbf{P}(\mathscr{E})$ admits a closed immersion into a standard projective space locally over S_G. Hence, $\mathbf{P}(\mathscr{E})$ is S-proper since this property is clear when S is k-affinoid, so it holds locally for the G-topology on S in general, and properness is local for this topology [T2, 5.6].

Corollary 9.4. *Let $f : X \to S$ be a proper map of k-analytic spaces and $\mathscr{L}$ be an invertible $\mathscr{O}_{X_G}$-module.*

(1) *The set $U_{\mathscr{L}}$ of $s \in S$ such that $\mathscr{L}_s$ is ample on the $\mathscr{H}(s)$-analytic space X_s is open and its formation commutes with k-analytic base change on S and with any analytic extension of the ground field.*
(2) *If $\mathscr{L}$ is relatively ample then locally on S there exists $n_0 > 0$ such that $f^*(f_*(\mathscr{L}^{\otimes n})) \to \mathscr{L}^{\otimes n}$ is surjective and the natural map $\iota_n : X \to \mathbf{P}(f_*(\mathscr{L}^{\otimes n}))$ is a closed immersion for all $n \geq n_0$.*

Proof. The crucial fact we have to show is that if $S = \mathscr{M}(k)$ and K/k is an analytic extension field then $\mathscr{L}$ is ample on X if and only if the associated coherent pullback $\mathscr{L}_K$ is ample on X_K. A ground field extension does not affect whether or not a map between coherent sheaves for the G-topology is surjective, and by Theorem 9.2 the property of a morphism being a closed immersion is likewise unaffected. Hence, the only problem is to show that for a coherent $\mathscr{O}_{X_G}$-module $\mathscr{F}$ (such as $\mathscr{L}^{\otimes n}$ for a fixed $n > 0$) the natural map $K \otimes_k \Gamma(X_G, \mathscr{F}) \to \Gamma((X_K)_G, \mathscr{F})$ is an isomorphism. More generally, we claim that $K \otimes_k \mathrm{H}^i(X_G, \mathscr{F}) \to \mathrm{H}^i((X_K)_G, \mathscr{F}_K)$ is an isomorphism for any $i \geq 0$. Observe that the ordinary tensor products here may be replaced with completed tensor products, since the cohomology is finite-dimensional. It suffices to prove that the maps in the Čech complex associated to a finite affinoid covering of a proper analytic space and a coherent sheaf for the G-topology are admissible (in the sense of having closed images whose subspace and quotient topologies coincide). This property is unaffected by a ground field extension (using completed tensor products), so by the relationship between strictly analytic spaces and rigid spaces [Ber2, 1.6.1] we may pass to the strictly analytic case and hence to coherent sheaves on proper rigid spaces (the equivalence with rigid-analytic properness is [T2, 4.5]). In this case the desired property of the Čech complex was proved by Kiehl [K, 2.5ff.] in his proof of coherence of higher direct images.

Now we prove the first part of the corollary. It follows from the invariance under a ground field extension that the formation of the set $U_{\mathscr{L}}$ is compatible with a ground field extension K/k in the sense that $\pi^{-1}(U_{\mathscr{L}}) = U_{\mathscr{L}_K}$ where $\pi : S_K \to S$ is the canonical map. Since π is also topologically a quotient map (it is even a compact surjection), it therefore suffices to solve the problem after a ground field extension. The formation of $U_{\mathscr{L}} \subseteq S$ is certainly local for the G-topology on S, so by using a compact k-analytic neighborhood of an arbitrary point $s \in S$ we see that it suffices to treat the case when S is compact (so X is compact). Hence, by using a ground field extension we can assume that $|k^{\times}| \neq \{1\}$ and X and S are strictly k-analytic. In this case there is a proper map of quasi-compact and quasi-separated rigid spaces $f_0 : X_0 \to S_0$ corresponding to f and an invertible sheaf $\mathscr{L}_0$ on X_0 corresponding to $\mathscr{L}$. By [C, 3.2.9], there is a subset $U_{\mathscr{L}_0} \subseteq S_0$ that is a Zariski-open subset in a canonical Zariski-open subset $W_{\mathscr{L}_0} \subseteq S$ such that the points of $U_{\mathscr{L}_0}$ are exactly the $s \in S$ such that $\mathscr{L}_0$ has ample pullback to $(X_0)_s = (X_s)_0$ (in the sense of rigid geometry) and such that the formation of $W_{\mathscr{L}_0}$ and $U_{\mathscr{L}_0}$ is compatible with arbitrary ground field extension K/k. Since ampleness on a fiber is unaffected by passage between the rigid-analytic and k-analytic categories, it follows that if we let $W_{\mathscr{L}} \subseteq S$ be the Zariski-open subset corresponding to $W_{\mathscr{L}_0} \subseteq S_0$ then the Zariski-open subset of $W_{\mathscr{L}}$ corresponding to $U_{\mathscr{L}_0}$ is equal to $U_{\mathscr{L}}$. This establishes the openness of $U_{\mathscr{L}}$, and so finishes the proof of the first part.

To prove the second part we may again reduce to the case when S is compact. The formation of higher direct images with respect to f (using the G-topology) is compatible with any ground field extension, by essentially the same argument we used above for cohomology over a field: we may pass to the case of an affinoid base, and we use that Kiehl's results on Čech complexes are valid in the relative setting over an affinoid base (not just over a ground field as base). Thus, once again using Theorem 9.2 for the property of being a closed immersion, we may assume $|k^{\times}| \neq \{1\}$ and that S and X are strictly k-analytic. The analogue of our desired result was proved locally on S_0 in the rigid-analytic case in [C, 3.1.4, 3.2.4, 3.2.7]. (The ability to get the closed immersion property for all large n is shown in the proof of [C, 3.2.7].) Since S_0 is quasi-compact, we therefore get a single n_0 such that $\mathscr{L}_0^{\otimes n}$ is generated by $(f_0)_*(\mathscr{L}_0^{\otimes n}) = (f_*(\mathscr{L}^{\otimes n}))_0$ and the resulting map $(\iota_n)_0$ is a closed immersion for all $n \geq n_0$. Passing back to the k-analytic category gives the desired result over S. ■

An immediate consequence of the second part of the preceding corollary is that locally on S_G (or locally on S when S is good) a high power of a relatively ample line bundle is the pullback by $\mathscr{O}(1)$ relative to a closed immersion into a standard projective space over the base. In particular, locally on S a sufficiently high power of a relatively ample line bundle satisfies the familiar cohomological vanishing and generation properties for higher direct images against a fixed coherent sheaf on X_G. In the rigid-analytic case this is [C, 3.2.4], but note that the present approach does not reprove this result in the rigid-analytic case since that result is a crucial part of the rigid-analytic ingredients used in the proof of Corollary 9.4.

Appendix A. Flatness in k-analytic geometry

A.1. **Flatness.** In this appendix we recall some facts from a theory of flatness for k-analytic maps that was developed very recently by A. Ducros in [Duc]. Note also that for quasi-finite morphisms this theory was developed by Berkovich already in [Ber2], and there was an unpublished work by Berkovich where some results about flat morphisms between good spaces were obtained, including the theorem that boundaryless flat morphisms are preserved under base changes.

Let $f : X \to Y$ be a morphism between k-analytic spaces, $\mathscr{F}$ be a coherent $\mathcal{O}_X$-module, $x \in X$ be a point with $y = f(x)$. If f is good then we say that $\mathscr{F}$ (resp. f) is *naively Y-flat* (Ducros says Y-flat) at

x if $\mathscr{F}_x$ (resp. $\mathcal{O}_{X,x}$) is a flat $\mathcal{O}_{Y,y}$-module. We say that $\mathscr{F}$ or f is *naively Y-flat* if it is so at all points of X. Unfortunately, this definition does not make too much sense in general since the naive flatness can be destroyed after a base change $Y' \to Y$ with a good source Y', and, even worse, one can built such examples with $Y' \to Y$ being an embedding of an analytic subdomain, see [Duc, 2.18]. For this reason, the only reasonable notion is that of universal naive flatness which we will simply call flatness: we say that an f as above is flat (or universally flat in [Duc]) if each its good base change is naively flat (and similarly for coherent sheaves). By [Duc, 2.9] flatness is of G-local nature, namely $\mathscr{F}$ is Y-flat at x if and only if there exists a pair of good analytic domains $U \hookrightarrow X$ and $V \hookrightarrow Y$ with $x \in U$ and $f(U) \subset V$ and such that $\mathscr{F}|_U$ is naively V-flat at x, and then $\mathscr{F}|_{U'}$ is naively V'-flat for any choice of such a pair U', V'. Because of G-locality flatness globalizes to all morphisms of k-analytic spaces: given a morphism $f : X \to Y$ a coherent $\mathcal{O}_{Y_G}$-module $\mathscr{F}_G$ is Y-flat at a point $x \in X$ if there exist good domains $x \in U \hookrightarrow X$ and $f(U) \subset V \hookrightarrow Y$ such that $(\mathscr{F}_G)|_U$ is V-flat at x. An important difficult theorem [Duc, 3.12.3] by Ducros asserts that for extendable (in particular, for boundaryless) morphisms between good spaces flatness is equivalent to the naive one, and hence can be checked on stalks. Recall that a morphism $X \to Y$ is called *extendable* at a point $x \in X$ if it factors into a composition of embedding of an analytic domain $X \hookrightarrow X'$ and a morphism $X' \to Y$ with $x \in \mathrm{Int}(X'/Y)$. We list below few basic properties of flatness proved in [Duc].

Lemma A.2. *(i) Flatness is preserved under base changes, ground field extensions and compositions.*

(ii) Flatness of a morphism $f : X \to Y$ is G-local on X and Y, and, moreover, if $f|_U$ is flat for an analytic subdomain $U \hookrightarrow X$ then each point $x \in U$ possesses a neighborhood U_x in X with flat $f|_{U_x}$.

Note that it is the second property that earlier allowed us to define flatness at a point for morphisms between not good spaces. In principle, it is not automatic that a G-locally defined property makes sense in the usual topology on an analytic space.

A.3. ***G*-smoothness.** Smooth morphisms were introduced by Berkovich in [Ber2, §3.5]. They are boundaryless morphisms defined as compositions of étale morphisms and projections $\mathbf{A}^n_X \to X$. This choice of the definition resulted from the fact that flatness was defined in loc.cit. only for quasi-finite morphisms and that sufficed only for introducing étale morphisms. A partial G-localization of smoothness is the notion of quasi-smooth morphisms defined as compositions of smooth morphisms and embeddings of analytic subdomains. The latter notion is however restricted to extendable morphisms. Having general flatness now available we can simply make the following definition: a morphism $f : X \to Y$ of k-analytic spaces is *G-smooth* if it is flat and the sheaf $\Omega^1_{Y_G/X_G}$ defined in [Ber2, 3.3] is a locally free sheaf whose dimension at a point $x \in X$ equals to the dimension of f at x. Similarly, we say that f is *G-étale* if it is flat and unramified (i.e. $\Omega^1_{Y_G/X_G} = 0$).

Lemma A.4. *(i) G-smoothness and G-étaleness are preserved under base changes, ground field extensions and compositions.*

(ii) G-smoothness (resp. G-étaleness) of a morphism $f : X \to Y$ is G-local on X and Y, and, moreover, if $f|_U$ is G-smooth (resp. G-étale) for an analytic subdomain $U \hookrightarrow X$ then each point $x \in U$ possesses a neighborhood U_x in X with G-smooth (resp. G-étale) $f|_{U_x}$.

The claim (ii) allows to define G-smoothness (resp. G-étaleness) at a point x (similarly to flatness at a point).

Proof. (i) follows from A.2(i) and basic properties of the sheaves $\Omega^1_{Y_G/X_G}$ established in Propositions 3.3.2 and 3.3.3 from [Ber2]. To prove (ii) we will need the following claim about a coherent $\mathcal{O}_{X_G}$-sheaf $\mathscr{F}_G$ which follows easily from the fact that any embedding of a subdomain $\mathscr{M}(\mathscr{A}) \hookrightarrow \mathscr{M}(\mathscr{B})$ corresponds to a flat homomorphism $\mathscr{B} \to \mathscr{A}$: if $\mathscr{F}_G$ is locally free (resp. vanishes) on a subdomain $U \hookrightarrow X$, then each point $x \in U$ possesses a neighborhood U_x in X such that $\mathscr{F}_G$ is locally free on U_x (resp. vanishes). (We use here coherence of $\mathscr{F}_G$ in an essential way. Note, that even the analogous vanishing property fails for locally constant étale sheaves, for example.) Now, (ii) follows from A.2(ii) and the above claim applied to $\mathscr{F}_G = \Omega^1_{Y_G/X_G}$. ■

Since we do not need that in this paper, we postpone a study of connections between G-smoothness, quasi-smoothness and smoothness, and between their étale analogs until another paper. Here we only state without proofs what we know about the relations and what we expect from them.

(i) Obviously, a morphism is étale if and only if it is G-étale and boundaryless.

(ii) One can show that G-étale morphisms are extendable. In particular, it follows that a morphism is quasi-étale if and only if it is G-étale and extendable.

(iii) We expect that if Y is good, then the analogs of (i) and (ii) hold for smoothness. Namely, $f : X \to Y$ is smooth if and only if it is G-smooth and boundaryless, and f is quasi-smooth if and only if it is G-smooth and extendable.

(iv) If (iii) holds then G-smooth boundaryless morphisms form a smallest class of morphisms which includes smooth morphisms and is G-local on the base, and we expect that the class of smooth morphisms is not G-local on the base.

(v) We expect that there are non-extendable G-smooth morphisms.

Conjecture (iv) would imply that the class of smooth morphisms between non-good spaces is not a reasonable class, and it is more natural to consider the family of all G-smooth boundaryless morphisms instead. This explains, in particular, why we studied descent of G-smoothness but skipped descent of smoothness as it is defined in [Ber2].

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