ERRATUM TO MODULAR CURVES AND RAMANUJAN'S CONTINUED FRACTION

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In [1, Lemma D.3], we made the following claim:

Lemma 1. *Let a and k be integers with $k > 0$, and let p be a prime that divides k. Assume $p \nmid a$, and let ζ be a primitive kth root of unity in $\mathbf{C}$. Define*

$$c_k(m) = \sum_{\substack{h \in (\mathbf{Z}/k\mathbf{Z})^\times \\ h \equiv a \bmod p}} \zeta^{hm}.$$

Then

$$c_k(m) = \begin{cases} (\mu(k/(m,k))\varphi(k))/((p-1)\varphi(k/(m,k))) & \text{if } \operatorname{ord}_p(k) \leq \operatorname{ord}_p(m), \\ 0 & \text{otherwise.} \end{cases}$$

This lemma is already false for $k = p$ quite generally, and our error in the proof of the lemma occurs in the line beginning "Evidently…" (the formula given for $(k, m + jk/p)$ in that line is wrong for $j \not\equiv 0 \bmod p$). A corrected version of the lemma, sufficient for our single application of Lemma D.3 in [1] is:

Lemma 2. *With the notation of Lemma 1, we have*

$$|c_k(m)| \leq p \cdot (k, m).$$

Proof. Since $\zeta^{k/p}$ is a primitive pth root of unity, we may write

$$\begin{aligned} c_k(m) &= \frac{1}{p} \sum_{\zeta_0^p = 1} \sum_{h \in (\mathbf{Z}/k\mathbf{Z})^\times} \zeta_0^{-(a-h)} \zeta^{mh} \\ &= \frac{1}{p} \sum_{j \in \mathbf{Z}/p\mathbf{Z}} \zeta^{-kja/p} \sum_{h \in (\mathbf{Z}/k\mathbf{Z})^\times} \zeta^{(m+jk/p)h}. \end{aligned}$$

We can remove the a in the first exponent since $p \nmid a$. Now the standard evaluation

$$\sum_{h \in (\mathbf{Z}/k\mathbf{Z})^\times} \zeta^{\ell h} = \mu(k/(k,\ell)) \cdot \frac{\varphi(k)}{\varphi(k/(k,\ell))}$$

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for $\ell \in \mathbf{Z}$, together with the bound

$$\varphi(ab) \leq a\varphi(b)$$

for all positive integers a, b yields the estimate

$$\left| \sum_{h \in (\mathbf{Z}/k\mathbf{Z})^\times} \zeta^{(m+jk/p)h} \right| \leq \frac{\varphi(k)}{\varphi(k/(k, m+jk/p))} \leq (k, m+jk/p).$$

Obviously,

$$(k, m+jk/p) = (k, m+jk/p, pm) \leq (k, pm) \leq p \cdot (k, m);$$

the Lemma follows. □

The only application of Lemma 1 in [1] is in the proof of Lemma 7.7, directly above and below (7.19), with $p = 5$. In the application we may absorb the additional factor of 5 into the O-constants, so Lemma 7.7 remains valid as stated.

References

[1] Cais, Bryden and Conrad, Brian. *Modular curves and Ramanujan's continued fraction*, J. Reine Angew. Math. **597** (2006), 27–104.

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