ARITHMETIC PROPERTIES OF THE SHIMURA-SHINTANI-WALDSPURGER CORRESPONDENCE

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with an appendix by BRIAN CONRAD [1]

Abstract

We prove that the theta correspondence for the dual pair $(\widetilde{\mathrm{SL}_2}, PB^\times)$, for B an indefinite quaternion algebra over $\mathbb{Q}$, acting on modular forms of odd square-free level, preserves rationality and p-integrality in both directions. As a consequence, we deduce the rationality of certain period ratios of modular forms and even p-integrality of these ratios under the assumption that p does not divide a certain L-value. The rationality is applied to give a direct construction of isogenies between new quotients of Jacobians of Shimura curves, completely independent of Faltings isogeny theorem.

Contents

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1. Introduction

In his seminal paper [26], Shimura initiated the systematic study of holomorphic modular forms of half-integral weight and showed that one could associate to a Hecke eigenform h of half-integral weight $k+\frac{1}{2}$ a Hecke eigenform f of integral weight $2k$ such that the p^2th Fourier coefficient of h is closely related to the pth Fourier coefficient of f. The correspondence which associates f to h is often described as the Shimura correspondence, and f is called the Shimura lift of h. Later, Shintani [33] described a method to go in the other direction, namely construct modular forms of half-integral weight beginning with forms of integral weight using the theta correspondence. At around the same time, Niwa [21] also explained the original Shimura lift in terms of theta lifts. (In the case of Maass forms, there is a much earlier construction due to Maass [20] of the lift to forms of half-integral weight; see [8] for an exposition.)

The relation between f and the square-free Fourier coefficients $a_\nu(h)$ of h remained highly mysterious, but for a suggestion of Shimura ([31]) that these should somehow be related to special values of L-functions associated to f. In two remarkable articles ([36], [37]) Waldspurger settled this question, showing (roughly) that $a_\nu(h)^2$ is proportional (as ν varies) to the value $L(k, f \otimes \chi_\nu)$ where χ_ν is the quadratic character associated to the field $\mathbb{Q}(\sqrt{\nu})$. A central tool that Waldspurger employs is the theta correspondence between the groups $\widetilde{\mathrm{SL}_2}$ and PGL_2 as in the work of Shintani and Niwa. In a later article ([38]), Waldspurger also studied the theta correspondence for the pair $(\widetilde{\mathrm{SL}_2}, PB^\times)$ for B a quaternion algebra, and its relation to the Jacquet-Langlands correspondence between PGL_2 and $PB^\times$.

Waldspurger's results are representation-theoretic in nature. In particular, he does not study the arithmetic properties of the theta-lifts in either direction. This issue was however considered by Shimura [32], who showed that (for suitable choices of theta function) the theta lift from $\widetilde{\mathrm{SL}_2}$ to $PB^\times$ is algebraic and further, in the opposite direction, there is a canonical transcendental period modulo which the theta lift is algebraic. In this article, we will prove analogs of Shimura's results for rationality over specified number fields and also p-adic integrality. As a consequence we deduce several results relating periods of modular forms on different Shimura curves. These results, in fact, constituted the main motivation for this article and we begin by describing them in more detail.

Let $N = N^+N^-$ be an odd square-free integer with N^- a product of an even number of primes. Let f be a holomorphic newform of even weight $2k$ on $\Gamma_0(N)$, g a holomorphic newform with respect to the unit group of an Eichler order $\mathcal{O}$ of level N^+ in the indefinite quaternion algebra B ramified at the primes dividing N^-, and with the same Hecke eigenvalues as f. Let (F_0, Φ) be a pair consisting of a Galois extension of $\mathbb{Q}$ that splits B along with a suitable splitting $\Phi : B \otimes F_0 \simeq M_2(F_0)$ (see Sec. 2.2.1). Set $\tilde{F}_0 = \mathbb{Q}$ if $2k = 2$ and $\tilde{F}_0 = F_0$ otherwise. Let F be any number field containing $\tilde{F}_0$ and all the Hecke eigenvalues of f, let p be a prime not dividing N and λ a prime in F lying over p. As shown in [22] and as will be recalled below, f and g may be normalized canonically up to λ-adic units in F. One has attached to f and g, canonical fundamental periods $u_\pm(f, F, \lambda)$ and $u_\pm(g, F, \lambda)$, well defined up to λ-adic units in F. For $\sigma \in Aut(\mathbb{C}/\tilde{F}_0)$, let $u_\pm(f^\sigma, F^\sigma, \lambda^\sigma)$

and $u_\pm(g^\sigma, F^\sigma, \lambda^\sigma)$ be the fundamental periods attached to f^σ and g^σ. These periods are chosen such that the period pair $(u_\pm(f, F, \lambda), u_\pm(f^\sigma, F^\sigma, \lambda^\sigma))$ gives a well defined element in $(\mathbb{C}^\times \times \mathbb{C}^\times)/(1, \sigma)F^\times$ (and likewise with f replaced by g). To begin with, we have the following theorem on rationality of period ratios.

Theorem 1.1.

$$\left(\frac{u_\pm(f, F, \lambda)}{u_\pm(g, F, \lambda)}\right)^\sigma = \frac{u_\pm(f^\sigma, F^\sigma, \lambda^\sigma)}{u_\pm(g^\sigma, F^\sigma, \lambda^\sigma)}.$$

In the special case $k = 2$, the above theorem can be used to construct directly isogenies defined over $\mathbb{Q}$ between quotients of Jacobians of different Shimura curves, without the crutch of Faltings' isogeny theorem. This application is treated in the last section of the article. (The idea that one should be able to construct such isogenies by proving the rationality of period ratios was suggested by Shimura [32].) In the case of higher weight, one might be able to use Thm. 1.1 to derive relations between the motives associated to the forms f and g, but we have not pursued this theme further in this article.

Indeed, our main interest is in integrality results for the ratios appearing above. With this in mind, let us define $u_\pm(f)$ (resp. $u_\pm(g)$) to be $u_\pm(f, F, \lambda)$ (resp. $u_\pm(g, F, \lambda)$ for any choice of F, so that both periods are well defined up to λ-adic units. Let ν be a quadratic discriminant and χ_ν the quadratic character $\left(\frac{\nu}{\cdot}\right)$. It is known under rather general conditions (see [34]) that $A(f, \nu) := |\nu|^{k-1}\mathfrak{g}(\chi_\nu)(2\pi i)^{-k}L(k, f, \chi_\nu)/u_\pm(f) = |\nu|^{k-1}\mathfrak{g}(\chi_\nu)(2\pi i)^{-k}L(\frac{1}{2}, \pi_f \otimes \chi_\nu)/u_\pm(f)$ is a λ-adic integer, where $\mathfrak{g}(\chi_\nu)$ is the Gauss sum attached to χ_ν and the $\pm$ sign holds according as $\chi_\nu(-1)\cdot(-1)^k = \pm 1$. Here π_f denotes the automorphic representation of PGL_2 attached to f and the L-function is being evaluated at the center of the critical strip, this being the point $s = k$ in the classical normalization and $s = 1/2$ in the automorphic normalization.

The integrality result we have in mind is motivated by the following observation. If f has weight 2, and λ is not Eisenstein for f (i.e. the mod λ Galois representation associated to f is irreducible), one may show, again using Faltings' isogeny theorem that $u_\pm(f)/u_\pm(g)$ is a λ-adic *unit*. So it is reasonable to ask if such a result holds for arbitrary even weights. The following theorem provides a conditional result in that direction.

Theorem 1.2. *Suppose $p > 2k + 1$ and $p \nmid \tilde{N} := \prod_{q|N} q(q+1)(q-1)$. Let χ_ν be the quadratic character associated to an odd fundamental quadratic discriminant ν and set $\epsilon = \mathrm{sign}((-1)^k\nu)$. Suppose $A(f, \nu) \not\equiv 0 \mod \lambda$. Then*

$$v_\lambda\left(\frac{u_\epsilon(f)}{u_\epsilon(g)}\right) \geq 0.$$

It is naturally of interest then to ask if there always exists a quadratic discriminant ν with prescribed sign and parity such that $A(f, \nu) \not\equiv 0 \mod \lambda$. This question in general seems to be extremely hard. However, as mentioned above, in the case of weight 2 (for instance for elliptic curves) and non-Eisenstein primes λ, we know *a priori* from Faltings' isogeny theorem that $u_\epsilon(f)/u_\epsilon(g)$ is a λ-adic unit. Feeding this information into the methods and results of this article, one obtains interesting applications to questions about the p-divisibility of the central values of quadratic twists of f. Assuming the exact form of the Birch-Swinnerton Dyer conjecture for elliptic curves of rank 0, one further gets applications to questions about p-torsion of Tate-Shafarevich groups. These applications are treated in a subsequent article ([24]), in which we also explain an intriguing relation between the

Waldspurger packet on $\widetilde{\mathrm{SL}_2}$ and congruences of modular forms of integral and half-integral weight.

The reader will note that the statements of Thms. 1.1 and 1.2 do not involve forms of half-integral weight. Nevertheless, their proof depends crucially on arithmetic properties of the Shimura correspondence and of forms of half-integral weight. We now give an introduction to our main theorems regarding the Shimura correspondence and the methods of this article.

Suppose $\boldsymbol{\chi}$ is a character of conductor N' dividing $4N$ with $\boldsymbol{\chi}(-1) = 1$ and set $M = \mathrm{lcm}(4, NN')$. Set $\chi_0 = \boldsymbol{\chi} \cdot \left(\frac{-1}{\cdot}\right)^k$, let $\chi = \boldsymbol{\chi} \cdot \left(\frac{-1}{\cdot}\right)^{k+\tau}$ (where $\tau = 0$ or 1) be such that χ is unramified at the prime 2, and use the same symbols χ_0 and χ to denote the associated adelic characters. Also suppose f_χ and g_χ are newforms in $\pi \otimes \chi$ and $\pi' \otimes \chi$ respectively where π and π' are the automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ and $B^\times(\mathbb{A})$ associated to f and g.

It follows then from work of Waldspurger that the space $S_{k+\frac{1}{2}}(M, \boldsymbol{\chi}, f_\chi)$ consisting of holomorphic forms of weight $k + \frac{1}{2}$ on $\Gamma_0(M)$ with character $\boldsymbol{\chi}$, and whose Shimura lift is f_χ, is two dimensional. Further this space has a unique one dimensional subspace, called the Kohnen subspace $S^+_{k+\frac{1}{2}}(M, \boldsymbol{\chi}, f_\chi)$, consisting of forms whose only non vanishing Fourier coefficients a_ξ are (possibly) those such that $(-1)^\tau \xi$ is congruent to $0, 1 \mod 4$. Let us denote by h_χ a nonzero vector in this subspace with algebraic Fourier coefficients. We may normalize h_χ to have all its Fourier coefficients be λ-adic integers in $\mathbb{Q}(f, \chi)$, and further so that at least one is a λ-adic unit. Here $\mathbb{Q}(f, \chi)$ is the field generated over $\mathbb{Q}$ by the Hecke eigenvalues of f and the values of the character χ.

The form h_χ may in fact be obtained as a theta lift from $PB^\times$ as follows. For $q \mid N$, denote by w_q and w'_q the signs of the Atkin-Lehner involutions acting on f and g respectively, so that $w_q = \pm w'_q$, the $+$ (resp. $-$) sign holding exactly when B is unramified (resp. ramified) at q. Fix ν, an odd quadratic fundamental discriminant such that $(-1)^\tau = \mathrm{sign}(\nu)$ and such that the following local conditions are satisfied at the primes dividing N:

(a) If $q \mid N$ but $q \nmid \nu$, $\chi_{0,q}(-1) = w'_q \cdot \chi_{\nu,q}(q)$.

(b) If $q \mid N$ and $q \mid \nu$, $\chi_{0,q}$ is ramified exactly when $q \mid N^-$ and for such q, $\chi_{0,q}(-1) = -1$.

Let us denote by g' the form $g_\chi \otimes (\chi\chi_\nu \circ \mathrm{Nm})^{-1} \in \pi' \otimes \chi_\nu$. One now considers the theta correspondence for the dual pair $(\widetilde{\mathrm{SL}_2}, PB^\times)$. It is shown in Sec. 3 below that the conditions (a) and (b) above imply (again from work of Waldspurger) that the form h_χ occurs in the theta lift $\Theta(\pi' \otimes \chi_\nu, \psi')$ where $\psi' = \psi^{1/|\nu|}$ and ψ is the usual additive character on $\mathbb{Q} \setminus \mathbb{A}_\mathbb{Q}$. Let V be the subspace of B consisting of the trace 0 elements. For an appropriate explicit choice of Schwartz function $\varphi \in V(\mathbb{A})$ (see Sec. 3), one has $\theta_\varphi(g') = \alpha_0 h_\chi$ and $\theta^t_\varphi(h_\chi) = \beta g'$ for scalars α_0 and β. The arithmetic properties of the complex numbers α_0 and β are then of crucial importance. It will turn out that β is algebraic, while α_0 is an algebraic multiple of the period $u_\epsilon(g)$ where $\epsilon = \mathrm{sign}((-1)^k\nu)$. In fact it is natural to write $\alpha_0 = \boldsymbol{\alpha}\mathfrak{g}(\chi)u_\epsilon(g)$, and $i^{k+\tau}\beta = \mathfrak{g}(\chi)^{-1}\boldsymbol{\beta}$, where $\mathfrak{g}(\chi)$ is the Gauss sum attached to χ. The following is one of our main theorems regarding the Shimura-Shintani-Waldspurger correspondence.

Theorem 1.3. *The complex numbers $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are algebraic, and $\boldsymbol{\alpha} \in F(\chi), \boldsymbol{\beta} \in \mathbb{Q}(f, \chi)$. Further, assuming $p > 2k + 1$ and $p \nmid \tilde{N}$, we have*
(a) $v_\lambda(\boldsymbol{\alpha}) \geq 0$.
(b) $v_\lambda(\boldsymbol{\beta}) \geq 0$.

The algebraicity of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is due to Shimura [32]; our contribution is the rationality of these over $F(\chi), \mathbb{Q}(f, \chi)$ respectively and the λ-adic integrality. It turns out that the theorem for $\boldsymbol{\alpha}$ is quite easy and with adequate preparation, is almost tautological (see Section 4). On the other hand, the rationality and integrality of $\boldsymbol{\beta}$ is much harder and requires the very detailed analysis of Section 5. Here is a brief description of the ideas involved. To check for rationality or integrality of $\boldsymbol{\beta}$, it suffices to evaluate $\theta_\varphi(h_\chi) = \beta g'$ at specific CM points $j : K \hookrightarrow B$ associated to an imaginary quadratic field K and check that the resulting values are rational or integral multiples of appropriate CM periods. From a computational point of view, it is easier to compute a sum of values at all Galois conjugates of a Heegner point, twisted by a Hecke character η'; the resulting sum is interpreted as a *period integral* $L_{\eta'}$ on a torus. Now one applies see-saw duality. It turns out that this is rather subtle, involving the choice of two characters κ, μ depending on η'. Here κ is a Hecke character of K of weight $(k, 0)$ at infinity, while μ is a finite order character of $\mathbb{Q}_{\mathbb{A}}^\times$. Further the pair (μ, η) is only well defined up to replacement by $(\kappa\cdot(\omega\circ \mathrm{Nm}_{K/\mathbb{Q}}), \mu\cdot\omega^2)$ for any finite order character ω of $\mathbb{Q}_{\mathbb{A}}^\times$. Let π_μ denote the automorphic theta representation of $\widetilde{\mathrm{SL}_2}(\mathbb{A})$ associated to μ and π_κ the automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ associated to the Hecke character κ. Then by an application of see-saw duality one gets roughly an expression for $L_{\eta'}$ as a triple integral

$$L_{\eta'}(\theta_\varphi^t(h_\chi)) = \int_{\mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A})} h_\chi(\sigma)\theta_\mu(\sigma)\overline{\theta_\kappa(\sigma)}d\sigma, \tag{1.1}$$

for some vectors θ_μ and θ_κ lying in π_μ and π_κ respectively. Let K^0 be the trace 0 elements of K, and $K^\perp$ the orthogonal complement to K for the norm form on B. With respect to the decomposition $V = K^0 + K^\perp$, the Schwartz function $\varphi \in V(\mathbb{A})$ splits up as a sum $\sum_{i\in I} \varphi_{1,i} \otimes \varphi_{2,i}$ over an indexing set I. More precisely, what one gets then is not a single integral of the form (1.1) but in fact a sum of such integrals indexed by the set I and depending on the splitting of the pure tensor φ as a sum of pure tensors. The data of such splitting is in general highly ramified, as are the local representations involved, and so one needs an elaborate argument to show that the sum of integrals so obtained may indeed be replaced by a single integral with convenient choices of vectors in π_μ and π_κ. This argument occupies all of Sec. 5.2. We should remark here that the weights of h_χ, θ_μ and θ_κ are $k + \frac{1}{2}, \frac{1}{2}$ and $k + 1$ respectively. As for the possibilities for the local representations at non-archimedean primes, many different types of ramification could occur, including for instance the possibility that π_μ and π_κ are both supercuspidal, even though we have restricted the ramification of π_f to be at worst Steinberg. [2]

The upshot of the argument is that one has an expression for the period integral as $c \cdot \langle H, \theta_\kappa\rangle$ for some constant c (that depends on f, χ, κ, μ) and a modular form H of weight $k+1$ with coefficients that are λ-integral and lie in $\mathbb{Q}(f, \chi)$, $\langle\cdot,\cdot\rangle$ being the usual Petersson inner product. (It is at this point we make use of the appendix due to Brian Conrad; indeed the form H is naturally presented as $w_Q H_0$ for a form H_0 with λ-integral Fourier coefficients and an Atkin-Lehner operator w_Q with $Q \mid N^2$. The main theorem of the appendix guarantees then that H has λ-integral Fourier coefficients as well.) Now one applies an argument similar to that of the authors' previous article [22] to show that $c \cdot \langle H, \theta_\eta\rangle/\Omega$ is a λ-adic integer

[2] If μ is the trivial character, θ_μ is an Eisenstein series. In this case, the integral (1.1) is identified with the values at $s = k$ (in the classical normalization) of the Rankin-Selberg Dirichlet series $D(s, h_\chi, \theta_\kappa)$ associated by Shimura to the cusp forms h_χ and θ_κ of weights $k + \frac{1}{2}$ and $k + 1$ respectively.

for a suitable CM period Ω. One needs to use here a refined study of congruences between θ_κ and other forms as well as the main conjecture of Iwasawa theory for the imaginary quadratic field K, which is a deep theorem of Rubin [25]. The constant c above arises from the delicate computations with the local integrals mentioned above, and is a p-integer but not necessarily a p-unit. Miraculously, its p-adic valuation turns out to be exactly what is needed to make the argument using Iwasawa theory and congruences go through. One needs to be particularly careful here since the choice of auxiliary quadratic discriminant ν introduces extra level structure into the problem, and with an eye on applications, one does not want to make any assumptions on ν other than those in Thm. 1.2. The rationality proceeds somewhat differently: the CM period must be chosen more carefully (to depend on κ), and one then needs to apply the rationality results of Blasius [2] for the special values of L-functions of Grossencharacters of K.

To use the integrality of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ we need several formulas. In what follows we will use the symbol $\sim$ to denote equality up to less important factors, and refer the reader to the main text of the article for more explicit equations. Crucial to us is a formula for the Fourier coefficients of the theta lift $\theta_\varphi(g')$ that is proved in [23]. This formula states roughly that

$$|a_\xi(\theta_\varphi(g'))|^2 \sim L(\frac{1}{2}, \pi \otimes \chi_\nu) L(\frac{1}{2}, \pi \otimes \chi_{\xi_0}) \frac{\langle g, g\rangle}{\langle f, f\rangle}. \tag{1.2}$$

for $\xi_0 = (-1)^\tau \xi$ satisfying a particular set of congruence conditions. This formula is used in two ways. Firstly it shows that the theta lift $\theta_\varphi(g')$ is nonvanishing for the particular choice of Schwartz function φ since $L(\frac{1}{2}, \pi \otimes \chi_\nu) \neq 0$ and we can find a ξ such that $L(\frac{1}{2}, \pi \otimes \chi_{\xi_0}) \neq 0$. Secondly, comparing it with the following formula of Baruch-Mao [1] which is proved using the relative trace formula of Jacquet,

$$\frac{|a_\xi(h_\chi)|^2}{\langle h, h\rangle} \sim \frac{L(\frac{1}{2}, \pi \otimes \xi_0)}{\langle f, f\rangle}, \tag{1.3}$$

and applying see-saw duality

$$\langle \theta_\varphi(g'), h_\chi\rangle = \langle g', \theta^t_\varphi(h_\chi)\rangle, \tag{1.4}$$

one obtains the following important formula

$$L(\frac{1}{2}, \pi \otimes \chi_\nu) \sim \boldsymbol{\alpha}\boldsymbol{\beta} u_\epsilon(g). \tag{1.5}$$

The integrality of $u_\epsilon(f)/u_\epsilon(g)$ follows immediately from (1.5) using the integrality of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ and the assumption on $A(f, \nu)$ being a p-unit. As a bonus, if one combines (1.5) with (1.4), one gets

$$\langle \theta^t_\varphi(h_\chi), \theta^t_\varphi(h_\chi)\rangle \sim L(\frac{1}{2}, \pi \otimes \chi_\nu) \langle h_\chi, h_\chi\rangle. \tag{1.6}$$

which is nothing but the explicit version of the Rallis inner product formula in this situation, obtained in a completely different way than the original method of Rallis!

It would be very interesting to generalize the results of this article to totally real fields other than $\mathbb{Q}$, but this seems to be much harder. For instance, for a real quadratic field, one would like integral period relations between the periods usually denoted u_{++}, u_{+-}, u_{-+} and u_{--}. Another interesting question is to study the integrality properties of theta lifts from $\widetilde{\mathrm{SL}_2}$ to $PB^\times$ for B a definite quaternion algebra over $\mathbb{Q}$. Very surprisingly, this seems harder than the indefinite case: the reader may find a discussion of the issues involved in the article [24].

The article is organized as follows. Sec. 2 contains preliminaries on modular forms of integral and half-integral weight and some results extracted from Waldspurger's article [37]. In Sec. 3, we work out, using the results of Waldspurger's article [38], some facts regarding the theta correspondence for $(\widetilde{\mathrm{SL}_2}, PB^\times)$ and study the same for a certain explicit choice of theta function. Sections 4 and 5 are devoted to proving the rationality and integrality of the Shintani and Shimura lifts respectively. Finally, in Sec. 6 we explain in more detail the various formulas mentioned above, and discuss the applications to arithmeticity of period ratios and isogenies between new-quotients of Jacobians of Shimura curves.

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2. Modular forms of integral and half-integral weight

2.1. **Preliminaries.**

2.1.1. *Metaplectic groups.* Here we follow the exposition and notations of [37] II § 4. If v is a place of $\mathbb{Q}$, let $\widetilde{\mathrm{S}}_v$ denote the metaplectic (degree 2) cover of $\mathrm{SL}_2(\mathbb{Q}_v)$. Likewise, let $\widetilde{\mathrm{S}}_{\mathbb{A}}$ denote the metaplectic (degree 2) cover of $\mathrm{SL}_2(\mathbb{A})$. We may identify $\widetilde{\mathrm{S}}_v$ (resp. $\widetilde{\mathrm{S}}_{\mathbb{A}}$) with $\mathrm{SL}_2(\mathbb{Q}_v) \times \{\pm 1\}$ (resp. $\mathrm{SL}_2(\mathbb{A}) \times \{\pm 1\}$), the product of two elements $(\sigma, \epsilon), (\sigma', \epsilon')$ being given by

$$(\sigma, \epsilon)(\sigma', \epsilon') = (\sigma\sigma', \epsilon\epsilon'\beta(\sigma, \sigma')),$$

where β_v is defined as follows. For $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Q}_v)$, let $x(\sigma) = c$ if $c \neq 0$, $x(\sigma) = d$, if $c = 0$. For v real, let $s_v(\sigma) = 1$. For $v = q$ a finite place, let $s_v(\sigma) = (c, d)_v$ if $cd \neq 0$ and $v_q(c)$ is odd, $s_v(\sigma) = 1$ otherwise. Here $(\cdot, \cdot)_v$ denotes the Hilbert symbol. Then

$$\beta_v(\sigma, \sigma') = (x(\sigma), x(\sigma'))_v(-x(\sigma)x(\sigma'), x(\sigma\sigma'))_v s_v(\sigma)s_v(\sigma')s_v(\sigma\sigma').$$

If $\sigma \in \mathrm{SL}_2(\mathbb{Q}_v)$, we denote also by the same symbol σ the element $(\sigma, 1) \in \widetilde{\mathrm{S}}_v$. The map $\sigma \mapsto (\sigma, \prod_v(s_v(\sigma))), \sigma \in \mathrm{SL}_2(\mathbb{Q})$ is a homomorphism of $\mathrm{SL}_2(\mathbb{Q})$ into $\widetilde{\mathrm{S}}_{\mathbb{A}}$, the image of which we denote by the symbol $\mathrm{S}_{\mathbb{Q}}$.

For $x \in \mathbb{Q}_v$, $\alpha \in \mathbb{Q}_v^\times$, define $\overline{\mathbf{n}}(x), \underline{\mathbf{n}}(x)$ and $\mathbf{d}(\alpha)$ to be the elements of $\widetilde{\mathrm{S}}_v$ given by

$$\overline{\mathbf{n}}(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad \underline{\mathbf{n}}(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, \quad \mathbf{d}(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}.$$

Let $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \widetilde{\mathrm{S}}_v$ and notice that

$$\underline{\mathbf{n}}(x) = \mathbf{d}(-1) \cdot w \cdot \overline{\mathbf{n}}(-x) \cdot w, \tag{2.1}$$

in $\widetilde{\mathrm{S}}_v$, a relation that we will use repeatedly.

2.1.2. For $t \in \mathbb{Q}_v^\times$ and ψ an additive character of $\mathbb{Q}_v$, let $\gamma_\psi(t)$ be the constant associated by Weil to the character ψ and the quadratic form tx^2. Recall that for $v = q$, a finite prime, $\gamma_\psi(t)$ may be computed to be

$$\gamma_\psi(t) = \lim_{n\to\infty} \int_{q^{-n}\mathbb{Z}_q} \psi(tx^2) d_t x, \tag{2.2}$$

where $d_t x$ is Haar measure chosen to be autodual with respect to the pairing $(x, y) \mapsto \psi(txy)$. We denote $\gamma_\psi(1)$ simply by the symbol γ_ψ. Define

$$\mu_\psi(t) = (t,t)_v \gamma_\psi(t)\gamma_\psi(1)^{-1} = \gamma_\psi(1)\gamma_\psi(t)^{-1}. \tag{2.3}$$

Then one has the equalities:

$$\begin{aligned} \mu_\psi(tt') &= (t,t')_v \mu_\psi(t)\mu_\psi(t'). \\ \mu_\psi(t^2) &= 1. \end{aligned}$$

Thus μ_ψ defines a genuine character of $\overline{\mathbb{Q}_v^\times}$, the extension of $\mathbb{Q}_v^\times$ by $\{\pm 1\}$ given by the Hilbert symbol. For $\alpha \in \mathbb{Q}_v^\times$, let ψ^α denote the character defined by $\psi^\alpha(x) = \psi(\alpha x)$. One checks easily that

$$\mu_{\psi^\alpha}(t) = (\alpha, t)_v \mu_\psi(t).$$

2.1.3. Let $(V, \langle , \rangle)$ be a quadratic space over $\mathbb{Q}_v$ and ψ an additive character of $\mathbb{Q}_v$. Suppose $Q(x) := \frac{1}{2}\langle x, x\rangle = \sum_{i=1}^d a_i x_i^2$ in terms of an orthogonal basis for V, where $d = \dim(V)$. Set

$$\gamma_{\psi,Q} := \prod_{i=1}^d \gamma_{\psi^{a_i}}.$$

$$D_Q := \begin{cases} (-1)^{(d-1)/2} \prod_{i=1}^d a_i \text{ if } d \text{ is odd,} \\ (-1)^{d/2-1} \prod_{i=1}^d a_i \text{ if } d \text{ is even.} \end{cases}$$

Then there exists a representation r_ψ of $\widetilde{\mathrm{S}}_v$ on $\mathcal{S}_\psi(V)$, the Schwartz space of V, called the *Weil representation*, which is characterized by

$$r_\psi(\mathbf{n})\varphi(x) = \psi(nQ(x))\varphi(x), \tag{2.4}$$

$$r_\psi(\mathbf{d}(\alpha))\varphi(x) = \mu_\psi(\alpha)^d (\alpha, D_Q)_v |\alpha|^{d/2}\varphi(\alpha x), \tag{2.5}$$

$$r_\psi(w)\varphi(x) = \gamma_{\psi,Q}\mathcal{F}_\psi(\varphi), \tag{2.6}$$

$$r_\psi(1,\epsilon)\varphi(x) = \epsilon^d \varphi(x), \tag{2.7}$$

where $\mathcal{F}_\psi$ denotes the Fourier transform with respect to the pairing $(x_1, x_2) \mapsto \psi(\langle x_1, x_2\rangle)$, the Haar measure on V being chosen such that $\mathcal{F}_\psi(\mathcal{F}_\psi(\varphi))(x) = \varphi(-x)$ for all $\varphi \in \mathcal{S}_\psi(V)$.

2.1.4. Suppose q is an odd prime. Let $\hat{\psi}$ be the character on $\mathbb{Z}/q\mathbb{Z}$ given by $\hat{\psi}(1) = e^{-2\pi i/q}$ and $\hat{\chi}$ any character on $(\mathbb{Z}/q\mathbb{Z})^\times$, extended to $\mathbb{Z}/q\mathbb{Z}$ by setting $\hat{\chi}(0) = 0$. Define the Gauss sum

$$G(\hat{\chi}, \hat{\psi}^a) = \sum_{\delta \in (\mathbb{Z}/q\mathbb{Z})^\times} \hat{\chi}(\delta) e^{-2\pi i a\delta/q},$$

so that $G(\hat{\chi}, \hat{\psi}^a) = \hat{\chi}^{-1}(a) G(\eta, \hat{\psi})$. If ϱ is the unique nontrivial quadratic character of $(\mathbb{Z}/q\mathbb{Z})^\times$,

$$G(\varrho, \hat{\psi}) = \begin{cases} \sqrt{q}, & \text{if } q \equiv 1 \mod 4. \\ i\sqrt{q}, & \text{if } q \equiv 3 \mod 4. \end{cases}$$

Hence $G(\varrho, \hat{\psi})^2 = \varrho(-1)q$.

2.1.5. Let q be a fixed finite prime and ψ the character on $\mathbb{Q}_q$ with kernel $\mathbb{Z}_q$ such that $\psi(\frac{1}{q}) = e^{-2\pi i/q}$. If $q \neq 2$ and $t \in \mathbb{Z}_q^\times$, one easily computes that $\gamma_\psi(t) = 1$ and $\mu_\psi(t) = 1$. Thus

$$\mu_{\psi^\alpha}(t) = (\alpha, t)_v, \tag{2.8}$$

for any $\alpha \in \mathbb{Q}_q^\times$. If $q = 2$, $\mu_\psi(t) = \frac{1}{2}[1 - i + (1+i)\chi_{-1,2}(t)]$ for $t \in \mathbb{Z}_2^\times$. In particular, $\mu_\psi(-1) = -i$. Note that $\mu_{\psi^\alpha}(-1) = (-1, \alpha)_2 \cdot i$ and $\mu_\psi(\alpha)^3 = (\alpha, \alpha)_2 \mu_\psi(\alpha) = (-1, \alpha)_2 \mu_\psi(\alpha)$.
Suppose now that q is odd , and $\psi' = \psi^\alpha$ with $v_q(\alpha) = -1$, $q\alpha \equiv a \mod q, a \in (\mathbb{Z}/q\mathbb{Z})^\times$. Then set $G(\hat{\chi}, \psi') := G(\hat{\chi}, \hat{\psi}^a)$. One computes from (2.2) that

$$\gamma_{\psi'} = q^{-1/2} G(\varrho, \hat{\psi}^a) = q^{-1/2} G(\varrho, \psi') = \varrho(a) q^{-1/2} G(\varrho, \hat{\psi}). \tag{2.9}$$

If $q = \infty$, and $\psi(x) = e^{2\pi i x}$ we have $\mu_\psi(-1) = i$.

2.1.6. Let $\underline{\chi}$ be a Dirichlet character of conductor M. We denote by χ the associated Grossencharacter of $\mathbb{Q}_\mathbb{A}^\times$, satisfying $\chi_q(q) = \underline{\chi}(q)$ for almost all q. If χ_q is a character of $\mathbb{Q}_q^\times$ of conductor q, we denote (in Sec. 3.2 alone) by $\hat{\chi}_q$ the induced character on $\mathbb{Z}_q^\times/(1+q\mathbb{Z}_q) \simeq (\mathbb{Z}/q\mathbb{Z})^\times$.

2.1.7. *Measures.* We use the same conventions here as in [22]. In the interest of brevity, the reader is referred to §1 of that article for the measure normalizations used on the different local and adelic groups, the only difference being that the indefinite quaternion algebra is called D in [22] as opposed to B in the present article.

2.2. **Modular forms of integral weight on an indefinite quaternion algebra.** There is nothing original in this section, the only purpose of which is to set up notation.

2.2.1. *Classical and adelic modular forms.* Let B be an indefinite quaternion algebra over $\mathbb{Q}$ with discriminant N^-, and $\mathcal{O}$ a maximal order in B. As in [15] we pick once and for all a finite Galois extension $F_0/\mathbb{Q}$ (contained in $\mathbb{C}$) that splits B and an isomorphism $\Phi : B \otimes F_0 \simeq \mathrm{M}_2(F_0)$ such that $\Phi(B) \subseteq \mathrm{M}_2(F_0 \cap \mathbb{R})$ and $\Phi(\mathcal{O}) \subseteq \mathrm{M}_2(R)$ where R is the ring of integers of F_0. Thus Φ induces an isomorphism $\Phi_\infty : B \otimes \mathbb{R} \simeq \mathrm{M}_2(\mathbb{R})$. Let Nm denote the reduced norm on B. Via Φ_∞, the group of reduced norm 1 elements in $B \otimes \mathbb{R}$ is identified with $\mathrm{SL}_2(\mathbb{R})$, hence acts in the usual way on the complex upper half-plane $\mathfrak{H}$, the action being $\gamma \cdot z = (az+b)/(cz+d)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}), z \in \mathfrak{H}$. Set $J(\gamma, z) = cz+d$ and $j(\gamma, z) = (\det \gamma)^{-1/2}(cz+d)$.

To define adelic modular forms, let ω be a finite order character and use the same symbol ω to denote the associated Grossencharacter of $\mathbb{Q}_{\mathbb{A}}^{\times}$. We view $B^{\times}$ as an algebraic group over $\mathbb{Q}$; $B_{\mathbb{A}}^{\times}$, $B_{\mathbb{A}_f}^{\times}$, $B_{\mathbb{Q}}^{\times}$ will denote its group of adelic points, points over the finite adeles and rational points respectively. Let $L^2(B_{\mathbb{Q}}^{\times} \setminus B_{\mathbb{A}}^{\times}, \omega)$ be the space of functions $s : B_{\mathbb{A}}^{\times} \to \mathbb{C}$ satisfying $s(\gamma z \beta) = \omega(z) s(\beta)$ $\forall \gamma \in B_{\mathbb{Q}}^{\times}$, $z \in \mathbb{Q}_{\mathbb{A}}^{\times}$ and having finite norm under the inner product $\langle s_1, s_2 \rangle = \frac{1}{2} \int_{\mathbb{Q}_{\mathbb{A}}^{\times} B_{\mathbb{Q}}^{\times} \setminus B_{\mathbb{A}}^{\times}} s_1(\beta) \overline{s_2(\beta)} d^{\times}\beta$. Also let $\mathcal{A}_0(\omega) = L_0^2(B_{\mathbb{Q}}^{\times} \setminus B_{\mathbb{A}}^{\times}, \omega) \subseteq L^2(B_{\mathbb{Q}}^{\times} \setminus B_{\mathbb{A}}^{\times}, \omega)$ be the closed subspace consisting of cuspidal functions.

For U any open compact subgroup of $B_{\mathbb{A}_f}^{\times}$ and $\tilde{\omega}$ any character of U whose restriction to $U \cap \mathbb{Q}_{\mathbb{A}_f}^{\times}$ equals $\omega|_{U \cap \mathbb{Q}_{\mathbb{A}_f}^{\times}}$, denote by $S_k(U, \tilde{\omega})$ the set of $s \in \mathcal{A}_0(\omega)$ satisfying $s(xu\kappa_\theta) = s(x)\tilde{\omega}(u) e^{ik\theta}$ for $u \in U$, $\kappa_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. By strong approximation for $B_{\mathbb{A}}^{\times}$, there exist $t_i \in B_{\mathbb{A}_f}^{\times}$, $i = 1, \ldots, h_U$, such that

$$B_{\mathbb{A}}^{\times} = \sqcup_{i=1}^{h_U} B^{\times} t_i U (B_{\infty}^{\times})^{+}, \tag{2.10}$$

where h_U is the cardinality of $\mathbb{Q}^{\times} \setminus \mathbb{Q}_{\mathbb{A}}^{\times} / \mathrm{Nm}(U)(\mathbb{Q}_{\infty}^{\times})^{+}$. Let $\Gamma_i(U) = B_{\mathbb{Q}}^{\times} \cap t_i U (B_{\infty}^{\times})^{+} t_i^{-1}$ and define ω_i to be the character on $\Gamma_i(U)$ defined by $\omega_i(\gamma) = \tilde{\omega}^{-1}(t_i^{-1}\gamma t_i)$. One defines the space $S_k(\Gamma_i, \omega_i)$ to consist of holomorphic functions $f : \mathfrak{H} \to \mathbb{C}$ satisfying

(i) $g(\gamma z) = j(\gamma, z)^k \omega_i(\gamma) g(z)$,

(ii) g vanishes at the cusps of $\Gamma_i(U)$.

If $\tilde{\omega}$ (resp. ω_i) is the trivial character, we write simply $S_k(U)$ (resp. $S_k(\Gamma_i(U))$). Also, if $h_U = 1$, we simply write $\Gamma(U)$ instead of $\Gamma_1(U)$. Given a collection of elements $\mathbf{g} = \{g_i\}, g_i \in S_k(\Gamma_i(U), \omega_i)$, define $s_{\mathbf{g}} \in S_k(U, \tilde{\omega})$ by $s_{\mathbf{g}}(\beta) = g_i(\beta_\infty(\imath)) j(\beta_\infty, \imath)^{-k} \tilde{\omega}(u)$, if $\beta = \gamma t_i u \beta_\infty$, $\gamma \in B_{\mathbb{Q}}^{\times}, u \in U, \beta_\infty \in (B_{\infty}^{\times})^{+}$. This is easily seen to be independent of the choice of the decomposition $\beta = \gamma t_i u \beta_\infty$. The assignment $\mathbf{g} \longmapsto s_{\mathbf{g}}$ gives an isomorphism $\oplus_i S_k(\Gamma_i(U), \omega_i) \simeq S_k(U, \tilde{\omega})$.

Remark 2.1. Suppose $B = \mathrm{M}(\mathbb{Q})$, ω has conductor M and $U = \prod U_q$ where

$$U = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_q), c \equiv 0 \mod M \right\}.$$

Then $h_U = 1$, $\Gamma(U) = \Gamma_0(M)$, and the character ω on $\Gamma(U)$ is identified with the character $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \omega(d)$ on $\Gamma_0(M)$. Thus $S_k(U, \tilde{\omega}) \simeq S_k(\Gamma(U), \omega) = S_k(\Gamma_0(M), \omega)$.

2.2.2. *Shimura curves.* Let $\mathfrak{H}^* = \mathfrak{H}$ if $B \neq \mathrm{M}_2(\mathbb{Q})$ and $\mathfrak{H}^* = \mathfrak{H} \cup \mathbb{Q} \cup \infty$ if $B = \mathrm{M}_2(\mathbb{Q})$. Consider the analytic space

$$Y_U^{\mathrm{an}} = B^{\times} \setminus B_{\mathbb{A}}^{\times} / U \cdot \mathbb{R}^{\times} \mathrm{SO}_2(\mathbb{R}) = B^{\times} \setminus \mathfrak{H} \times B_{\mathbb{A}_f}^{\times} / U,$$

and its compactification

$$X_U^{\mathrm{an}} = B^{\times} \setminus \mathfrak{H}^* \times B_{\mathbb{A}_f}^{\times} / U = \sqcup_{i=i}^{h_U} \Gamma_i(U) \setminus \mathfrak{H}^*,$$

the last equality corresponding to the decomposition in (2.10). Shimura has shown that X_U^{an} is the analytic space associated to a smooth curve X_U defined over $\mathbb{Q}$. The curve X_U

is possibly disconnected, each component being defined over the class field of $\mathbb{Q}$, denoted $\mathbb{Q}_U$, corresponding to the open subgroup $\mathbb{Q}^\times \operatorname{Nm}(U)(\mathbb{R}^\times)^+$ of $\mathbb{Q}_\mathbb{A}^\times$. The set of components of X_U is canonically identified with $\operatorname{Gal}(\mathbb{Q}_U/\mathbb{Q})$.

Suppose $\mathbf{g} = \{g_i\} \in \oplus_i S_{2k}(\Gamma_i(U))$. For each i, the differential form $(2\pi\imath dz)^{\otimes k} g_i(z)$ is $\Gamma_i(U)$ invariant, hence descends to a section of Ω^k on $\Gamma_i(U) \setminus \mathfrak{H}^*$ (by the cuspidality of g_i), which we denote by $\tilde{g}_i$. Let $\tilde{\mathbf{g}}$ be the section of Ω^k on X_U that equals $\tilde{g}_i$ on the component $\Gamma_i(U) \setminus \mathfrak{H}^*$. The assignment $\mathbf{g} \mapsto \tilde{\mathbf{g}}$ gives an isomorphism

$$\oplus_i S_{2k}(\Gamma_i(U)) \simeq H^0(X_{U,\mathbb{C}}, \Omega^k).$$

2.2.3. *Automorphic representations and newforms.* Let π be any irreducible representation of the Hecke algebra of $B_\mathbb{A}^\times$ that occurs in $\mathcal{A}_0(\omega)$. It is well known that π factors as an infinite tensor product $\pi = \otimes_{q \le \infty} \pi_q$, where π_q is an irreducible representation of (the Hecke algebra of) $B^\times(\mathbb{Q}_q)$. In this article, we will only consider those π that satisfy the following two conditions:

(*) π_∞ is the weight-$2k$ discrete series representation $\sigma(|\cdot|^{\frac{2k-1}{2}}, |\cdot|^{\frac{2k-1}{2}})$ of $\mathrm{GL}_2(\mathbb{R})$.

(**) If $q \mid N^-$, π_q is a one-dimensional representation of $B^\times(\mathbb{Q}_q)$.

In this case, one may pick a distinguished line in π, defined to be the span of a vector $v = \otimes_q v_q$ where the v_q are defined as follows:
(a) For any finite $q \nmid N^-$, by a theorem of Casselman [3], there exists a unique power q^{n_q} such that the space of vectors in π_q that is invariant under

$$\left\{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_q), c \equiv 0 \mod q^{n_q}, d \equiv 1 \mod q^{n_q}\right\}$$

is one-dimensional. We take v_q to be any such non-zero vector. Note that if $n_q \ge 1$, v_q is the unique vector up to multiplication by a scalar that transforms under

$$\left\{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_q), c \equiv 0 \mod q^{n_q}\right\}$$

by the character $\gamma \mapsto \omega_q(d)$.
(b) For $q \mid N^-$, we take v_q to be any non-zero vector in the one-dimensional representation π_q.
(c) For $q = \infty$, the restriction of π_∞ to $\mathrm{SL}_2(\mathbb{R})$ splits as the direct sum of the weight-$2k$ holomorphic and antiholomorphic discrete series, and we take v_∞ to correspond to a lowest weight vector in the former.

Any multiple of v will be called a newform in π.

2.2.4. *Some relevant open compact subgroups.* We now pick some specific examples of open compact U that will play an important role in this article. We fix once and for all isomorphisms $\Phi_q : B \otimes \mathbb{Q}_q \to \mathrm{M}_2(\mathbb{Q}_q)$ for $q \nmid N^-$ such that $\Phi_q(\mathcal{O} \otimes \mathbb{Z}_q) = \mathrm{M}_2(\mathbb{Z}_q)$. Let N^+ be an integer coprime to N^- and $\mathcal{O}'$ the unique Eichler order of level N^+ in B such that for $q \nmid N^-$

$$\Phi_q(\mathcal{O}' \otimes \mathbb{Z}_q) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}_q), c \equiv 0 \mod N^+\right\},$$

and for $q \mid N^-$, $\mathcal{O}' \otimes \mathbb{Z}_q = \mathcal{O} \otimes \mathbb{Z}_q$.

Set $N = N^+N^-$. Let χ be a character of conductor N_χ dividing N. Let $\mathcal{O}'(\chi)$ be the unique Eichler order in B such that $\mathcal{O}'(\chi) \otimes \mathbb{Z}_q = \mathcal{O}' \otimes \mathbb{Z}_q$, unless $q \mid N_\chi$ and $q \mid N^+$, in which case

$$\Phi_q(\mathcal{O}'(\chi)\otimes\mathbb{Z}_q)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{M}_2(\mathbb{Z}_q), c\equiv 0 \mod q^2\right\}.$$

We now define the following open compact subgroups of $B^\times_{\mathbb{A}_f}$.

(1) $U_0=\prod_q U_{0,q}$ where $U_{0,q}=(\mathcal{O}'\otimes\mathbb{Z}_q)^\times$.
(2) $U_0(\chi)=\prod_q U_{0,q}(\chi)$, where $U_{0,q}(\chi)=(\mathcal{O}'(\chi)\otimes\mathbb{Z}_q)^\times$.
(3) $U_1(\chi)=\prod_q U_{1,q}(\chi)$, where $U_{1,q}(\chi)=U_{0,q}=U_{0,q}(\chi)$ if $q\nmid N_\chi$ or if $q\nmid N^+$, and

$$U_{1,q}(\chi) = \left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in U_{0,q}(\chi),\ d\equiv 1 \mod q\right\} \text{ if } q\mid N_\chi, \text{ and } q\mid N^+.$$

Let $\omega_\chi=\chi^2$. We define below a character $\tilde{\omega}_\chi$ on $U_0(\chi)$ such that $\tilde{\omega}_\chi|_{\hat{\mathbb{Z}}^\times}=\omega_\chi|_{\hat{\mathbb{Z}}^\times}$. Firstly, for each q define $\tilde{\omega}_{\chi,q}$ on $U_{0,q}(\chi)$ as follows:

- For $q\nmid N_\chi$, $\tilde{\omega}_{\chi,q}(u)=1$ for any $u\in U_{0,q}(\chi)$.
- For $q\mid N_\chi$ and $q\mid N^+$, $\tilde{\omega}_{\chi,q}(u)=\chi_q(d)^2$ for $u=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in U_{0,q}(\chi)$.
- For $q\mid N_\chi$ and $q\mid N^-$, $\tilde{\omega}_{\chi,q}(u)=\chi_q(\mathrm{Nm}(u))$ for $u\in U_{0,q}(\chi)$.

Then, set $\tilde{\omega}_\chi=\prod_q\tilde{\omega}_{\chi,q}$ on $U_0(\chi)$. Now letting Γ (resp. Γ_χ) be the group of norm 1 units in $\mathcal{O}'$ (resp. $\mathcal{O}'(\chi)$), we see from the previous section that we have canonical isomorphisms

$$\begin{aligned} S_{2k}(\Gamma_\chi,\chi') &\simeq S_{2k}(U_0(\chi),\tilde{\omega}_\chi). \\ S_{2k}(\Gamma) &\simeq S_{2k}(U_0). \end{aligned} \tag{2.11}$$

where χ' is defined to be the restriction of $\tilde{\omega}_\chi^{-1}$ to $\Gamma_\chi\subseteq U_{0,q}$. (Note that in the case $B=\mathrm{M}_2(\mathbb{Q})$, $\chi'(\gamma)=\chi^2(d)$ for $\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma_\chi$.)

Let $\Gamma^1_\chi=B^\times\cap U_1(\chi)(B^\times_\infty)^+$. Since $B^\times_{\mathbb{A}}=B^\times(U_1(\chi)(B^\times_\infty)^+$, and $\chi'|_{\Gamma^1_\chi}$ is the trivial character, we have an isomorphism

$$S_{2k}(\Gamma^1_\chi)\simeq S_{2k}(U_1(\chi),\tilde{\omega}_\chi). \tag{2.12}$$

Let $g\in S_{2k}(\Gamma)=S_{2k}(U_0)$ be a newform. Denote by π_g the automorphic representation of $B^\times_{\mathbb{A}}$ generated by s_g. Since N is square-free, π_g satisfies both conditions (*) and (**), and s_g is a newform in π_g. For χ as above, we denote by $\pi_{g,\chi}$ the representation $\pi_g\otimes(\chi\circ\mathrm{Nm})$. It is clear that $\pi_{g,\chi}$ also satisfies conditions (*) and (**), and it follows from Casselman's theorem mentioned above that there is a vector $g_\chi\in S_{2k}(U_0(\chi),\tilde{\omega}_\chi)$, unique up to scalar multiplication, such that s_{g_χ} is a newform in $\pi_{g,\chi}$.

For the moment, g and g_χ are only well defined up to scalars, but we will see below that (at least for $p\nmid N$) they may be canonically normalized up to p-adic units in a suitable number field.

2.2.5. *Complex conjugation and action of an element of negative norm.* For δ any unit in $\mathcal{O}'(\chi)$ with reduced norm -1 and $g'\in S_{2k}(\Gamma_\chi,\chi')$ (resp. $g'\in\overline{S_{2k}}(\Gamma_\chi,\chi')$), denote by $g'|\delta$ the form given by $(g'|\delta)(z)=J(\delta,z)^{-2k}\chi'(\delta)\overline{g'(\delta\overline{z})}$ (resp. $(g'|\delta)(z)=J(\delta,\overline{z})^{-2k}\chi'(\delta)\overline{g'(\delta\overline{z})}$.) If δ' is any other such element, then $\gamma:=\delta\delta'^{-1}\in\Gamma_\chi$, hence $g'|\delta$ is independent of the choice of δ. Let $g'^c=\overline{g'|\delta}$ for any such choice of δ. If $g'\in S_{2k}(\Gamma_\chi,\chi')$ (resp. $\overline{S_{2k}}(\Gamma_\chi,\chi')$) then $g'|\delta\in S_{2k}(\Gamma_\chi,\overline{\chi'})$ (resp. $\overline{S_{2k}}(\Gamma_\chi,\overline{\chi'})$) and $g'^c\in\overline{S_{2k}}(\Gamma_\chi,\chi')$ (resp. $S_{2k}(\Gamma_\chi,\chi')$). It is easy to check that $(g'|\delta)|\delta=g'$ and $((g')^c)^c=g'$.

Let $\jmath$ denote the element $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in B_\infty^\times = \mathrm{GL}_2(\mathbb{R})$ and let $s_{g'}^{\jmath}$ be the element of $\pi_{g'}$ given by $s_{g'}^{\jmath}(\beta) = s_{g'}(\beta\jmath)$. Let $\beta \in B_{\mathbb{A}}^\times$ and suppose that $\beta = \gamma u \beta_\infty$ and $\beta\jmath = \gamma' u' \beta'_\infty$ be decompositions given by (2.10) with $U = U_{0,\chi}$. Thus $\gamma u = \gamma' u'$ and $\gamma\beta_\infty = \gamma'\beta'_\infty\jmath$. Let $\delta = (\gamma')^{-1}\gamma$, so that also $\delta = u'u^{-1} = \beta'_\infty\jmath\beta_\infty^{-1}$. Thus δ is a unit in $\mathcal{O}'_\chi$ of negative reduced norm, whence $\mathrm{Nm}(\delta) = -1$. Now, letting $z = \beta_\infty \cdot \imath$, we see that

$$\begin{aligned} s_{g'}(\beta\jmath) &= g'(\beta'_\infty(\imath))j(\beta'_\infty, \imath)^{-2k}\tilde{\omega}_\chi(u') = g'(\beta'_\infty(\imath))J(\beta'_\infty, \imath)^{-2k}\,\mathrm{Nm}(\beta'_\infty)^k\tilde{\omega}_\chi(u') \\ &= g'(\delta\beta_\infty\jmath(\imath))J(\delta\beta_\infty\jmath, \imath)^{-2k}\,\mathrm{Nm}(\beta_\infty)^k\tilde{\omega}_\chi(\delta u) \\ &= g'(\delta\bar{z})J(\delta, \bar{z})^{-2k}J(\beta_\infty, -\imath)^{-2k}\,\mathrm{Nm}(\beta_\infty)^k J(\jmath, \imath)^{-2k}(\chi')^{-1}(\delta)\tilde{\omega}_\chi(u). \end{aligned}$$

Thus

$$\overline{s_{g'}^{\jmath}(\beta)} = \overline{s_{g'}(\beta\jmath)} = J(\delta, z)^{-2k}\chi'(\delta)\overline{g'(\delta\bar{z})}j(\beta_\infty, \imath)^{-2k}(\tilde{\omega}_\chi)^{-1}(u) = s_{g'|\delta}(\beta),$$

so that $\overline{s_{g'}^{\jmath}} = s_{g'|\delta}$.

2.2.6. *Rational and integral structures.* Let $L := L_{g_\chi}$ be the field generated by the Hecke eigenvalues of g_χ and let p be a prime not dividing N. Fix once and for all an embedding $\lambda : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$. The inclusion $U_{1,\chi} \xrightarrow{\phi} U_{0,\chi}$ yields an inclusion $\phi^* : S_{2k}(U_0(\chi), \tilde{\omega}_\chi) \hookrightarrow S_{2k}(U_1(\chi), \tilde{\omega}_\chi) = S_{2k}(\Gamma^1_\chi) \simeq H^0(X_{U_1(\chi)}, \Omega^k)$. The curve $X := X_{U_1(\chi)}$ has good reduction over $\mathbb{Z}[\frac{1}{N}]$ and hence in particular at p. Let $\mathscr{X}$ be the minimal regular model of X over $\mathbb{Z}_p$. Thus we have inclusions

$$H^0(X_L, \Omega^{\otimes k}) \hookrightarrow H^0(X_{L_\lambda}, \Omega^{\otimes k}) \hookleftarrow H^0(\mathscr{X}_{\mathcal{O}_\lambda}, \Omega^{\otimes k}) =: \mathcal{M}_{L,\lambda}.$$

For any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, let $(g_\chi)^\sigma$ be the newform (defined up to a scalar) whose Hecke eigenvalues are obtained by applying σ to the Hecke eigenvalues of g_χ. We then normalize the collection $\{(g_\chi)^\sigma\}$ by requiring that $\widetilde{\phi^*(s_{(g_\chi)^\sigma})} \in H^0(X_{L^\sigma}, \Omega^{\otimes k})$, be a primitive element in the lattice $\mathcal{M}_{L^\sigma,\lambda}$, and further that the compatibility condition

$$\widetilde{\phi^*(g_\chi)^\sigma} = \widetilde{\phi^*(g_\chi)}^{\sigma}$$

be satisfied for all σ. This defines $s_{(g_\chi)^\sigma}$ up to an element of $(L^\sigma)^\times$ that is a unit at all primes above p.

When $B = \mathrm{M}_2(\mathbb{Q})$, the rational and integral structures defined above agree with the usual structures provided by the q-expansion principle. When $B \neq \mathrm{M}_2(\mathbb{Q})$, no q-expansions are available; however evaluating at CM points provides a suitable alternative criterion for rationality and integrality. (See Prop. 5.1 for an exact statement.)

2.3. **Modular forms of half-integral weight: review of Waldspurger's work.**

2.3.1. *Classical and adelic modular forms.* For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$ and $z \in \mathbb{C}$, define

$$\tilde{j}(\gamma, z) = \left(\frac{c}{d}\right)\mu_{\psi,2}(d)(cz+d)^{1/2},$$

so that $\tilde{j}(\gamma, z)^4 = j(\gamma, z)^2$. Here $\left(\frac{\cdot}{\cdot}\right)$ denotes the Kronecker symbol as in [26] p.442. Let M be a positive integer, divisible by 4, $\kappa = 2k+1$ be an odd positive integer and $\boldsymbol{\chi}$ a Dirichlet character modulo M such that $\boldsymbol{\chi}(-1) = 1$. Let $\chi_0 = \boldsymbol{\chi} \cdot \left(\frac{-1}{\cdot}\right)^k$ and use the same symbol χ_0 to denote the associated adelic character. We denote by $S_{\kappa/2}(M,\boldsymbol{\chi})$ the space of holomorphic functions h on $\mathfrak{H}$, that satisfy

$$h(\gamma(z)) = \tilde{j}(\gamma,z)^{\kappa}\boldsymbol{\chi}(d)h(z)$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M)$, and that vanish at the cusps of $\Gamma_0(M)$.

We now review the adelic definition of forms of half-integral weight. Let $\tilde{\rho}$ denote the right regular representation of the Hecke algebra of $\widetilde{\mathrm{S}}_{\mathbb{A}}$ on $\tilde{\mathcal{A}}_0$, the space of cusp forms on $\mathrm{S}_{\mathbb{Q}} \setminus \widetilde{\mathrm{S}}_{\mathbb{A}}$. Also let $\Gamma_q = \mathrm{SL}_2(\mathbb{Z}_q)$ and $\Gamma_q(n) = \{x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_q, d \equiv 0 \mod q^n\}$. We define, following Waldspurger [37], $\tilde{\mathcal{A}}_{\kappa/2}(M,\chi_0)$ to be the subspace of $\tilde{\mathcal{A}}_0$ consisting of elements t satisfying

(i) If $q \nmid M$ and $\sigma \in \Gamma_q$, $\tilde{\rho}_q(\sigma)t = t$;

(ii) If $q \mid M$, $q \neq 2$ and $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_q(v_q(M))$, $\tilde{\rho}_q(\sigma)t = \chi_{0,q}(d)t$;

(iii) For $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_2(v_2(M))$, $\tilde{\rho}_2(\sigma)t = \tilde{\epsilon}_2(\sigma)\chi_{0,2}(d)t$;

(iv) If $\theta \in \mathbb{R}$, $\tilde{\rho}_{\mathbb{R}}(\tilde{\kappa}(\theta))(t) = e^{i\kappa\theta/2}t$;

(v) $\tilde{\rho}_{\mathbb{R}}(\tilde{D})t = [\kappa(\kappa-4)/8]t$.

where $\tilde{\rho}_q$ denotes the restriction of ρ to $\widetilde{\mathrm{S}}_q$, $\tilde{D}$ is the Casimir element for $\widetilde{\mathrm{S}}_{\mathbb{R}}$ and $\tilde{\epsilon}_2(\sigma)$, $\tilde{\kappa}(\theta)$ are defined on p. 382 of [37]. For $z = u + iv \in \mathfrak{H}$, let $b(z) \in \widetilde{\mathrm{S}}_{\mathbb{A}}$ be the element which is 1 at all the non-archimedean places and equal to

$$\begin{pmatrix} v^{1/2} & uv^{-1/2} \\ 0 & v^{-1/2} \end{pmatrix}$$

at the real place. If $h \in S_{\kappa/2}(M)$, there exists a unique continuous function t_h on $\mathrm{S}_{\mathbb{Q}} \setminus \widetilde{\mathrm{S}}_{\mathbb{A}}$, such that for all $z \in \mathfrak{H}, \theta \in \mathbb{R}$,

$$t_h(b(z)\tilde{\kappa}(\theta)) = v^{\kappa/4}e^{i\kappa\theta/2}h(z).$$

Proposition 2.2. *(*[37] *Prop. 3) If $h \in S_{\kappa/2}(M,\boldsymbol{\chi})$, $t_h \in \tilde{\mathcal{A}}_{\kappa/2}(M,\chi_0)$. The assignment $h \mapsto t_h$ gives an isomorphism $S_{\kappa/2}(M,\boldsymbol{\chi}) \simeq \tilde{\mathcal{A}}_{\kappa/2}(M,\chi_0)$.*

Remark 2.3. (a) Our $\boldsymbol{\chi}$ and χ_0 play the role of the symbols χ and χ_0 respectively of Waldspurger's article [37]. We will also use the symbol χ below, but for a character that does not play any role in [37].
(b) When convenient, we identify $S_{\kappa/2}(M,\boldsymbol{\chi})$ and $\tilde{\mathcal{A}}_{\kappa/2}(M,\chi_0)$ via the isomorphism above.

2.3.2. *Fourier coefficients: rational and integral structures.* Let $h \in S_{\kappa/2}(M)$. Then h has a familiar q-expansion

$$h = \sum_{\xi \in \mathbb{Z}, \xi > 0} a_{\xi}(h)q^{\xi}$$

where $q = e^{2\pi iz}$. We say that h is algebraic (resp. F-rational, resp. λ-integral) if for all ξ the coefficients $a_{\xi}(h)$ are algebraic (resp. lie in F, resp. are λ-integral.) Further, h is said to be λ-adically normalized if it is λ-integral and if at least one Fourier coefficient of h is a unit at λ.

Let $t \in \tilde{\mathcal{A}}_0$. Let ψ denote the usual additive character of $\mathbb{Q} \setminus \mathbb{A}$ i.e. $\psi_\infty(x) = e^{2\pi i x}$ and ψ_q is the unique character on $\mathbb{Q}_q$ with kernel $\mathbb{Z}_q$ such that $\psi_q(x) = e^{-2\pi i x}$ for $x \in \mathbb{Z}[\frac{1}{q}]$. Define the ψ^ξ-th Fourier coefficient of t to be the function on $\widetilde{\mathrm{S}}_{\mathbb{A}}$ given by

$$W(t, \psi^\xi, \sigma) = \int_{\mathbb{Q}\setminus\mathbb{A}} t(\mathbf{n}\sigma)\psi^\xi(-n)dn.$$

The relation between the classical and adelic Fourier coefficients is

Proposition 2.4. *([37], Lemma 3) Let $h \in S_{\kappa/2}(M)$. Then*

$$a_\xi(h) = v^{-\kappa/4} e^{2\pi\xi v} W(t_h, \psi^\xi, \mathbf{d}_{\mathbb{R}}(v^{1/2})).$$

2.4. **The Shimura correspondence.** We now assume that N is odd and fix, as in the introduction, a holomorphic newform $f \in S_{2k}(\Gamma_0(N))$. The following proposition can be extracted from the main result of [37]. (The form f_χ that occurs below is a newform in $\pi_f \otimes \chi$ as defined in section 2.2.4. Also the reader is referred to [37] Sec. I.2 for the definition of the space $S_{k+\frac{1}{2}}(M, \boldsymbol{\chi}, f_\chi)$ in the statement of the proposition.)

Proposition 2.5. *Let $\boldsymbol{\chi}$ be a character of conductor dividing $4N$ with $\boldsymbol{\chi}(-1) = 1$, $N' :=$ cond$(\boldsymbol{\chi})$, $M := \mathrm{lcm}(4, N'N)$, and suppose $\chi := \boldsymbol{\chi} \cdot \left(\frac{-1}{\cdot}\right)^{k+\tau}$ is unramified at 2. Then $S_{k+\frac{1}{2}}(M, \boldsymbol{\chi}, f_\chi) \subseteq \tilde{\mathcal{A}}_{k+\frac{1}{2}}(M, \chi_0)$ is two dimensional. Further, it admits a unique one-dimensional subspace $S^+_{k+\frac{1}{2}}(M, \boldsymbol{\chi}, f_\chi)$, called the Kohnen subspace, consisting of forms h, all whose nonzero Fourier coefficients $a_\xi(h)$ satisfy $\chi_{0,2}(-1)\xi \equiv 0, 1 \mod 4$ i.e. $(-1)^\tau\xi \equiv 0, 1$ mod 4.*

More precisely, if h_χ denotes a non-zero vector in $S^+_{k+\frac{1}{2}}(M, \boldsymbol{\chi}, f_\chi)$, w_q the eigenvalue of the Atkin-Lehner involution (at q) acting on f, $\xi_0 = (-1)^\tau\xi$ and $a_\xi(h_\chi)$ denotes the ξth Fourier coefficient of h_χ, then $a_\xi(h_\chi) = 0$ unless the following conditions are satisfied:

(a) For all $q \mid N, q \nmid N'$, $\left(\frac{\xi_0}{q}\right) \neq -w_q$.

(b) For all $q \mid N'$, $\left(\frac{\xi_0}{q}\right) = \chi_{0,q}(-1)w_q = \chi_q(-1)w_q$.

(c) $\xi_0 \equiv 0, 1 \mod 4$.

If (a),(b),(c), are satisfied, and ξ_0 is a fundamental quadratic discriminant, then

$$a_\xi(h_\chi)^2 = A \cdot |\xi|^{k-1/2} L(\frac{1}{2}, \pi \otimes \chi_{\xi_0}),$$

for a nonzero constant A depending on f, χ and the choice of h_χ.

Proof: For the benefit of the reader, we indicate how this may be deduced from [37]. We refer the reader to Sec. VIII of the same article for the notations used in this proof. Recall that f_χ is the newform of character χ^2 associated to the representation $\pi \otimes \chi$. Then cond$(f_\chi) = M/4$. Waldspurger has defined for each q and each integer e, a set $U_q(e, f_\chi)$ consisting of functions on $\mathbb{Q}_q^\times$ with support in $\mathbb{Z}_q^\times$ and invariant by $(\mathbb{Z}_q^\times)^2$. Let E be any integer and $e_q = v_q(E)$. For $\underline{A}$ any function on the square-free integers and $\underline{c}_E = (c_q) \in \prod_q U_q(e_q, f_\chi)$, let

$$\begin{aligned} h(\underline{c}_E, \underline{A})(z) &= \sum_{n=1}^{\infty} a_n(\underline{c}_E, \underline{A}) e^{2\pi i n z}, \\ a_n(\underline{c}_E, \underline{A}) &= \underline{A}(n^{sf}) n^{(2k-1)/4} \prod_q c_q(n), \end{aligned}$$

where n^{sf} denotes the square-free part of n. Let $\overline{U}(E, f_\chi, \underline{A})$ be the span of all such functions $h(\underline{c}_E, \underline{A})$ as $\underline{c}_E$ varies. The main result of [37], Thm. 1, p. 378, states that for any integer M',

$$S_{k+\frac{1}{2}}(M', \boldsymbol{\chi}, f_\chi) = \oplus_{\frac{M}{4}|E|M'} \overline{U}(E, f_\chi, \underline{A}^{f_\chi}),$$

where $\underline{A}^{f_\chi}$ is a function on the square-free positive integers satisfying

$$\underline{A}^{f_\chi}(\xi)^2 = L(1/2, f_\chi \otimes \chi_0^{-1}\chi_\xi)\varepsilon(1/2, \chi_0^{-1}\chi_\xi) = L(1/2, f \otimes \chi_{\xi_0})\varepsilon(1/2, \chi_0^{-1}\chi_\xi).$$

It follows from this and the computations below (at the prime 2) that $S_{k+\frac{1}{2}}(M, \boldsymbol{\chi}, f_\chi) = \overline{U}(M, f_\chi, \underline{A}^{f_\chi})$. To check that $S_{k+\frac{1}{2}}(M, \boldsymbol{\chi}, f_\chi)$ is two dimensional, it is sufficient to check (with $E = M$) that $U_q(e_q, f_\chi)$ has cardinality equal to 1 for all $q \neq 2$ and $U_2(e_2, f_\chi)$ has cardinality equal to 2. As for the statement about the Fourier coefficients one needs to review carefully the definition of the sets $U_q(e_q, f_\chi)$ which may be found on p. 454-455 of [37]. We consider various cases:

Case A: If $q \neq 2, q \mid N$, $q \nmid N'$, we are in Case (4) of [37]: $\tilde{n}_q = m_q = e = 1, \lambda'_q = -q^{-1/2}\chi_q(q)w_q$. Then $U_q(e, f_\chi) = \{c_q^s[\lambda'_q]\}$. If $u \in \mathbb{Z}_q^\times$,

$$c_q^s[\lambda'_q](u) = \begin{cases} 2^{1/2} \text{ if } (q,u)_q = -q^{1/2}\chi_{0,q}(q^{-1})\lambda'_q \text{ i.e if } (q, (-1)^\tau u)_q = w_q, \\ 0, \text{ otherwise, i.e. if } (q, (-1)^\tau u)_q = -w_q. \end{cases}$$

If $u \in q\mathbb{Z}_q^\times$, then $c_q^s[\lambda'_q](u) = 1$. Thus $U_q(e, f_\chi)$ indeed consists of a single element c_q and $c_q(\xi) \neq 0$ if and only if ξ satisfies condition (a) of the proposition.

Case B: If $q \neq 2, q \mid N'$, we are in Case (1) of [37]: $m_q = 2, \underline{\lambda}'_q = 0, e = \tilde{n}_q = 2$. Let ϵ be a unit in $\mathbb{Z}_q$ which is not a square. Note that $\chi_q(-1) = \chi_{0,q}(-1)$. By [37] Prop. 19, p. 480,

$$\omega_q(f_\chi) = \begin{cases} (\mathbb{Q}_q^\times/(\mathbb{Q}_q^\times)^2) \setminus (-1)^\tau\epsilon(\mathbb{Q}_q^\times)^2 \text{ if } \chi_q(-1) = \chi_{0,q}(-1) = 1, w_q = 1, \\ (\mathbb{Q}_q^\times/(\mathbb{Q}_q^\times)^2) \setminus (-1)^\tau(\mathbb{Q}_q^\times)^2 \text{ if } \chi_q(-1) = \chi_{0,q}(-1) = 1, w_q = -1, \\ (-1)^\tau\epsilon(\mathbb{Q}_q^\times)^2 \text{ if } \chi_q(-1) = \chi_{0,q}(-1) = -1, w_q = 1, \\ (-1)^\tau(\mathbb{Q}_q^\times)^2 \text{ if } \chi_q(-1) = \chi_{0,q}(-1) = -1, w_q = -1. \end{cases}$$

Hence $U_q(e, f_\chi) = \{\gamma[0,\nu]; \nu \in \omega_q(f_\chi),\ v_q(\nu) \equiv 0(2)\} = \{\gamma[0,u], v_q(u) = 0, ((-1)^\tau u, q)_q = \chi_{0,q}(-1) \cdot w_q\}$. Thus $U_q(e, f_\chi)$ consists of a single element c_q and $c_q(\xi) \neq 0$ if and only if ξ satisfies condition (b) of the proposition.

Case C: $q = 2$. We are now in Case (8) of [37]: $m_2 = 0, \tilde{n}_2 = 2$ and we only need to consider $e = 2$. If $\alpha_2 \neq \alpha'_2$, $U_2(e, f_\chi)$ consists of two elements $\delta_1 = c'_2[\alpha_2], \delta_2 = c'_2[\alpha'_2]$. If $c = \delta_1 - \delta_2$, one checks that $c(u) = 0$ unless $(-1)^\tau u \equiv 0, 1 \mod 4$, and that any linear combination of δ_1, δ_2 with this property must be a scalar multiple of c. If $\alpha_2 = \alpha'_2 = \alpha$, say, $U_2(e, f_\chi)$ consists again of two elements $\gamma_1 = c'_2[\alpha], \gamma_2 = c''_2[\alpha]$. Now one checks that γ_2 satisfies $\gamma_2(u) = 0$ unless $(-1)^\tau u \equiv 0, 1 \mod 4$, and that this is the only linear combination of γ_1 and γ_2 with this property. ■

3. Explicit theta correspondence

3.1. **Theta correspondence for the pair $(\widetilde{\mathrm{SL}}_2, PB^\times)$.** Let ψ' be any character of $\mathbb{Q}\setminus\mathbb{A}$. Let $V \subset B$ be the subspace of trace 0 elements, thought of as a quadratic space with $Q(x) = -\mathrm{Nm}(x)$ and let $\langle , \rangle$ denote the associated bilinear form, $\langle x, y\rangle = -(xy^i + yx^i)$, i being the main involution. The metaplectic cover $\widetilde{\mathrm{Sp}}(W \otimes V)$ splits over the orthogonal group $\mathrm{O}(V)$ whose identity component is identified with $PB^\times$, the action of $\beta \in PB^\times$ on V being given by $R(\beta)(v) = \beta v \beta^{-1}$. Thus, for each place v of $\mathbb{Q}$, the Weil representation of $\widetilde{\mathrm{Sp}}(W \otimes V)_v$ yields a representation of $\widetilde{\mathrm{S}}_v \times PB_v^\times$ on $\mathcal{S}_{\psi'}(V \otimes \mathbb{Q}_v)$ denoted $\omega_{\psi'}$. The restriction of $\omega_{\psi'}$ to $\widetilde{\mathrm{S}}_v$ is a genuine representation of $\widetilde{\mathrm{S}}_v$, denoted $r_{\psi'}$, satisfying

$$r_{\psi'}(\mathbf{n})\varphi(x) = \psi'(nQ(x))\varphi(x), \tag{3.1}$$

$$r_{\psi'}(\mathbf{d}(a))\varphi(x) = \mu_{\psi'}(a)(a,-1)_v|a|^{3/2}\varphi(ax), \tag{3.2}$$

$$r_{\psi'}(w,\epsilon)\varphi(x) = \epsilon\gamma_{\psi',Q}\mathcal{F}_{\psi'}(\varphi). \tag{3.3}$$

where we write ψ' instead of ψ'_v and the sign in (3.3) is $+$ or $-$ depending on whether v is unramified or ramified in B. The Haar measure on $V \otimes \mathbb{Q}_v$ is picked to be autodual with respect to the pairing $(x_1, x_2) \mapsto \psi'(\langle x_1, x_2\rangle)$. Further, $\omega_{\psi'}(\sigma, \beta) = r_{\psi'}(\sigma)R(\beta)$, where $R(\beta)\varphi(x) = \varphi(\beta^{-1}x\beta)$.

For $s \in \mathcal{A}_0$, $t \in \tilde{\mathcal{A}}_0$, $\varphi \in \mathcal{S}_{\psi'}(V_{\mathbb{A}})$, define

$$\begin{aligned}\theta(\psi', \varphi, \sigma, \beta) &= \sum_{x \in V} r_{\psi'}(\sigma)R(\beta)\varphi(x),\\ t_{\psi'}(\varphi, \sigma, s) &= \int_{PB_{\mathbb{Q}}^\times \backslash PB_{\mathbb{A}}^\times} \theta(\psi', \varphi, \sigma, \beta)s(\beta)d^\times\beta,\\ T_{\psi'}(\varphi, \beta, t) &= \int_{\mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A})} \overline{\theta(\psi', \varphi, \sigma, \beta)}t(\sigma)d\sigma.\end{aligned}$$

If $\mathcal{V}$, $\tilde{V}$ denote representations of the Hecke algebra of $PB_{\mathbb{A}}^\times, \widetilde{\mathrm{S}}_{\mathbb{A}}$ respectively, we set

$$\begin{aligned}\Theta(\mathcal{V}, \psi') &= \{t_{\psi'}(\varphi, \cdot, s); s \in \mathcal{V}, \varphi \in \mathcal{S}_{\psi'}(V_{\mathbb{A}})\},\\ \Theta(\tilde{V}, \psi') &= \{T_{\psi'}(\varphi, \cdot, t); t \in \tilde{V}, \varphi \in \mathcal{S}_{\psi'}(V_{\mathbb{A}})\},\end{aligned}$$

these being representation spaces for the Hecke algebras of $\widetilde{\mathrm{S}}_{\mathbb{A}}, PB_{\mathbb{A}}^\times$ respectively. If we need to work with $PB^\times$ and PGL_2 simultaneously, we write Θ_0 instead of Θ for the lifts between $\widetilde{\mathrm{SL}_2}$ and PGL_2 to distinguish these from the lifts between $\widetilde{\mathrm{SL}_2}$ and $PB^\times$.

Let ν be an odd quadratic discriminant, $\delta = \nu/|\nu|$ and set $\psi' = \psi^{1/|\nu|} = \psi^{\delta/\nu}$. (In future we will write $\mathcal{F}(\varphi)$ for $\mathcal{F}_{\psi'}(\varphi)$. with this choice of ψ'.) Also let τ be such that $\delta = (-1)^\tau$. For f as in the previous section let π denote the automorphic representation of PGL_2 corresponding to f and π' the corresponding representation of $PB^\times$ associated by Jacquet-Langlands. Thus $\pi' = \pi_g$ for a newform $g \in S_{2k}(\Gamma)$. We normalize g as in Sec. 2.2.6. We now compute the central character of $\tilde{\pi} := \Theta(\pi' \otimes \chi_\nu, \psi')$ using results in [38].

Lemma 3.1. *Let γ_q be defined by $\varepsilon(\pi_q \otimes \chi_{\nu,q}, 1/2) = \gamma_q\chi_{\nu,q}(-1)\varepsilon(\pi_q, 1/2)$. Then*

$$\gamma_q = \begin{cases} 1 \text{ if } q \nmid N,\\ \chi_{\nu,q}(q) \text{ if } q \mid N, q \nmid \nu,\\ w_q \text{ if } q \mid N, q \mid \nu,\\ \mathrm{sign}(\nu) \text{ if } q = \infty.\end{cases}$$

Proof: For $q \nmid N$, this is easy to check. Assume $q \mid N$. Let $\pi_q \simeq \sigma(\mu, \mu^{-1})$ and $\{1, \eta, q, \eta q\}$ with η a unit in $\mathbb{Z}_q$ be a set of coset representatives for $\mathbb{Q}_q^\times/(\mathbb{Q}_q^\times)^2$. If $\mu \neq |\cdot|^{1/2}$, then $w_q = 1$, and $\mathbb{Q}_q(\pi_q) = \mathbb{Q}_q^\times \setminus \eta(\mathbb{Q}_q^\times)^2$ by [38] Lemme 1, p. 226. (We refer the reader to the same article for the definition of $\mathbb{Q}_v(\pi_v)$.) If $\mu = |\cdot|^{1/2}$, then $w_q = -1$ and $\mathbb{Q}_q(\pi_q) = (\mathbb{Q}_q^\times)^2$ by the same lemma. Finally, $\mathbb{Q}_\infty(\pi_\infty) = \mathbb{R}_+^*$. By [38] Thm. 2,

$$\varepsilon(\pi_q \otimes \chi_{\nu,q}, 1/2) = \pm\chi_{\nu,q}(-1)\varepsilon(\pi_q, 1/2),$$

where the + or − sign holds according as $\nu \in \mathbb{Q}_q(\pi_q)$ or not. The lemma follows. ■

Proposition 3.2. *Let* $\alpha_q := \pm 1$ *according as* $q \mid N^+$ *or* $q \mid N^-$. *Then*

$$\tilde{\pi}_q(-\mathbf{1}) = \begin{cases} 1 \text{ if } q \nmid 2N, \\ \chi_{\nu,q}(q)w'_q \text{ if } q \mid N, q \nmid \nu, \\ \alpha_q \text{ if } q \mid N, q \mid \nu, \\ (-1)^k i \text{ if } q = \infty, \\ -\delta i \text{ if } q = 2. \end{cases}$$

Proof: Let $\tilde{\pi}_q = \Theta(\pi'_q \otimes \chi_{\nu,q}, \psi'_q)$. For convenience of notation we drop the subscript q in the equations below.

$$\begin{aligned} \tilde{\pi}(-\mathbf{1}) &= \varepsilon(\tilde{\pi}, \psi')\mu_{\psi'}(-1) \\ &= \varepsilon(\Theta(\pi' \otimes \chi_\nu, \psi'), \psi') \cdot (\delta\nu, -1) \cdot \mu_\psi(-1) \\ &= \alpha \cdot (\delta\nu, -1) \cdot \mu_\psi(-1) \cdot \varepsilon(\Theta(\pi \otimes \chi_\nu, \psi'), \psi') \text{ ([37] Thm. 2, p. 277)} \\ &= \alpha \cdot (\delta\nu, -1) \cdot \mu_\psi(-1) \cdot \varepsilon(\pi \otimes \chi_\nu, 1/2) \text{ ([37] Lemme 6, p. 234)} \\ &= \alpha \cdot \gamma \cdot (\delta, -1) \cdot \mu_\psi(-1) \cdot \varepsilon(\pi, 1/2) = \alpha \cdot \gamma \cdot (\delta, -1) \cdot \mu_\psi(-1) \cdot w \\ &= \gamma \cdot (\delta, -1) \cdot \mu_\psi(-1)w'. \end{aligned}$$

Note that for $q = 2$, $\gamma = 1, \mu_\psi(-1) = -i$ and $w'_2 = w_2 = 1$. The proposition is now immediate from the preceding lemma. ■

We can now show that the form h_χ can be constructed as a theta lift from $PB^\times$. Indeed, we have the following proposition.

Proposition 3.3. *Suppose that* $L(\frac{1}{2}, \pi \otimes \chi_\nu) \neq 0$ *and that* $\boldsymbol{\chi}$ *is a character of conductor dividing* $4N$ *with* $\boldsymbol{\chi}(-1) = 1$ *and satisfying the following conditions:*
(a) If $q \mid N, q \nmid \nu$, $\chi_{0,q}(-1) = \chi_{\nu,q}(q)w'_q (= \alpha_q \chi_{\nu,q}(q)w_q)$.
(b) If $q \mid N, q \mid \nu$, $\chi_{0,q}(-1) = \alpha_q$.
Then for $\chi = \boldsymbol{\chi} \cdot \left(\frac{-1}{\cdot}\right)^{k+\tau}$, *one has that* χ *is unramified at* 2 *and* $S_{k+\frac{1}{2}}(M, \boldsymbol{\chi}, f_\chi) \subseteq \Theta(\pi' \otimes \chi_\nu, \psi')$. *In fact if* $\tilde{\pi}$ *denotes this last representation, we have* $S_{k+\frac{1}{2}}(M, \boldsymbol{\chi}, f_\chi) = S_{k+\frac{1}{2}}(M, \boldsymbol{\chi}, \tilde{\pi})$ *(notation as in* [37] *p.391.)*

Proof: We shall see below that χ is unramified at 2 and hence $S_{k+\frac{1}{2}}(M, \boldsymbol{\chi}, f_\chi)$ is one dimensional by Prop. 2.5. Assuming this for the moment, let h be any non zero form in $S_{k+\frac{1}{2}}(M, \boldsymbol{\chi}, f_\chi)$ and denote by T the automorphic representation of $\widetilde{\mathrm{S}}_{\mathbb{A}}$ generated by h. By [37] Prop.4 (p. 391), $\mathcal{V}'(\psi, T) = V_0 \otimes \chi_0^{-1}$ where V_0 is the automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ generated by f_χ. (See [36], p.99 for the definition of $\mathcal{V}'(\psi, T)$.) If $\tilde{V}$ is the automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$ generated by f, we see that $V_0 \otimes \chi_0^{-1} = \tilde{V} \otimes \chi_{-1}^\tau$. By the definition of $\mathcal{V}'(\psi, T)$, there exists $\alpha \in \mathbb{Q}^\times$ such that $\Theta_0(T, \psi^\alpha) = \tilde{V} \otimes \chi_{-1}^\tau \otimes \chi_\alpha$. (Here Θ_0 denotes the lift to PGL_2.) Hence $\Theta_0(\tilde{V} \otimes \chi_{-1}^\tau \otimes \chi_\alpha, \psi^\alpha) = T$. Then $\tilde{\pi} = \Theta(\pi' \otimes \chi_{-1}^\tau \otimes \chi_{|\nu|}, \psi^{|\nu|}) = \Theta(\pi' \otimes \chi_\nu, \psi')$ is non-zero by [38] Prop. 22, p.295 and is in the same

Waldspurger packet as T. By [38] Thm. 3, p. 381, to show that $\tilde{\pi} = T$, it suffices to show that their central characters agree i.e. that the central character of h is equal to the central character of $\tilde{\pi}$. This is clear at the finite places q, $q \neq 2$ and for $q = \infty$ from the previous proposition and from conditions (a) and (b). For $q = 2$ one notes that

$$\begin{aligned} \varepsilon(\pi \otimes \chi_\nu, \frac{1}{2}) &= \prod_q \varepsilon(\pi_q \otimes \chi_{\nu,q}, \frac{1}{2}) \\ &= \prod_{q|N, q\nmid\nu} (\nu, -q)_q w_q \cdot \prod_{q|N,q|\nu} (-1, q)_q \cdot \chi_{\nu,2}(-1) \cdot (-1)^k \\ &= \prod_{q|N, q\nmid\nu} (\nu, -q)_q \alpha_q w_q \cdot \prod_{q|N,q|\nu} (-1, q)_q \alpha_q \cdot \chi_{\nu,2}(-1) \cdot (-1)^k \\ &= \prod_{q|N} \chi_{0,q}(-1) \cdot \prod_{q|2\nu} \chi_{\nu,q}(-1) \cdot \chi_{0,\infty}(-1) = \chi_{0,2}(-1) \cdot \chi_{\nu,\infty}(-1) \end{aligned}$$

Since $L(\frac{1}{2}, \pi \otimes \chi_\nu) \neq 0$, $\chi_{0,2}(-1) = \chi_{\nu,\infty}(-1) = \delta$. Thus $\tilde{\pi}(-1) = -\delta i = \tilde{\varepsilon}_2(-1)\chi_{0,2}(-1)$, as required. This shows that $T = \tilde{\pi}$ and hence $S_{k+\frac{1}{2}}(M, \boldsymbol{\chi}, f_\chi) \subseteq S_{k+\frac{1}{2}}(M, \boldsymbol{\chi}, \tilde{\pi})$. The other inclusion follows from [37] Prop. 4 (ii) (p. 391) since $\mathcal{V}'(\psi, \tilde{\pi}) = V_0 \otimes \chi_0^{-1}$. Finally, note that since $\chi = \chi_0 \cdot \left(\frac{-1}{\cdot}\right)^\tau$, $\chi_2(-1) = 1$ and χ is unramified at 2 as promised earlier in the proof. ■

In Sec. 3.2 we shall pick an explicit Schwartz function $\varphi \in \mathcal{S}_{\psi'}(V_{\mathbb{A}})$ and a vector $s \in \pi' \otimes \chi_\nu$ such that $t_{\psi'}(\varphi, \cdot, s)$ equals (some multiple of) h_χ.

3.2. **Explicit theta functions.** For $q \mid N^-$, let L_q be the unique unramified extension of $\mathbb{Q}_q$ of degree 2, π a uniformizer in $\mathbb{Z}_q$ and B_π be the quaternion algebra given by

$$\begin{aligned} B_\pi &= L_q + L_q u \\ um &= \overline{m} u \text{ for } m \in L \\ u^2 &= \pi \end{aligned}$$

Fix once and for all an isomorphism $B \otimes \mathbb{Q}_q \simeq B_\pi$. This isomorphism must necessarily identify $\mathcal{O}' \otimes \mathbb{Z}_q$ with $R_q + R_q u$, where R_q is the ring of integers of L_q. Also fix once and for all a unit $\omega \in R_q$ with $\omega^2 \in \mathbb{Z}_q$, such that $R_q = \mathbb{Z}_q + \mathbb{Z}_q \omega$. Hence $R_q^0 = \mathbb{Z}_q \omega$.

Let $\boldsymbol{\chi}, \nu, \chi_0, \chi, \psi'$ be as in the previous section. Let s_{g_χ} be a newform in $\pi' \otimes \chi = \pi_g \otimes \chi$, normalized as in Sec. 2.2.6, and $s_{g,\chi}$ the unique element of π_g such that $s_{g,\chi}(\beta) \cdot \chi(\mathrm{Nm}(\beta)) = s_{g_\chi}(\beta)$ i.e. $s_{g,\chi} \otimes (\chi \circ \mathrm{Nm}) = g_\chi$. Also set $s = s_{g,\chi} \otimes (\chi_\nu \circ \mathrm{Nm}) \in \pi_g \otimes \chi_\nu$.

We now make the following choice of Schwartz function $\varphi = \varphi_{f,\chi,\nu} \in \mathcal{S}_{\psi'}(V_{\mathbb{A}})$: $\varphi = \prod_v \varphi_v$ where:

(a) If q is odd and $q \nmid \nu N^+ N^-$, $\varphi_q = \mathbb{I}_{\{x \in \mathcal{O}' \otimes \mathbb{Z}_q, tr(x) = 0\}}$.

(b) If $q \mid \nu$, $q \nmid N$, $\varphi_q \begin{pmatrix} b & -a \\ c & -b \end{pmatrix} = 0$, unless $a, b, c \in \mathbb{Z}_q, b^2 - ac \in q\mathbb{Z}_q$, in which case

$$\varphi_q \begin{pmatrix} b & -a \\ c & -b \end{pmatrix} = \begin{cases} \chi_{\nu,q}(a) (\text{resp. } \chi_{\nu,q}(c)), \text{ if } v_q(a) = 0 \text{ (resp. } v_q(c) = 0), \\ 0, \text{ otherwise i.e. if both } v_q(a) = 0 \text{ and } v_q(c) = 0. \end{cases}$$

(c1) If $q \mid N^+$, $q \nmid \nu$, and $\chi_{0,q}$ is unramified, $\varphi_q = \mathbb{I}_{\{x \in \mathcal{O}' \otimes \mathbb{Z}_q, tr(x) = 0\}}$.

(c2) If $q \mid N^+$, $q \nmid \nu$, and $\chi_{0,q}$ is ramified, $\varphi_q \begin{pmatrix} b & -a \\ c & -b \end{pmatrix} = 0$, unless $a \in \frac{1}{q}\mathbb{Z}_q, b \in \mathbb{Z}_q, c \in q^2\mathbb{Z}_q$ in which case

$$\varphi_q \begin{pmatrix} b & -a \\ c & -b \end{pmatrix} = \begin{cases} (\chi_{\nu,q}\chi_{0,q}^{-1})(a') = \chi_{0,q}^{-1}(a') \text{ if } v_q(a) = -1, a = a'/q, \\ 0, \text{ if } v_q(a) \geq 0. \end{cases}$$

(c3) If $q \mid N^+, q \mid \nu$, and $\chi_{0,q}$ is unramified, $\varphi_q \begin{pmatrix} b & -a \\ c & -b \end{pmatrix} = 0$, unless $a \in \mathbb{Z}_q, b \in q\mathbb{Z}_q, c \in q\mathbb{Z}_q$ in which case

$$\varphi_q \begin{pmatrix} b & -a \\ c & -b \end{pmatrix} = \begin{cases} (\chi_{\nu,q}\chi_{0,q}^{-1})(a) = \chi_{\nu,q}(a) \text{ if } v_q(a) = 0, \\ 0, \text{ if } v_q(a) \geq 1. \end{cases}$$

(d1) If $q \mid N^-$, $q \nmid \nu$, $\varphi_q(a + bu) = 0$ unless $a \in R_q^0, b \in R_q$ in which case

$$\varphi_q(a + bu) = \begin{cases} (\chi_{\nu,q}\chi_{0,q}^{-1})(a'), \text{if } \chi_{0,q} \text{ is ramified, and } v_q(a) = 0, \\ 0, \text{ if } \chi_{0,q} \text{ is ramified and } v_q(a) \geq 1, \\ 1, \text{ if } \chi_{0,q} \text{ is unramified.} \end{cases}$$

(d2) If $q \mid N^-$, $q \mid \nu$, set $\pi = \nu\epsilon$ where ϵ is chosen to be a unit in $\mathbb{Z}_q^\times$ with $(\epsilon, q) = w_q' = -w_q$. Then $\varphi_q(a + bu) = 0$ unless $a \in qR_q$, $\mathrm{Nm}(b) \in (\mathbb{Z}_q^\times)^2$. In that case, write $b = c \cdot \frac{\bar{e}}{e}$ for any $c \in \mathbb{Z}_q^\times$ and $e \in R_q^\times$. Then set $\varphi_q(a+bu) = (\chi_{\nu,q}\chi_{0,q}^{-1})(c) \cdot \chi_\nu(\mathrm{Nm}(e))$. If $b = c \cdot \frac{\bar{e}}{e} = c_1 \cdot \frac{\overline{e_1}}{e_1}$, then setting $e' = e_1/e$, $c' = c_1/c$, we see that $c' = \overline{e'}/e'$, hence $(c')^2 = \mathrm{Nm}(c') = 1 \Rightarrow c' = \pm 1$. If $c' = 1$, then $\overline{e'} = e' \Rightarrow e' \in \mathbb{Z}_q \Rightarrow \chi_{\nu,q}(\mathrm{Nm}(e')) = 1$. If $c' = -1$, $(\chi_{\nu,q}\chi_{0,q}^{-1})(c') = -\chi_{\nu,q}(-1)$. Also $\overline{e'} = -e' \Rightarrow e' \in \mathbb{Z}_q^\times \omega \Rightarrow \mathrm{Nm}(e') \in (\mathbb{Z}_q^\times)^2 \,\mathrm{Nm}(\omega) = -(\mathbb{Z}_q^\times)^2\omega^2 \Rightarrow \chi_{\nu,q}(\mathrm{Nm}(e')) = \chi_{\nu,q}(-\omega^2) = -\chi_{\nu,q}(-1)$. In any case, we see that φ_q is well defined, i.e. independent of the choice of decomposition $b = c \cdot \frac{\bar{e}}{e}$. Further, by a similar argument we may check that for $a \in qR_q$, $\varphi_q(a + bu)$ depends only on the congruence class of $b \mod q$.

(e) $q = 2$. Set

$$\varphi_2 \begin{pmatrix} b & -a \\ c & -b \end{pmatrix} = \mathbb{I}_{\mathbb{Z}_2}(b)\mathbb{I}_{2\mathbb{Z}_2}(a)\mathbb{I}_{2\mathbb{Z}_2}(c).$$

(f) If $q = \infty$, set

$$\varphi_\infty \begin{pmatrix} b & -a \\ c & -b \end{pmatrix} = \frac{\pi}{|\nu|^{1/2}}(a - 2ib - c)^k e^{-\frac{2\pi}{|\nu|}(\frac{a^2}{2}+b^2+\frac{c^2}{2})}.$$

The choice of Schwartz function is crucial to what follows and is inspired by computations in Shintani [33], Kohnen [18] and Waldspurger [37].

Proposition 3.4. *Let $t' = t_{\psi'}(\varphi, \sigma, s)$. We have*

(1) $t' \in \tilde{\mathcal{A}}_{k+\frac{1}{2}}(M, \chi_0)$.
(2) *Let $h' \in S_{k+\frac{1}{2}}$ be such that $t' = t_{h'}$. Then $h' \in S_{k+\frac{1}{2}}(M, \boldsymbol{\chi}, f_\chi)$.*

Proof: It suffices to show that $t' \in \tilde{\mathcal{A}}_{k+\frac{1}{2}}(M, \chi_0)$. For then from the result of Prop. 3.3, $t' = t_{h'} \in \tilde{\mathcal{A}}_{k+\frac{1}{2}}(M, \chi_0) \cap \tilde{\pi}$, hence $h' \in S_{k+\frac{1}{2}}(M, \boldsymbol{\chi}, \tilde{\pi}) = S_{k+\frac{1}{2}}(M, \boldsymbol{\chi}, f_\chi)$. Let D denote the usual Casimir operator for $\mathrm{PGL}_2(\mathbb{R})$. By [36] Lemma 42, p.73-74, $R_\infty(D)\varphi_\infty =$

$4r_{\psi',\infty}(\tilde{D})\varphi_\infty + \frac{3}{2}\varphi_\infty$, hence $r_{\psi',\infty}(\tilde{D})(t') = [\kappa(\kappa-4)/8]t'$. It is enough then to check (i) - (iv) below.
(i) If $q \nmid M$ and $\sigma \in \Gamma_q$, $r_{\psi',q}(\sigma)(\varphi_q) = \varphi_q$;
(ii) If $q \mid M$, $q \neq 2$ and $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_q(v_q(M))$, $r_{\psi',q}(\sigma)\varphi_q = \chi_{0,q}(d)\varphi_q = \chi_q(d)\varphi_q$;
(iii) If $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_2(v_2(M))$, $r_{\psi',2}(\sigma)\varphi_2 = \chi_{0,2}(d)\varphi_2$;
(iv) If $\theta \in \mathbb{R}$, $r_{\psi',\infty}(\tilde{\kappa}(\theta))(\varphi_\infty) = e^{i\kappa\theta/2}\varphi_\infty$.

We may verify these using (3.1) - (3.3). We begin with the following observation which will be used repeatedly in what follows: for $n \geq 1$,

(3.4) $\quad \Gamma_q(n)$ is generated by $\overline{\mathbf{n}}(x), \mathbf{d}(\alpha), \underline{\mathbf{n}}(y)$, for $x \in \mathbb{Z}_q, \alpha \in \mathbb{Z}_q^\times, y \in q^n\mathbb{Z}_q$,

and for $n = 0$,

(3.5) $\quad \Gamma_q(0) = \Gamma_q$ is generated by $\overline{\mathbf{n}}(x), \mathbf{d}(\alpha), w$, for $x \in \mathbb{Z}_q, \alpha \in \mathbb{Z}_q^\times$.

(i) This is immediate for $q \nmid \nu M$ by (3.5), noting that $\mathcal{F}\varphi_q = \varphi_q$ for $q \nmid \nu M$. For $q \mid \nu$, $q \nmid N$, one first computes $\mathcal{F}\varphi_q$:

$$\mathcal{F}\varphi_q \begin{pmatrix} b & -a \\ c & -b \end{pmatrix} = q^{3/2} \int_{\mathbb{Z}_q^3} \varphi_q \begin{pmatrix} y & -x \\ z & -y \end{pmatrix} \psi'(2by - az - cx) dxdydz.$$

Let $\overline{a} = \begin{pmatrix} b & -a \\ c & -b \end{pmatrix}$ and $\overline{x} = \begin{pmatrix} y & -x \\ z & -y \end{pmatrix}$. Since φ_q is invariant under the transformation $x \mapsto x+q, y \mapsto y, z \mapsto z$ and under the symmetric transformations sending $y \mapsto y+q$ and $z \mapsto z+q$, one sees that $\mathcal{F}\varphi_q(\overline{a}) = 0$ unless $a, b, c \in \mathbb{Z}_q$. Thus letting $a, b, c \in \mathbb{Z}_q$,

$$\begin{aligned} \mathcal{F}\varphi_q(\overline{a}) &= q^{3/2} \sum_{\alpha,\beta,\gamma \in \mathbb{Z}/q\mathbb{Z}} \int_{x,y,z \in \mathbb{Z}_q, x \equiv \alpha, y \equiv \beta, z \equiv \gamma (q)} \varphi_q(\overline{x}) \psi'(2by - az - cx) dxdydz \\ &= q^{-3/2} \sum_{\alpha,\beta,\gamma \in \mathbb{Z}/q\mathbb{Z}} \psi'(2b\beta - a\gamma - c\alpha) \varphi_q \begin{pmatrix} \beta & -\alpha \\ \gamma & -\beta \end{pmatrix} \\ &= q^{-3/2} \Bigg[\sum_{\alpha \in (\mathbb{Z}/q\mathbb{Z})^\times} \psi'(-c\alpha)\varphi_q \begin{pmatrix} 0 & -\alpha \\ 0 & 0 \end{pmatrix} + \sum_{\gamma \in (\mathbb{Z}/q\mathbb{Z})^\times} \psi'(-a\gamma)\varphi_q \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix} \\ &\qquad + \sum_{\alpha,\beta,\gamma \in (\mathbb{Z}/q\mathbb{Z})^\times} \psi'(2b\beta - a\gamma - c\alpha)\varphi_q \begin{pmatrix} \beta & -\alpha \\ \gamma & -\beta \end{pmatrix} \Bigg] \\ &= q^{-3/2} \Bigg[\sum_{\gamma \in (\mathbb{Z}/q\mathbb{Z})^\times} \psi'(-a\gamma)\chi_{\nu,q}(\gamma) + \sum_{\alpha \in (\mathbb{Z}/q\mathbb{Z})^\times} \psi'(-c\alpha)\chi_{\nu,q}(\alpha) \\ &\qquad + \sum_{\alpha,\delta \in (\mathbb{Z}/q\mathbb{Z})^\times} \psi'(2b\alpha\delta - a\alpha\delta^2 - c\alpha)\chi_{\nu,q}(\alpha) \Bigg] \\ &= q^{-3/2} G(\varrho, \psi') \left[\varrho(-a) + \varrho(-c) + \sum_{\delta \in (\mathbb{Z}/q\mathbb{Z})^\times} \varrho(-a\delta^2 + 2b\delta - c) \right], \end{aligned}$$

where ϱ denotes the unique nontrivial quadratic character of $(\mathbb{Z}/q\mathbb{Z})^\times$. Using the fact that

$$\sum_{\delta\in\mathbb{Z}/q\mathbb{Z}} \varrho(\delta^2+x) = \begin{cases} -1, & \text{if } x\neq 0,\\ q-1, & \text{if } x=0,\end{cases}$$

we see that $\mathcal{F}\varphi_q = \varrho(-1)q^{-1/2}G(\varrho,\psi')\varphi_q$. Now, from (2.9),

$$r_{\psi'}(w)(\varphi_q) = \gamma_{\psi'^{-1}}\gamma_{\psi'}^2\cdot\varrho(-1)q^{-1/2}G(\varrho,\psi')\varphi_q = \varphi_q.$$

Since $q\neq 2$, $r_{\psi'}(\mathbf{d}(\alpha))(\varphi_q) = \gamma_{\psi'}(\alpha)\chi_{\nu,q}^{-1}(\alpha)\varphi_q = \varphi_q$ (by (2.8).) Finally, $r_{\psi'}(\overline{\mathbf{n}}(x))\varphi_q = \varphi_q$ for $x\in\mathbb{Z}_q$. Thus, φ_q is indeed invariant under Γ_q as required.

(ii) We need to work through the cases (c1)-(c4) and (d1)-(d2).
Case (c1): $q\mid N^+$, $q\nmid\nu$ and $\chi_{0,q}$ unramified;

$$\mathcal{F}\varphi_q\begin{pmatrix} b & -a\\ c & -b\end{pmatrix} = \begin{cases} 1, & \text{if } v_q(a)\geq -1, v_q(b)\geq 0, v_q(c)\geq 0,\\ 0, & \text{otherwise.}\end{cases}$$

Thus $\mathcal{F}\varphi_q$ is invariant by $\underline{\mathbf{n}}(y)$ for $y\in q\mathbb{Z}_q$, hence using (2.1) and (3.4) one sees that φ_q is invariant by $\Gamma_q(1)$.
Case (c2): $q\mid N^+$, $q\nmid\nu$ and $\chi_{0,q}$ ramified;

$$\mathcal{F}\varphi_q\begin{pmatrix} b & -a\\ c & -b\end{pmatrix} = \begin{cases} G(\hat{\chi}_q^{-1},(\psi')^c), & \text{if } v_q(c)=0, v_q(b)\geq 0, v_q(a)\geq -2,\\ 0, & \text{otherwise.}\end{cases}$$

Thus $\mathcal{F}\varphi_q$ is invariant by $\underline{\mathbf{n}}(x)$ for $v_q(x)\geq 2$. Since $r_{\psi'}(\mathbf{d}(\alpha))(\varphi_q) = \chi_{0,q}^{-1}(\alpha)\varphi_q$ and φ_q is invariant by $\overline{\mathbf{n}}(x)$ for $x\in\mathbb{Z}_q$, we see that φ_q transforms as required under $\Gamma_q(2)$.
Case (c3): $q\mid N^+$, $q\mid\nu$ and $\chi_{0,q}$ unramified;

$$\mathcal{F}\varphi_q\begin{pmatrix} b & -a\\ c & -b\end{pmatrix} = \begin{cases} G(\hat{\chi}_{\nu,q}^{-1},(\psi')^c), & \text{if } v_q(c)=0, v_q(b)\geq 0, v_q(a)\geq 0,\\ 0, & \text{otherwise.}\end{cases}$$

$\mathcal{F}(\varphi_q)$ is invariant by $\underline{\mathbf{n}}(x)$ for $v_q(x)\geq 1$. Since φ_q is invariant by $\overline{\mathbf{n}}(x)$ for $x\in\mathbb{Z}_q$ and $r_{\psi'}(\mathbf{d}(\alpha))(\varphi_q) = \chi_{\nu,q}(\alpha)\varphi_q(\alpha\cdot) = \varphi_q$, we see that φ_q transforms as required under $\Gamma_q(1)$.
Case (d1): For $q\mid N^-, q\nmid\nu$ and $\chi_{0,q}$ unramified, φ_q is invariant under $\mathbf{d}(\alpha)$ and $\overline{\mathbf{n}}(x)$, $x\in\mathbb{Z}_q$. Since

$$\mathcal{F}\varphi_q(a+bu) = \mathbb{I}_{R_q^0+\frac{1}{q}R_q}(a+bu),$$

we see that $r_{\psi',q}(\overline{\mathbf{n}}(-y))\mathcal{F}\varphi_q = \mathcal{F}\varphi_q$ for $y\in q\mathbb{Z}_q$, and consequently, φ_q is invariant under $\Gamma_q(1)$. Next let $q\mid N^-, q\nmid\nu$ with $\chi_{0,q}$ ramified. Clearly φ_q is invariant under $\overline{\mathbf{n}}(x)$ and transforms under $\mathbf{d}(\alpha)$ by $\chi_{0,q}(\alpha^{-1})$. One easily computes that

$$\mathcal{F}\varphi(a+bu) = \begin{cases} G(\hat{\chi}_q^{-1},(\psi')^{2a'\omega^2}), & \text{if } a\in \frac{1}{q}R_q^0\setminus R_q^0, a=\frac{a'}{q}\omega, b\in\frac{1}{q}R_q,\\ 0, & \text{otherwise.}\end{cases}$$

so that $r_{\psi',q}(\overline{\mathbf{n}}(-y))\mathcal{F}\varphi_q = \mathcal{F}\varphi_q$ for $y\in q^2\mathbb{Z}_q$, which shows that φ_q transforms as required under $\Gamma_q(2)$.
Case (d2): $q\mid N^-, q\mid\nu$. In this case, necessarily $\chi_{0,q}$ is ramified since $\chi_{0,q}(-1)=-1$. φ_q is invariant under $\overline{\mathbf{n}}(x), x\in\mathbb{Z}_q$ and transforms under $\mathbf{d}(\alpha)$ by $\chi_{0,q}^{-1}(\alpha)$. One checks also that $\mathcal{F}\varphi_q(a+bu)=0$ unless $a\in R_q$, $b\in\frac{1}{q}R_q$. Thus $r_{\psi',q}(\overline{\mathbf{n}}(-y))\mathcal{F}\varphi_q = \mathcal{F}\varphi_q$ for $y\in q^2\mathbb{Z}_q$, whence φ_q transforms in the required manner under $\Gamma_q(2)$.

(iii) We have $\tilde{\epsilon}_2(\overline{\mathbf{n}}(x)) = \tilde{\epsilon}_2(\underline{\mathbf{n}}(y)) = 1$ for $x \in \mathbb{Z}_2, y \in 2^2\mathbb{Z}_2$. Also $\tilde{\epsilon}_2(\mathbf{d}(\alpha)) = \mu_\psi(\alpha^{-1})$, and $r_{\psi'}(\mathbf{d}(\alpha))\varphi_2(x) = \mu_{\psi'}(\alpha)^3\varphi_2(\alpha x)$. Note that

$$\mu_{\psi'}(\alpha)^3 = ((\nu_0,\alpha)_2\mu_\psi(\alpha))^3 = (\nu_0,\alpha)_2\mu_\psi(\alpha^{-1}).$$

Thus in any case $r_{\psi'}(\mathbf{d}(\alpha))\varphi_2 = \tilde{\epsilon}_2(\mathbf{d}(\alpha))(\nu_0,\alpha)\varphi_2(\alpha\cdot) = \tilde{\epsilon}_2(\mathbf{d}(\alpha))(\nu_0,\alpha)\chi_{\nu,2}(\alpha)\varphi_2$ $= \tilde{\epsilon}_2(\mathbf{d}(\alpha))((-1)^\tau,\alpha)_2\varphi_2 = \chi_{0,2}(\alpha)\tilde{\epsilon}_2(\mathbf{d}(\alpha))\varphi_2$. Since

$$\mathfrak{F}\varphi_2\begin{pmatrix} b & -a \\ c & -b \end{pmatrix} = \mathbb{I}_{\frac{1}{2}\mathbb{Z}_2}(a)\mathbb{I}_{\frac{1}{2}\mathbb{Z}_2}(b)\mathbb{I}_{\frac{1}{2}\mathbb{Z}_2}(c),$$

$r_{\psi'}(\overline{\mathbf{n}}(x))(\varphi_2) = \varphi_2$ and $r_{\psi'}(\overline{\mathbf{n}}(-y))\mathfrak{F}\varphi_2 = \mathfrak{F}\varphi_2$ for $x \in \mathbb{Z}_2, y \in 2^2\mathbb{Z}_2$.
(iv) See [33], Remark 2.1, p. 105. ■

We will show later in Sec. 4 (see Prop. 4.2 and the paragraph following Thm. 4.5) that $h', t' \neq 0$ and also that some nonzero scalar multiple of h' has Fourier coefficients in $\mathbb{Q}(f,\chi)$, the field generated over $\mathbb{Q}$ by the eigenvalues of f and the values of χ. Assuming this for the moment, let h_χ be a scalar multiple of h' with Fourier coefficients in $\mathbb{Q}(f,\chi)$ and suppose that we have chosen h_χ to be λ-adically normalized i.e. the ideal generated by the Fourier coefficients of h_χ is an integral ideal in $\mathbb{Q}(f,\chi)$ and prime to λ. (Thus h_χ is only well defined up to a λ-adic unit in $\mathbb{Q}(f,\chi)$.) Let $t = t_{h_\chi}$ and set $s' = T_{\psi'}(\varphi, g, t)$.

Proposition 3.5. *$s' = \beta s$ for some scalar β.*

Proof: By [38] (proof of Prop. 22, p. 295), one knows that $\Theta(\tilde{\pi},\psi') = \pi' \otimes \chi_\nu = \pi_g \otimes \chi_\nu$, hence $s' \in \pi_g \otimes \chi_\nu$. Recall that s was defined to be the unique vector in $\pi_g \otimes \chi_\nu$ satisfying $s \otimes (\chi_\nu^{-1} \circ \mathrm{Nm}) \otimes (\chi \circ \mathrm{Nm}) = s_{g_\chi}$ where s_{g_χ} is a λ-adically primitive newform in $\pi \otimes \chi$. Recall also that s_{g_χ} may be characterized (up to a scalar) as the unique vector $v = \otimes_q v_q$ where $v_q \in \pi_{f,q} \otimes \chi_q$ satisfies

(a) v_∞ is a lowest weight vector in the holomorphic discrete series representation of weight $2k$;

(b) For finite q, v_q transforms under $U_{0,q}(\chi)$ by $\tilde{\omega}_{\chi,q}$.

It is easy to check that for $\kappa_\theta := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, $R(\kappa_\theta)\overline{\varphi}_\infty = e^{2ik\theta}\overline{\varphi}_\infty$. Thus to establish the proposition, it suffices to show that $((\chi_{\nu,q}\overline{\chi}_q)\circ\mathrm{Nm}(u))\cdot R(u)\varphi = \overline{\tilde{\omega}(u)}\cdot\varphi$ for $u \in U_{0,q}(\chi)$ i.e. for all finite q,

$$((\chi_{\nu,q}\overline{\chi}_q)\circ\mathrm{Nm}(u))\cdot R(u)\varphi_q = \overline{\tilde{\omega}_{\chi,q}(u)}\cdot\varphi_q \text{ for } u \in U_{0,q}(\chi). \tag{3.6}$$

One checks that
(i) For $q \nmid \nu N$, $u \in U_{0,q}(\chi)$, one has $(\chi_\nu\overline{\chi})(\mathrm{Nm}(u)) = 1$, $\tilde{\omega}_{\chi,q}(u) = 1$ and $R(u)\varphi_q = \varphi_q$.
(ii) For $q \mid \nu$, $q \nmid N$, $u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_{0,q}(\chi)$, $(\chi_{\nu,q}\overline{\chi}_q)(\mathrm{Nm}(u)) = \chi_{\nu,q}(\mathrm{Nm}(u))$, $R(u)\varphi_q = \chi_{\nu,q}(\mathrm{Nm}(u))\varphi_q$, $\tilde{\omega}_{\chi,q}(u) = 1$.
(iii) For $q \mid N^+$, $u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_{0,q}(\chi)$, $(\chi_{\nu,q}\overline{\chi}_q)(\mathrm{Nm}(u)) = \chi_{\nu,q}(ad)\chi_q^{-1}(ad)$, $R(u)\varphi_q = (\chi_{\nu,q}\chi_q^{-1})(d/a)\varphi_q$, $\tilde{\omega}_{\chi,q}(u) = \chi_q(d)^2$.

(iv) For $q \mid N^-$, suppose $u' = \alpha + \beta' u \in U_q$. Then $\tilde{\omega}_{\chi,q}(u') = \chi_q(\mathrm{Nm}(u'))$. Also

$$\begin{aligned}(u')^{-1}(a+bu)u' &= \frac{1}{\mathrm{Nm}(u')}(\overline{\alpha} - \beta' u)(a+bu)(\alpha + \beta' u)\\ &= \frac{1}{\mathrm{Nm}(u')}(\mathrm{Nm}(\alpha)a + (\mathrm{Nm}(\beta') + b\overline{\alpha}\bar{\beta}' - \overline{b}\alpha\beta')\pi) + \\ &\qquad (2\overline{\alpha}\beta' a + \overline{\alpha}^2 b - \beta'^2 \pi \bar{b})u).\end{aligned}$$

Since $\mathrm{Nm}(u') = \mathrm{Nm}(\alpha) - \pi\,\mathrm{Nm}(\beta')$, both $\mathrm{Nm}(u')$ and $\mathrm{Nm}(\alpha)$ are units. Now, if $q \mid \nu$, $R(u')\varphi_q(a+bu) = 0$ unless $a \in qR_q$ and $b \in R_q$. Since $\overline{\alpha}^2/\mathrm{Nm}(u') = \mathrm{Nm}(\alpha)/\mathrm{Nm}(u') \cdot \overline{\alpha}/\alpha$ and $\chi_{\nu,q}, \chi_q$ both have conductor q, we see that

$$\begin{aligned}R(u')\varphi_q &= (\chi_{\nu,q}\chi_q^{-1})(\mathrm{Nm}(\alpha)/\mathrm{Nm}(u')) \cdot \chi_{\nu,q}(\mathrm{Nm}(\alpha))\varphi_q\\ &= \chi_{\nu,q}(\mathrm{Nm}(\alpha))\varphi_q = \chi_{\nu,q}(\mathrm{Nm}(u'))\varphi_q.\end{aligned}$$

The verification that $R(u')\varphi_q = \chi_{\nu,q}(\mathrm{Nm}(u'))\varphi_q$ in the case $q \nmid \nu$ is simpler and is left to the reader.
(v) If $q = 2$, $R(u)\varphi_2 = \varphi_2$, $\chi_2(\mathrm{Nm}(u)) = \chi_\nu(\mathrm{Nm}(u)) = 1$ and $\tilde{\omega}_2(u) = 1$.
We see in each case that (3.6) is verified. ■

It will be important for us to know that $\beta \neq 0$. This will be established in Prop. 4.2(modulo the proof of Theorem 4.1, which appears in [23].)

4. Arithmetic properties of the Shintani lift

4.1. **Period integrals à la Shintani and Shimura.** For $w \in \mathbb{C}$ and $\alpha \in V \otimes_{\mathbb{R}} \mathbb{C}$, define

$$\begin{aligned}[\alpha, w] &= \begin{pmatrix} w & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \alpha \begin{pmatrix} w \\ 1 \end{pmatrix}\\ &= cw^2 - 2bw + a,\end{aligned}$$

if $\alpha = \begin{pmatrix} b & -a \\ c & -b \end{pmatrix}$. For $x \in V$ and any subgroup $\Gamma' \subset B^{(1)}$, let $G_x = \{h \in \mathrm{SL}_2(\mathbb{R}), h^{-1}xh = x\}$, $\Gamma'_x = G_x \cap \Gamma'$. Suppose that $g' \in S_k(\Gamma', \omega)$, and $\omega|_{\Gamma'_x}$ is the trivial character. Then put, as in ([32] (2.5); see same reference for normalization of the measure below)

$$P(g', x, \Gamma') = \int_{\Gamma'_x \backslash G_x} [x, hw]^k g'(hw) d(\Gamma'_x h)$$

for any $w \in \mathfrak{H}$. Denote by V^* the set of $x \in V$ such that $\mathrm{Nm}(x) < 0$ (i.e. $Q(x) = -\mathrm{Nm}(x) > 0$.) By [32] Lemma 2.1, $P(g', x, \Gamma')$ is independent of the choice of w and is equal to 0 unless $x \in V^*$. Let $R(\Gamma')$ be the set of equivalence classes in V^* for the conjugation action of Γ' and for $\mathcal{C} \in R(\Gamma')$, set $N(\mathcal{C}) = N(x)$ for any choice of $x \in \mathcal{C}$. By [32] (2.6), $P(g', x, \Gamma')$ only depends on the class of x in $R(\Gamma')$. Thus for $\mathcal{C} \in R(\Gamma')$ we may set $P(g', \mathcal{C}, \Gamma') = P(g', x, \Gamma')$ for any choice of $x \in \mathcal{C}$.

4.2. **Fourier coefficients and nonvanishing of the Shintani lift.** Let $\xi \in \mathbb{Q}$. We now compute the ψ^ξ-th Fourier coefficient of $t' = t_{\psi'}(\varphi, \sigma, s)$. As in [38] (p. 291), this is given by

$$\begin{aligned} W(t',\psi^\xi,\sigma) &= W(t',(\psi')^{|\nu|\xi},\sigma) = \int_{Z_{\mathbb{A}}B_{\mathbb{Q}}^\times\backslash B_{\mathbb{A}}^\times} s(\beta)\sum_{x\in V,q(x)=|\nu|\xi} r_{\psi'}(\sigma)R(\beta)\varphi(x)d^\times\beta \\ &= \int_{Z_{\mathbb{A}}B_{\mathbb{Q}}^\times\backslash B_{\mathbb{A}}^\times} g_\chi(\beta)(\chi_\nu\chi^{-1})(\mathrm{Nm}(\beta))\sum_{x\in V,q(x)=|\nu|\xi} r_{\psi'}(\sigma)R(\beta)\varphi(x)d^\times\beta. \end{aligned}$$

Since $B_{\mathbb{A}}^\times = B_{\mathbb{Q}}^\times\cdot(U_0(\chi)\times(B_\infty^\times)^+)$ and $g_\chi(\beta)(\chi_\nu\chi^{-1})(\mathrm{Nm}(\beta))R(\beta)\varphi$ is invariant under $\beta\mapsto\beta u$ for $u\in U_0(\chi)$,

$$\begin{aligned} W(t',\psi^\xi,\sigma) &= \mathrm{vol}(U_0(\chi))\int_{\Gamma_\chi\backslash SL_2(\mathbb{R})} g_\chi(\beta_\infty)\cdot\sum_{x\in V,q(x)=|\nu|\xi} r_{\psi'}(\sigma)R(\beta_\infty)\varphi(x)d^{(1)}\beta_\infty \\ &= \mathrm{vol}(U_0(\chi))\sum_{\mathcal{C}\in R(\Gamma_\chi),q(\mathcal{C})=|\nu|\xi}\int_{\Gamma_\chi\backslash SL_2(\mathbb{R})} g_\chi(\beta_\infty)\sum_{x\in\mathcal{C}} r_{\psi'}(\sigma)R(\beta_\infty)\varphi(x)d^{(1)}\beta_\infty. \end{aligned}$$

Now, put $\sigma = d_{\mathbb{R}}(y^{1/2})$. Since $\mathrm{vol}(U_0(\chi)) = C/\pi^2$ for $C = 6[U_0 : U_0(\chi)]\prod_{q|N^+}(q+1)^{-1}\prod_{q|N^-}(q-1)^{-1}$, we get

$$\begin{aligned} W(t',\psi^\xi,\sigma) &= C\pi^{-2}\sum_{\mathcal{C}\in R(\Gamma_\chi),q(\mathcal{C})=|\nu|\xi}\sum_{x\in\mathcal{C}}\varphi_{fin}(x)\int_{\Gamma_\chi\backslash SL_2(\mathbb{R})} g_\chi(\beta_\infty)\cdot \\ &\qquad\qquad r_{\psi'}(d_{\mathbb{R}}(y^{1/2}))R(\beta_\infty)\varphi_\infty(x)d^{(1)}\beta_\infty \\ &= C\sum_{\mathcal{C}\in R(\Gamma_\chi),q(\mathcal{C})=|\nu|\xi}\varphi_{fin}(x)I(x), \end{aligned} \tag{4.1}$$

where x is any element in $\mathcal{C}$, and

$$I(x) = \frac{1}{\pi^2}\int_{\Gamma_{\chi,x}\backslash SL_2(\mathbb{R})} g_\chi(\beta_\infty)r_{\psi'}(d_{\mathbb{R}}(y^{1/2}))R(\beta_\infty)\varphi_\infty(x)d^{(1)}\beta_\infty. \tag{4.2}$$

Since $\varphi_{fin}(\gamma^{-1}x\gamma) = \chi'(\gamma)\varphi_{fin}(x)$, we see that χ' restricted to $\Gamma_{\chi,x}$ is the trivial character if $\varphi_{fin}(x)\neq 0$, so that the integrand in (4.2) is indeed $\Gamma_{\chi,x}$ invariant, and the product $\varphi_{fin}(x)I(x)$ is independent of the choice of $x\in\mathcal{C}$.

By [33] (Sublemma on p. 102) and [32] (2.23) (and taking into account that our additive character is ψ' instead of ψ),

$$\begin{aligned} I(x) &= \frac{1}{\pi^2}e^{-2\pi i\xi u}\int_{\Gamma_{\chi,x}\backslash SL_2(\mathbb{R})} g_\chi(\beta_\infty)r_{\psi'}(b(z)))R(\beta_\infty)\varphi_\infty(x)d^{(1)}\beta_\infty \\ &= (|\nu|\xi)^{-1/2}v^{(2k+1)/4}e^{-2\pi\xi v}P(g_\chi,x,\Gamma_\chi). \end{aligned} \tag{4.3}$$

The formulas (4.1) and (4.3) above can be used to relate the Fourier coefficients $a_\xi(h')$ to certain period integrals of g_χ along tori embedded in $B^\times$. Applying the method of Waldspurger [39], one can show the following

Theorem 4.1. *If $a_\xi(h')\neq 0$, then the following conditions must be satisfied:*

(a) For all $q\mid N, q\nmid N'$, $\left(\frac{\xi_0}{q}\right)\neq -w_q$.

(b) For all $q\mid N'$, $\left(\frac{\xi_0}{q}\right) = \chi_{0,q}(-1)w_q$.

(c) $\xi_0\equiv 0$ or $1 \mod 4$.

Suppose that conditions (a), (b), (c) are satisfied. Then

$$|a_\xi(h')|^2 = C(f,\chi,\nu)\pi^{-2k}|\nu\xi|^{k-\frac{1}{2}}L(\frac{1}{2},\pi_f\otimes\chi_\nu)L(\frac{1}{2},\pi_f\otimes\chi_{\xi_0})\cdot\frac{\langle g_\chi,g_\chi\rangle}{\langle f_\chi,f_\chi\rangle},$$

where $C(f,\chi,\nu)\in\mathbb{Q}$ *and is a p-adic unit if* $p\nmid\tilde{N}:=\prod_{q|N}q(q+1)(q-1)$. *(Recall that* f_χ *is the Jacquet-Langlands lift of* g_χ *to* GL_2*, normalized to have its first Fourier coefficient equal to* 1.*)*

The proof of the above theorem will appear in another article [23], since it uses methods very different from those of the present article.

Let us set $h'=\alpha_0 h_\chi$. Then we have

Proposition 4.2. $\alpha_0,\beta\neq 0$.

Proof: One knows from Waldspurger [37] that there exists ξ such that $L(\frac{1}{2},\pi_f\otimes\chi_{\xi_0})\neq 0$. Further $L(\frac{1}{2},\pi_f\otimes\chi_\nu)\neq 0$. Hence $|a_\xi(h')|\neq 0$ for some ξ whence $h',t'\neq 0$ and $\alpha_0\neq 0$. By see-saw duality (see [19]),

$$\langle\alpha_0 h_\chi,h_\chi\rangle=\langle g_\chi,\beta g_\chi\rangle,$$

so that $\beta\neq 0$ also. ■

4.3. **Fundamental periods of modular forms on quaternion algebras.** Let $n=2k-2$, so that n is a nonnegative integer. Set $\tilde{F}_0=\mathbb{Q}$ if $n=0$ and $\tilde{F}_0=F_0$ if $n>0$. For A any $\mathcal{O}_{\tilde{F}_0}$-algebra contained in $\mathbb{C}$, let $L(n,A)$ be the A-module of homogenous polynomials in two variables (X,Y) of degree n with coefficients in A. There is a natural action of Γ^1_χ on $L(n,A)$ given by

$$(\sigma_n(\gamma)P)(X,Y)=P(aX+cY,bX+dY)\text{ if }\Phi(\gamma)=\begin{pmatrix}a&b\\c&d\end{pmatrix}.$$

Thus we can define the (parabolic) cohomology group $H^1_p(\Gamma^1_\chi,L(n,A))$, following Shimura. Let $\overline{S_{n+2}}(\Gamma^1_\chi)$ denote the space of antiholomorphic cusp forms of weight $n+2$ on Γ^1_χ. The theory of Eichler-Shimura gives for every such n, a canonical isomorphism

$$\mathfrak{c}:S_{n+2}(\Gamma^1_\chi)\oplus\overline{S_{n+2}}(\Gamma^1_\chi)\simeq H^1_p(\Gamma^1_\chi,L(n,\mathbb{C})). \tag{4.4}$$

We recall the definition of the map $\mathfrak{c}$ above. Put $\omega(g')=g'(z)(Xz+Y)^n dz$ for $g'\in S_{n+2}(\Gamma^1_\chi)$ and $\omega(g')=g'(z)(X\bar{z}+Y)^n d\bar{z}$ for $g'\in\overline{S_{n+2}}(\Gamma^1_\chi)$. Define for any such g',

$$\mathfrak{c}(g',\gamma)=\int_{z_0}^{\gamma z_0}\omega(g')$$

for some choice of $z_0\in\mathfrak{H}$. The cohomology class of the map $\gamma\to\mathfrak{c}(g',\gamma)$ does not depend on the choice of z_0, and is denoted $\mathfrak{c}[g']$.

Suppose now that $g'=g_\chi$. Let $\mathbb{T}$ denote the Hecke algebra associated to the group Γ^1_χ. Both sides of (4.4) carry a natural action of $\mathbb{T}$ and the isomorphism (4.4) is in fact $\mathbb{T}$-equivariant. In addition, both sides of (4.4) carry natural involutions $x\mapsto x^c$. On the left, this is defined in Sec. 2.2.5. On the right, this may be defined as follows. First pick a unit $\delta\in\mathcal{O}(\chi)$ of norm -1 and such that $\Phi_q(\delta)\equiv\begin{pmatrix}1&0\\0&1\end{pmatrix}$ mod q for $q\mid\gcd(N_\chi,N^+)$. Such a unit is known to exist by work of Eichler. Then for $\mathfrak{c}\in Z(\Gamma^1_\chi,L(n,A))$, define $(\mathfrak{c}|\delta)(\gamma)=-\sigma_n(\delta)\mathfrak{c}(\delta^{-1}\gamma\delta)$. The asignment $\mathfrak{c}\mapsto\mathfrak{c}|\delta$ preserves $B(\Gamma^1_\chi,L(n,A))$ hence induces an involution of $H^1_P(\Gamma^1_\chi, L(n,A))$, also denoted by the symbol δ. If δ' is any other choice of δ, then $\delta' = \delta\alpha$ for some $\alpha \in \Gamma^1_\chi$. Now, writing $\gamma' = \delta^{-1}\gamma\delta$, $\sigma_n(\delta')\mathfrak{c}((\delta')^{-1}\gamma\delta') = \sigma_n(\delta\alpha)\mathfrak{c}(\alpha^{-1}\delta^{-1}\gamma\delta\alpha) = \sigma_n(\delta)\sigma_n(\alpha)\mathfrak{c}(\alpha^{-1}\gamma'\alpha)$. But

$$\begin{aligned}
\sigma_n(\alpha)\mathfrak{c}(\alpha^{-1}\gamma'\alpha) &= \sigma_n(\alpha)[\mathfrak{c}(\alpha^{-1}) + \sigma_n(\alpha^{-1})\mathfrak{c}(\gamma') + \sigma_n(\alpha^{-1}\gamma')\mathfrak{c}(\alpha)] \\
&= \sigma_n(\alpha)\mathfrak{c}(\alpha^{-1}) + \mathfrak{c}(\gamma') + \sigma_n(\gamma')\mathfrak{c}(\alpha) \\
&= [(\mathfrak{c}(1) - \mathfrak{c}(\alpha)] + \mathfrak{c}(\gamma') + \sigma_n(\gamma')\mathfrak{c}(\alpha) \\
&= \mathfrak{c}(\gamma') + (\sigma_n(\gamma') - 1)\mathfrak{c}(\alpha) \\
&= \mathfrak{c}(\gamma') + \sigma_n(\delta^{-1})(\sigma_n(\gamma) - 1)\sigma_n(\delta)\mathfrak{c}(\alpha),
\end{aligned}$$

since $\mathfrak{c}(1) = 0$. Thus

$$\sigma_n(\delta')\mathfrak{c}((\delta')^{-1}\gamma\delta') = \sigma_n(\delta)\mathfrak{c}(\delta^{-1}\gamma\delta) + (\sigma_n(\gamma) - 1)\sigma_n(\delta)\mathfrak{c}(\alpha),$$

whence the involution defined above on the cohomology group $H^1_p(\Gamma^1_\chi, L(n,A)$ is actually independent of the choice of δ. We denote it by the symbol c. If $g' \in S_{2k}(\Gamma^1_\chi)$ then

$$\begin{aligned}
\mathfrak{c}(\gamma, g'^c) &= \int_{z_0}^{\gamma z_0} g'(\delta\overline{z})J(\delta,\overline{z})^{-2k}(X\overline{z}+Y)^n\overline{dz} \\
&= \int_{\delta\overline{z_0}}^{\delta\gamma\overline{z_0}} g'(z)J(\delta,\delta^{-1}z)^{-2k}(X\delta^{-1}z+Y)^n J(\delta^{-1},z)^{-2}\,\mathrm{Nm}(\delta)dz \\
&= -\int_{\delta\overline{z_0}}^{\delta\gamma\delta^{-1}\cdot\delta\overline{z_0}} g'(z)\sigma_n(\delta^{-1})(Xz+Y)^n dz \\
&= \sigma_n(\delta^{-1})\mathfrak{c}(\delta\gamma\delta^{-1}, g').
\end{aligned}$$

Thus $[\mathfrak{c}(g'^c)] = [\mathfrak{c}(g')]^c$ for $g' \in S_{2k}(\Gamma^1_\chi)$. Likewise one may check that $[\mathfrak{c}(g'^c)] = [\mathfrak{c}(g')]^c$ for $g' \in \overline{S_{2k}}(\Gamma^1_\chi)$, whence the map (4.4) commutes with the involutions c. By multiplicity one, the maximal subspace of $S_{n+2}(\Gamma^1_\chi) \oplus \overline{S_{n+2}}(\Gamma^1_\chi)$ on which $\mathbb{T}$ acts by λ_{g_χ} is two dimensional, a basis for it being given by $\{g_\chi, g^c_\chi\}$. The involution c preserves this subspace and acts diagonally, with eigenvectors $\{g_\chi + g^c_\chi, g_\chi - g^c_\chi\}$, the corresponding eigenvalues being $1, -1$ respectively.

Since (4.4) commutes with the actions of $\mathbb{T}$ and c, the subspace $H^1_p(\Gamma^1_\chi, L(2k-2,\mathbb{C}))^{\pm,\lambda_{g_\chi}}$ of $H^1_p(\Gamma^1_\chi, L(2k-2,\mathbb{C})$ on which $\mathbb{T}$ acts by the eigencharacter λ_{g_χ} associated to g_χ and c acts by ± 1 is one-dimensional. Let A be any $\mathcal{O}_{\tilde{F}_0}$-algebra in $\mathbb{C}$ that is a principal ideal domain and contains all the eigenvalues of g_χ. Let $\xi_\pm(g_\chi, A)$ be any generator of the free rank one A-submodule $H^1_p(\Gamma^1_\chi, L(2k-2, A))^{\pm,\lambda_{g_\chi}}$. If $\sigma \in Aut(\mathbb{C}/\tilde{F}_0)$, then $\Gamma^1_{\chi^\sigma} = \Gamma^1_\chi$ and we may choose $\xi_\pm((g_\chi)^\sigma, A^\sigma) = (\xi_\pm(g_\chi, A))^\sigma$.

We can now define the fundamental periods $u_\pm(g_\chi, A)$ (and $u_\pm((g_\chi)^\sigma, A^\sigma)$) by

$$\begin{aligned}
\mathfrak{c}[g_\chi] \pm \mathfrak{c}[g^c_\chi] &= u_\pm(g_\chi, A)\xi_\pm(g_\chi, A), \\
\mathfrak{c}[(g_\chi)^\sigma] \pm \mathfrak{c}[\overline{((g_\chi)^\sigma)^c}] &= u_\pm((g_\chi)^\sigma, A^\sigma)\xi_\pm((g_\chi)^\sigma, A^\sigma).
\end{aligned}$$

Up to units in A, these periods are independent of the choice of Φ, g_χ and $\xi_\pm(g_\chi, A)$. For F any subfield of $\overline{\mathbb{Q}}$ containing $\tilde{F}_0$ and all the eigenvalues of g_χ, let $A_{F,\lambda}$ be the subring of elements in F with non-negative λ-adic valuation. Define $u_\pm(g_\chi, F, \lambda)$ to be equal to $u_\pm(g_\chi, A_{F,\lambda})$. Also define $u_\pm(g_\chi, \lambda)$ to be $u_\pm(g_\chi, F, \lambda)$ for any choice of F so that it is only well defined up to a λ-adic unit in $\overline{\mathbb{Q}}$.

4.3.1. *An auxiliary description of the fundamental periods.* Let us write

$$V_{\mathbb{R}} = \left\{ \left(\begin{array}{cc} r & s \\ t & -r \end{array} \right), r, s, t \in \mathbb{R} \right\}.$$

Denote by $\mathcal{P}_{\mathbb{R}}^{k-1}$ be the vector space over $\mathbb{R}$ of $\mathbb{R}$-valued homogeneous functions $\mathfrak{h}$ on $V_{\mathbb{R}}$ of degree $k-1$ satisfying $(\partial^2/\partial r^2 + 4\partial^2/\partial s \partial t)\mathfrak{h} = 0$. Let $\mathcal{P}_{\mathbb{C}}^{k-1} = \mathcal{P}_{\mathbb{R}}^{k-1} \otimes \mathbb{C}$ and ρ_{k-1} the representation of Γ_χ^1 on $\mathcal{P}_{\mathbb{C}}^{k-1}$ given by

$$[\rho_{k-1}(\gamma)\mathfrak{h}](x) = \mathfrak{h}(\gamma^i x \gamma).$$

Finally, let σ_{2k-2} be the representation of Γ_χ^1 on $L(2k-2, \mathbb{C})$ defined in the previous section. The following is well known.

Lemma 4.3. *For $\mathfrak{h} \in \mathcal{P}_{\mathbb{C}}^{k-1}$, define $p(\mathfrak{h}) \in L(2k-2, \mathbb{C})$ by*

$$p(\mathfrak{h})(X, Y) = \mathfrak{h}(\epsilon^{-1} \left[\begin{array}{c} X \\ Y \end{array} \right] \left[\begin{array}{cc} X & Y \end{array} \right]).$$

Then p gives an isomorphism of representations of Γ_χ^1, $(\rho_{k-1}, \mathcal{P}_{\mathbb{C}}^{k-1}) \simeq (\sigma_{2k-2}, L(2k-2, \mathbb{C}))$ sending $\mathcal{P}_{\mathbb{R}}^{k-1}$ to $L(2k-2, \mathbb{R})$. This induces an isomorphism of cohomology groups

$$p_* : H_p^1(\Gamma_\chi^1, \mathcal{P}_{\mathbb{C}}^{k-1}) \simeq H_p^1(\Gamma_\chi^1, L(2k-2, \mathbb{C})).$$

One may define an involution c on $H_p^1(\Gamma_\chi^1, \mathcal{P}_A^{k-1})$ as follows. For $\mathfrak{c}' \in Z(\Gamma_\chi^1, \mathcal{P}_A^{k-1})$ and $\xi \in V_{\mathbb{C}}$, set $\mathfrak{c}'(\gamma, \xi) = (\mathfrak{c}'(\gamma))(\xi)$. For δ any unit as in the previous section, and for $\mathfrak{c}' \in Z(\Gamma_\chi^1, \mathcal{P}_A^{k-1})$ define $\mathfrak{c}'|\delta$ by

$$(\mathfrak{c}'|\delta)(\gamma, \xi) = (-1)^k \mathfrak{c}'(\delta^{-1}\gamma\delta, \delta^{-1}\xi\delta).$$

Since $\epsilon^{-1}\delta^t\epsilon = \delta^i = -\delta^{-1}$, for $\mathfrak{c}' \in Z(\Gamma_\chi^1, \mathcal{P}_{\mathbb{C}}^{k-1})$, we get

$$\begin{aligned} ((p_*(\mathfrak{c}'))|\delta)(\gamma)(X, Y) &= -\sigma_n(\delta)(p_*\mathfrak{c}')(\delta^{-1}\gamma\delta, \left[\begin{array}{cc} X & Y \end{array} \right]) \\ &= -p_*\mathfrak{c}'(\delta^{-1}\gamma\delta, \left[\begin{array}{cc} X & Y \end{array} \right] \delta) \\ &= -\mathfrak{c}'(\delta^{-1}\gamma\delta, \epsilon^{-1}\delta^t \left[\begin{array}{c} X \\ Y \end{array} \right] \left[\begin{array}{cc} X & Y \end{array} \right] \delta) \\ &= -\mathfrak{c}'(\delta^{-1}\gamma\delta, -\delta^{-1}\epsilon^{-1} \left[\begin{array}{c} X \\ Y \end{array} \right] \left[\begin{array}{cc} X & Y \end{array} \right] \delta) \\ &= p_*(\mathfrak{c}'|\delta)(\gamma)(X, Y). \end{aligned}$$

Thus $p_*(\mathfrak{c}'|\delta) = (p_*(\mathfrak{c}')|\delta)$, whence the assignment $\mathfrak{c}' \mapsto \mathfrak{c}'|\delta$ induces an involution on the cohomology group $H_p^1(\Gamma_\chi^1, \mathcal{P}_{\mathbb{C}}^{k-1})$ that is independent of the choice of δ. We denote this involution also by the symbol c.

Given $z, z_0 \in \mathfrak{H}$, and $x \in V$, define

$$\begin{aligned} X(z, z_0, x, g_\chi) &= \int_{z_0}^{z} [x, w]^{k-1} g_\chi(w) dw, \\ \mathfrak{r}(\gamma, z_0, x, g_\chi) &= X(\gamma z_0, z_0, x, g_\chi). \end{aligned}$$

One checks easily that $\mathfrak{r}(\gamma, z_0, x, g_\chi)$ as a function of (γ, x) lies in $Z(\Gamma_\chi^1, \mathcal{P}_{\mathbb{C}}^{k-1})$ and its cohomology class in $H_p^1(\Gamma_\chi^1, \mathcal{P}_{\mathbb{C}}^{k-1})$ is independent of the choice of z_0. We denote this cohomology class by $\mathfrak{c}'[g_\chi]$ and note that $p_*(\mathfrak{c}'[g_\chi]) = \mathfrak{c}[g_\chi]$. Now let $\mathcal{P}_A^{k-1}$ denote the sub-A-module of

$\mathcal{P}_{\mathbb{C}}^{k-1}$ consisting of those $\mathfrak{h}$ whose coefficients lie in A and note that c preserves $H^1_p(\Gamma^1_\chi, \mathcal{P}_A^{k-1})$. We may thus define another set of fundamental periods $u'_\pm(g_\chi, A)$ (well defined up to elements of $A^\times$) by

$$\begin{aligned}\mathfrak{c}'[g_\chi] \pm \mathfrak{c}'[g_\chi^c] &= u'_\pm(g_\chi, A)\xi'_\pm(g_\chi, A),\\ \mathfrak{c}'[g_\chi^\sigma] \pm \mathfrak{c}'[((g_\chi)^\sigma)^c] &= u'_\pm(g_\chi^\sigma, A^\sigma)\xi'_\pm(g_\chi^\sigma, A^\sigma),\end{aligned}$$

where $\xi'_\pm(g_\chi, A)$ is a generator of the free rank one A-submodule $H^1_p(\Gamma^1_\chi, \mathcal{P}_A^{k-1})^{\pm,\lambda_{g_\chi}}$ and $\xi'_\pm(g_\chi^\sigma, A^\sigma) = (\xi'_\pm(g_\chi, A))^\sigma$.

We also have the following lemma whose proof we leave as an easy exercise for the reader.

Lemma 4.4. *1.*

$$p(\mathcal{P}_A^{k-1}) \subseteq L(n, A).$$

2. Suppose that all primes $q < 2k$ are invertible in A. Then

$$p(\mathcal{P}_A^{k-1}) = L(n, A).$$

It follows from the lemma that we may pick $\xi'_\pm(g_\chi, A)$ such that $p_*(\xi'_\pm(g_\chi, A)) = \xi_\pm(g_\chi, A)$. Then $u_\pm(g_\chi, A) = u'_\pm(g_\chi, A)$.

4.4. **Rationality and integrality of the Shintani lift.** Denote t' now by the symbol $t'_{g,\chi,\nu}$ and h_χ by $h_{g,\chi}$ to denote the dependence on g, χ and ν.

Theorem 4.5. *Write $t'_{g,\chi,\nu} = \alpha'(g, \chi, \nu, F, \lambda)u_+(g_\chi, F, \lambda)h_{g,\chi}$ for some non-zero constant $\alpha'(g, \chi, \nu, F, \lambda)$.*

(a) Let $\sigma \in Aut(\mathbb{C}/\tilde{F}_0)$. Then $(\alpha'(g, \chi, \nu, F, \lambda))^\sigma = \alpha'(g^\sigma, \chi^\sigma, \nu, F^\sigma, \lambda^\sigma)$. Thus $\alpha'(g, \chi, \nu, F, \lambda) \in F(\chi)$.

(b) $v_\lambda(\alpha'(g, \chi, \nu, F, \lambda)) \geq 0$.

Proof: With the preparation from the previous section, the proof is almost tautological. In fact we only need to copy the proof of [32] Prop. 4.5 (which proves that the Shintani lift is algebraic) with some care to take care of rationality and λ-adic integrality. Letting $C_1 = [\Gamma_\chi : \Gamma^1_\chi]$, we see by (4.3) that

$$\begin{aligned}a_\xi(t') &= v^{-\kappa/4}e^{2\pi\xi v}W(t', \psi^\xi, d_{\mathbb{R}}(v^{1/2}))\\ &= \sum_{\mathcal{C}\in R(\Gamma_\chi), q(\mathcal{C})=\nu\xi} \varphi_{fin}(x)(\nu\xi)^{-1/2}P(g_\chi, x, \Gamma_\chi)\\ &= C_1 \cdot \sum_{\mathcal{C}\in R(\Gamma_\chi), q(\mathcal{C})=\nu\xi} \varphi_{fin}(x)(\nu\xi)^{-1/2}P(g_\chi, x, \Gamma^1_\chi)\\ &= \frac{1}{2}C_1 \cdot (\nu\xi)^{-1/2} \sum_{\mathcal{C}\in R(\Gamma_\chi), q(\mathcal{C})=\nu\xi} [\varphi_{fin}(x)P(g_\chi, x, \Gamma^1_\chi) + \varphi_{fin}(\delta^{-1}x\delta)P(g_\chi, \delta^{-1}x\delta, \Gamma^1_\chi)]\\ &= \frac{1}{2}C_1 \cdot \sum_{\mathcal{C}\in R(\Gamma_\chi), q(\mathcal{C})=\nu\xi} [\varphi_{fin}(x)\mathfrak{r}(\gamma_x, x) + \varphi_{fin}(\delta^{-1}x\delta)\mathfrak{r}(\delta^{-1}\gamma_x\delta, \delta^{-1}x\delta)],\end{aligned}$$

where γ_x is any generator of the group $\Gamma^1_{\chi,x}\{\pm1\}/\{\pm1\}$. (Here $\mathfrak{r}(\gamma_x, x)$ is defined to be $\mathfrak{r}(\gamma_x, z_0, x, g_\chi)$ for any choice of z_0. This is independent of the choice of z_0 since γ_x fixes x.) Now, $\varphi_{fin}(\delta^{-1}x\delta) = \chi'(\delta)(\chi \cdot \chi_\nu)(-1)\varphi_{fin}(x) = (-1)^k\chi'(\delta)\varphi_{fin}(x)$ and $\mathfrak{r}(\delta^{-1}\gamma_x\delta, \delta^{-1}x\delta) =$

$(-1)^k\chi'(\delta)^{-1}(\mathfrak{r}|\delta)(\gamma_x,x)$. Let $I^{g,\chi}(x)=\varphi_{fin}(x)\mathfrak{r}(\gamma_x,x)+\varphi_{fin}(\delta^{-1}x\delta)\mathfrak{r}(\delta^{-1}\gamma_x\delta,\delta^{-1}x\delta)$ and $A=A_{F,\lambda}$. Then

$$\begin{aligned} I^{g,\chi}(x) &= \varphi_{fin}(x)[\mathfrak{r}(\gamma_x,x)+(\mathfrak{r}|\delta)(\gamma_x,x)] = \varphi_{fin}^{g,\chi,\nu}(x)u'_+(g_\chi,A)\xi'_+(g_\chi,A,\gamma_x,x) \\ &= \varphi_{fin}^{g,\chi,\nu}(x)u_+(g_\chi,A)\xi'_+(g_\chi,A,\gamma_x,x). \end{aligned}$$

where $\xi'_+(g_\chi,A,\gamma_x,x)$ is defined to be $\mathfrak{c}(\gamma_x,x)$ for any $\mathfrak{c}\in Z(\Gamma^1_\chi,\mathcal{P}_A^{k-1})$ in the class of $\xi'_+(g_\chi,A)$. Again this is independent of the choice of $\mathfrak{c}$ since γ_x fixes x. Thus $I^{g,\chi}(x)/u_+(g_\chi,A)=\varphi_{fin}^{g,\chi,\nu}(x)\xi'_+(g_\chi,A,\gamma_x,x)\in A$, which proves part (b) of the theorem. Finally,

$$\begin{aligned} \left(\frac{I^{g,\chi}(x)}{u_+(g_\chi,A)}\right)^\sigma &= (\varphi_{fin}^{g,\chi,\nu}(x)\xi'_+(g_\chi,A,\gamma_x,x))^\sigma = \varphi_{fin}^{g^\sigma,\chi^\sigma,\nu}(x)\xi'_+((g_\chi)^\sigma,A^\sigma,\gamma_x,x) \\ &= \left(\frac{I^{g^\sigma,\chi^\sigma}(x)}{u_+((g_\chi)^\sigma,A^\sigma)}\right), \end{aligned}$$

whence part (a) is established too. ■

The proof of the proposition shows that $t'/u_+(g_\chi)$ has its Fourier coefficients in $F(\chi)$. In particular, the form h_χ is definable over $F(\chi)$. Since h_χ may be obtained as a theta lift from PGL_2 (i.e. the special case $B=\mathrm{M}_2(\mathbb{Q})$) for an appropriate choice of ν, and since F may be taken to be $\mathbb{Q}(f)$ in this case, we see that some nonzero multiple of h_χ has all its Fourier coefficients in $\mathbb{Q}(f,\chi)$ as had been claimed in Sec. 3.2 (see the paragraph before Prop. 3.5.)

We now study the relation between the period $u_+(g_\chi)$ and $u_\epsilon(g)$ where $\epsilon:=\mathrm{sign}(\chi(-1))=(-1)^k\,\mathrm{sign}(\nu)$. For each $q\mid N_\chi$, let χ^q be the finite order character corresponding to the unique Grossencharacter that restricted to $\prod_l\mathbb{Z}_l^\times\times(\mathbb{R}^+)^\times$ is χ_q at the factor q and 1 at all other factors. Thus $\chi=\prod_{q|N_\chi}\chi^q$. For $\Pi\subseteq\{l;l\mid N_\chi\}$, set $\chi^\Pi=\prod_{l\in\Pi}\chi^l$.

Proposition 4.6. *Let $\gamma=u_+(g_\chi)/u_\epsilon(g)$.*
(a) $\gamma/\mathfrak{g}(\chi)\in F(\chi)$.
(b) $v_\lambda(\gamma)\geq 0$.
(c) If $B=M_2(\mathbb{Q})$, $v_\lambda(\gamma)=0$.

Proof: Let $U^\Pi=\prod_{l\notin\Pi}U_{0,l}\times\prod_{l\in\Pi}U_{1,l}(\chi)$. Also set $\Gamma^{1,\Pi}=B^\times\cap(U^\Pi\cdot(B_\infty^\times)^+)$. Suppose that $q\notin\Pi$ and $s'=s_{g'}$ is a newform in $S_{2k}(\Gamma^{1,\Pi})=S_{2k}(U^\Pi)$. Define $\gamma_q^\pm:=\frac{u_\pm(g'_{\chi^q})}{u_{\pm\epsilon_q}g'}$ with $\epsilon_q=\chi^q(-1)$ where g' and g'_{χ^q} are arithmetically normalized as in Sec. 2.2.6. We claim that the following statements hold:
(a)$'$ $\gamma_q^\pm/\mathfrak{g}(\chi^q)\in F(\chi)$, and
(b)$'$ $v_\lambda(\gamma_q^\pm)\geq 0$.

Clearly (a) follows from (a)$'$ and (b) from (b)$'$ since $\mathfrak{g}(\chi),\mathfrak{g}(\chi^q)$ are λ-adic units and $\mathfrak{g}(\chi)/\prod_{q|N_\chi}\mathfrak{g}(\chi^q)\in\mathbb{Q}(\chi)\subseteq F(\chi)$. First consider the case $q\nmid N^-$. We recall from [13] how one can construct in this case some multiple of g'_{χ^q} from g'. For $i=1,\ldots,q-1$, set

$$\sigma_i=\begin{pmatrix}1 & \frac{i}{q}\\ 0 & 1\end{pmatrix}\in(B\otimes\mathbb{Q}_q)^\times$$

and identify σ_i with the corresponding element of $B_{\mathbb{A}}^{\times}$ which is 1 at all other places. Now set

$$\mathcal{R}_{\chi,q}(s)(x) = \chi^q(\mathrm{Nm}(x)) \left(\sum_{i=1}^{q-1} \overline{\chi^q(i)} s(x\sigma_{i,q}) \right)$$

for any $s \in S_{2k}(U^{\Pi})$. Then $\mathcal{R}_{\chi,q}(s_{g'})$ is a nonzero scalar multiple of $s_{g'_{\chi^q}}$. Write $\sigma_i = t_i^{-1} \cdot u_i$ for $t_i \in B^{\times}, u_i \in U^1(\chi)$. If s' corresponds to the classical form g', $\mathcal{R}_{\chi,q}(s')$ corresponds to the classical modular form $\sum_{i=1}^{q-1} \overline{\chi_q(i)} g'|_{t_i^{-1}}$. We then have a commutative diagram

$$\begin{array}{ccc} S_{2k}(\Gamma^{1,\Pi}) & \longrightarrow & H^1_p(\Gamma^{1,\Pi}, L(n,\mathbb{C})) \\ \Big\downarrow \mathcal{R}_{\chi,q} & & \Big\downarrow \phi_{\chi,q} \\ S_{2k}(\Gamma^{1,\Pi\cup\{q\}}) & \longrightarrow & H^1_p(\Gamma^{1,\Pi\cup\{q\}}, L(n,\mathbb{C})) \end{array}$$

where

$$\phi_{\chi,q}(\mathfrak{r})(\gamma) = \sum_{i=1}^{q-1} \overline{\chi^q(i)} \sigma(t_i^{-1}) \mathfrak{r}(t_i^{-1}\gamma t_i)$$

and the horizontal maps are isomorphisms as in the previous section. Clearly,

$$\phi_{\chi,q}(H^1(\Gamma^{1,\Pi}, L(n, A_{F,\lambda}))) \subseteq H^1(\Gamma^{1,\Pi\cup\{q\}}, L(n, A_{F,\lambda})).$$

Suppose $\mathcal{R}_{\chi,q}(g') = \delta_q \mathfrak{g}(\chi^q)^{-1} g'_{\chi^q}$. To prove (a)$'$ and (b)$'$ it suffices then to show that $\delta_q \in F(\chi)$ and $v_\lambda(\delta_q) = 0$ i.e. we need to compare the arithmetic properties of the form $\mathcal{R}_{\chi^q}(g')$ with those of g'. We now apply the rationality and integrality criteria of [12] and [22], formulated more precisely in our context in Prop. 5.1 below. Since $\mathcal{R}_{\chi^q}(g')$ and g' are the same except at the prime q and since g' is arithmetically normalized, the criteria above reduce the problem to studying the rationality and λ-divisibility of a certain ratio of local integrals at q. This ratio (being defined purely locally) is independent of the choice of quaternion algebra and so to compute it we might as well assume that $B = \mathrm{M}_2(\mathbb{Q})$. But in this case, we may pick $t_{i,q} = \begin{pmatrix} 1 & -i/q \\ 0 & 1 \end{pmatrix}$, $g' = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$ and directly compute

$$\begin{aligned} \mathcal{R}_{\chi,q}(g') &= \sum_{i=1}^{q-1} \overline{\chi^q(i)} g|_{t_i^{-1}} = \sum_{i=1}^{q-1} \overline{\chi^q(i)} \sum_n a_n e^{2\pi i n (z+\frac{i}{q})} = \mathfrak{g}(\overline{\chi^q}) \sum_{(n,q)=1} \chi^q(n) a_n e^{2\pi i n z} \\ &= \mathfrak{g}(\overline{\chi^q}) g'_{\chi^q}(z), \end{aligned}$$

which proves what is required. The case $q \mid N^-$ is somewhat easier since in this case g'_{χ^q} is a scalar multiple of g'. To study the arithmetic properties of this scalar we again apply the criteria mentioned above, from which the desired result follows easily. (For (a)$'$, one needs to make the observation that the CM periods p_K appearing in the rationality criterion satisfy $p_K(\eta \cdot \chi_q \circ \mathrm{Nm}_{K/\mathbb{Q}}, 1)/p_K(\eta, 1)\mathfrak{g}(\chi_q) \in K(\eta, \chi_q)$ for any imaginary quadratic field K and Hecke character η of K.)

Finally, we prove (c) (which in fact we never use in this article.) By [34], there exists a character η such that $\mathfrak{g}(\eta^{-1})|\mathfrak{c}_\eta|^{k-1}(2\pi i)^{-1}L(1,f,\eta) \sim u_\epsilon(f)$ where we use the symbol $\sim$ to denote equality up to a λ-adic unit. On the other hand $L(1,f,\eta) \sim L(1,f_\chi,\chi^{-1}\eta)$ since $p \nmid \tilde{N}$ and $\mathfrak{g}(\eta^{-1}\chi)|\mathfrak{c}_{\eta\chi^{-1}}|^{k-1}(2\pi i)^{-1}L(1,f_\chi,\chi^{-1}\eta)/u_+(f_\chi)$ has nonnegative λ-adic valuation, again by [34]. Thus $v_\lambda(u_\epsilon(f)/u_+(f_\chi)) \geq 0$ and combining this with part (b) we see that $v_\lambda(\gamma) = 0$. ■

Corollary 4.7. *Let* $\alpha' = \alpha'(g, \chi, \nu, F, \lambda)$. *Set* $\alpha = \alpha'\gamma$ *and* $\boldsymbol{\alpha} := \alpha \cdot \mathfrak{g}(\chi)^{-1}$. *Then* $\boldsymbol{\alpha} \in F(\chi)$ *and* $v_\lambda(\boldsymbol{\alpha}), v_\lambda(\alpha) \geq 0$.

Finally, we specialize to $\chi = 1$. Writing $\boldsymbol{\alpha}(g, F, \lambda)$ in this case to express the dependence on g, F, λ, we have for all $\sigma \in \mathrm{Aut}(\mathbb{C}/\tilde{F}_0)$, (and from part (a) of Thm. 4.5)

Proposition 4.8.

$$(\boldsymbol{\alpha}(g, F, \lambda))^\sigma = \boldsymbol{\alpha}(g^\sigma, F^\sigma, \lambda^\sigma). \tag{4.5}$$

5. Arithmetic properties of the Shimura lift

In this section, we study the rationality and integrality of the Shimura lift i.e. of the constant β appearing in Prop. 3.5.

5.1. **CM periods and criteria for rationality and integrality.** Let K be an imaginary quadratic field unramified at the primes dividing N and $K \hookrightarrow B$ be a Heegner embedding for the order $\mathcal{O}'(\chi)$ i.e. an embedding of K in B such that $\mathcal{O}'(\chi) \cap K = \mathcal{O}_K$. Such an embedding exists exactly when K is inert at all primes dividing N^- and split at the primes dividing N^+. Let z be the associated Heegner point on $\mathfrak{H}$ (i.e. the unique fixed point on $\mathfrak{H}$ of $(K \otimes \mathbb{R})^\times$) and η' a Grossencharacter of K of infinity type $(-k, k)$ i.e. satisfying $\eta'(xx_\infty) = \eta'(x)x_\infty^k \overline{x}_\infty{}^{-k}$ for $x \in K_\mathbb{A}^\times, x_\infty \in K_\infty^\times$. Equivalently η' is the Grossencharacter corresponding to an algebraic Hecke character of type $(-k, k)$. Define

$$L_{\eta'}(s) = j(\alpha, i)^{2k} \int_{K^\times K_\infty^\times \backslash K_\mathbb{A}^\times} s(x\alpha)\eta'(x) d^\times x$$

for $s \in \pi' \otimes \chi$ and $\alpha \in \mathrm{SL}_2(\mathbb{R})$ being any element such that $\alpha(i) = z$ or equivalently, $\alpha \cdot \mathrm{SO}_2(\mathbb{R}) \cdot \alpha^{-1} = (K \otimes \mathbb{R})^{(1)}$. Of particular interest to us are characters of the following type. The inclusion $K_\mathbb{A}^\times \hookrightarrow B_\mathbb{A}^\times$ maps U_K into $U_0(\chi)$, where $U_K := \hat{\mathcal{O}}_K^\times$. Let Σ_K denote the set of Hecke characters of K of infinity type $(-k, k)$ whose restriction to U_K equals $\tilde{\omega}_\chi^{-1}|_{U_K}$. Clearly Σ_K has cardinality equal to the class number of K. There is some abuse of notation since Σ_K depends on the choice of Heegner point and not just on K. Note that for $\eta' \in \Sigma_K$, $\eta'|_{\mathbb{Q}_\mathbb{A}^\times} = \chi^{-2}$.

We now pick an element $\tilde{j} \in B$ such that $\tilde{j} \in N_{B^\times}(K^\times)$ and $B = K + K\tilde{j}$. Let $\mathcal{I}$ be the ideal in K given by $\mathcal{I} = \{x \in K; x\tilde{j} \in \mathcal{O}'(\chi)\}$. Since p is split in B and $\mathcal{O}'(\chi) \otimes \mathbb{Z}_p$ is the maximal order in $B \otimes \mathbb{Q}_p$, it is clear that we may pick $\tilde{j}$ such that $\mathcal{I}$ and (hence) $\mathrm{Nm}\,\tilde{j}$ are both prime to p. Let $\hat{\eta} = \eta'\mathbb{N}^{-k}$ (where $\mathbb{N}$ is the usual norm character) and denote by $\hat{\boldsymbol{\eta}}$ the algebraic Hecke character corresponding to $\hat{\eta}$. Also let $\Omega(\hat{\boldsymbol{\eta}}) = (2\pi i)^{2k} p_K(\hat{\boldsymbol{\eta}}, 1) \in \mathbb{C}^\times/\mathbb{Q}(\hat{\boldsymbol{\eta}})^\times$ where $p_K(\hat{\boldsymbol{\eta}}, 1)$ is the period defined in [10] and let Ω be the period defined in [22], Sec. 2.3.3. that is well defined up to a λ-adic unit. The following proposition is a mild strengthening of Prop A.9 of [12] Appendix, and Prop. 2.9 of [22]. (In the statement below, $(\eta')^\sigma$is the Grossencharacter associated to $\hat{\boldsymbol{\eta}}^\sigma\mathbb{N}^k$.)

Proposition 5.1. *Suppose* $s'' = \beta s_{g_\chi}$.
(a) $\beta \in \mathbb{Q}(f, \chi)$ *if and only if for all (or even infinitely many) Heegner points* $K \hookrightarrow B$ *and all* $\eta' \in \Sigma_K$,

$$(2\pi i)^k \{\pi J(\tilde{j}, z)\Im(z)\}^k L_{\eta'}(s'')/\Omega(\hat{\boldsymbol{\eta}}) \in \overline{\mathbb{Q}},$$

and for all $\sigma \in Gal(\overline{\mathbb{Q}}/K \cdot \mathbb{Q}(f,\chi))$,

$$\left(\frac{(2\pi i)^k \{\pi J(\tilde{j},z)\Im(z)\}^k L_{\eta'}(s'')}{\Omega(\hat{\boldsymbol{\eta}})} \right)^{\sigma} = \frac{(2\pi i)^k \{\pi J(\tilde{j},z)\Im(z)\}^k L_{(\eta')^{\sigma}}(s'')}{\Omega(\hat{\boldsymbol{\eta}}^{\sigma})}.$$

(b) Suppose $\beta \in \overline{\mathbb{Q}}$. *Then* $v_\lambda(\beta) \geq 0$ *if and only if for all Heegner points* $K \hookrightarrow B$ *with* $p \nmid h_K$ *(the class number of* K*), and all* $\eta' \in \Sigma_K$,

$$v_\lambda \left(\{\pi^2 J(\tilde{j},z)\Im(z)\}^k \cdot \frac{L_{\eta'}(s'')}{\Omega^k} \right) \geq 0.$$

Further, it suffices to check this last condition for any set of Heegner points that reduce mod p *to an infinite set of points on the special fiber of* $X_{U_1(\chi)}$.

In our case $s' = \beta s$ and $s'' = s' \otimes (\chi\chi_\nu \circ \mathrm{Nm})$. Note that for any $\eta' \in \Sigma_K$, the character $\eta' \cdot ((\chi\chi_\nu) \circ \mathrm{Nm})$ is trivial when restricted to $\mathbb{Q}_{\mathbb{A}}^{\times}$, hence there exists a Grossencharacter $\tilde{\eta}$ of K of infinity type $(0,k)$ such that $\tilde{\eta}(\tilde{\eta}^{\rho})^{-1} = \eta' \cdot ((\chi\chi_\nu) \circ \mathrm{Nm}_{K/\mathbb{Q}})$. (Here and henceforth, ρ denotes the complex conjugation of K.) Picking such a character $\tilde{\eta}$, we set $\eta = \tilde{\eta} \cdot \mathbb{N}^{-k/2}$ so that $\eta(\eta^{\rho})^{-1} = \eta' \cdot ((\chi\chi_\nu) \circ \mathrm{Nm}_{K/\mathbb{Q}})$ as well. In future, we will denote $\mathrm{Nm}_{K/\mathbb{Q}}$ simply by the symbol Nm, since it agrees with the reduced norm restricted to $K \hookrightarrow B$.

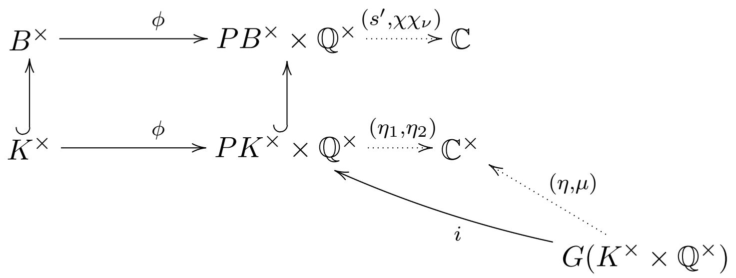
Let $B = K \oplus K^{\perp}$ be the orthogonal decomposition of B for the norm form, so that $V = K^0 \oplus K^{\perp}$. Set $V_1 = K^0$ and $V_2 = K^{\perp}$. Then $\mathrm{O}(V_1) = \{\pm 1\}$, $\mathrm{O}(V_2)^0 = K^{(1)}$. We will need to work below with the corresponding (connected components of) similitude groups. Note that $\mathrm{GO}(V)^0$ is identified with $PB^{\times} \times \mathbb{Q}^{\times}$, the action of $([x],a)$ being by $y \mapsto a \cdot (x^{-1}yx)$. Then we have the natural map $\phi : B^{\times} \to PB^{\times} \times \mathbb{Q}^{\times}$ given by $\phi(x) = ([x], \mathrm{Nm}\, x)$ and the form s'' on $B^{\times}$ is obtained by pulling back the form $(s', \chi\chi_\nu)$ on $PB^{\times} \times \mathbb{Q}^{\times}$. Let H be the group $\mathrm{G}(\mathrm{O}(V_1) \times \mathrm{O}(V_2))^0 = \mathrm{G}(\mathbb{Q}^{\times} \times K^{\times}) = \{(a,b) \in \mathbb{Q}^{\times} \times K^{\times}, a^2 = \mathrm{Nm}_{K/\mathbb{Q}}\, b\}$. For $(a,b) \in H$, we have $\mathrm{Nm}_{K/\mathbb{Q}}(a^{-1}b) = 1$, hence there exists $c \in K^{\times}$ such that $a^{-1}b = c^{\rho}/c$. Now the action of (a,b) on $y = y_1 + y_2\mathfrak{J} \in V$ is given by

$$y_1 + y_2\mathfrak{J} \mapsto ay_1 + by_2\mathfrak{J} = ay_1 + a\frac{c^{\rho}}{c}\mathfrak{J} = a \cdot c^{-1}(y_1 + y_2\mathfrak{J})c,$$

so that the natural inclusion $H \hookrightarrow \mathrm{GO}(V)^0$ is identified with $i : (a,b) \mapsto ([c^{-1}], a) \in PK^{\times} \times \mathbb{Q}^{\times} \subset PB^{\times} \times \mathbb{Q}^{\times}$. Set $\eta_2 = \chi^{-1}\chi_\nu$, $\eta_1 = \eta' \cdot (\chi\chi_\nu) \circ \mathrm{Nm}_{K/\mathbb{Q}}$, so that η' is the pullback of (η_1, η_2) via ϕ. Recall that η has been chosen such that $\eta_1 = \eta(\eta^{\rho})^{-1}$. Thus

$$((\eta_1,\eta_2) \circ i)(a,b) = \eta_1(c^{-1})\eta_2(a) = \eta(\frac{c^{\rho}}{c})\eta_2(a) = \eta(b)\mu(a),$$

where $\mu(a) = \eta^{-1}|_{\mathbb{Q}^{\times}}(a)\eta_2(a)$. Diagrammatically, we have



where the solid arrows denote maps of algebraic groups and the dotted arrows represent automorphic forms on the corresponding adelic groups.

Suppose that $\varphi_\infty(\alpha^{-1}\cdot\alpha)=\varphi_{1,\infty}\otimes\varphi_{2,\infty}\in\mathcal{S}_{\psi'}(V_1(\mathbb{R}))\otimes\mathcal{S}_{\psi'}(V_2(\mathbb{R}))$ and for finite primes q, $\varphi_q=\sum_{i_q\in I_q}\varphi_{1,i_q}\otimes\varphi_{2,i_q}\in\mathcal{S}_{\psi'}(V_1(\mathbb{Q}_q))\otimes\mathcal{S}_{\psi'}(V_2(\mathbb{Q}_q))$. By see-saw duality,

$$\begin{aligned}
j(\alpha,i)^{-2k}L_{\eta'}(s') &= \int_{H(\mathbb{Q})\backslash H(\mathbb{A})} T_{\psi'}(\varphi,g,h_\chi)\eta'(g)((\chi\chi_\nu)(\mathrm{Nm}(g))d^\times g\\
&= \int_{\mathrm{O}(V_1)\times\mathrm{O}(V_2)(\mathbb{Q})\backslash\mathrm{O}(V_1)\times\mathrm{O}(V_2)(\mathbb{A})} T_{\psi'}(\varphi,(g_1,g_2),h_\chi)\mu(g_1)\eta(g_2)d^\times g_1 d^\times g_2\\
&= \langle T_{\psi'}(\varphi,h_\chi)(g_1,g_2),\overline{\mu}(g_1)\overline{\eta}(g_2)\rangle\\
&= \sum_{i=(i_q)\in\prod_q I_q}\langle h_\chi, t_{\psi'}(\varphi,\overline{\mu})\cdot t_{\psi'}(\varphi,\overline{\eta})\rangle\\
&= \sum_{i=(i_q)\in\prod_q I_q}\int_{\mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{SL}_2(\mathbb{A})} h_\chi(\sigma)t_{\psi_0}(\otimes\overline{\varphi_{1,i_q}},\sigma,\mu)t_{\psi_0}(\otimes\overline{\varphi_{2,i_q}},\sigma,\eta)d^{(1)}\sigma,
\end{aligned}$$

where $\psi_0=\overline{\psi'}$.

In the following section, we will show that for the purposes of computing the integral above, we may alter $\overline{\varphi_q}$ so that it is a pure tensor of a particularly simple form. With this goal in mind, we set up some notation. Let q be a prime and suppose that we have fixed for all $l\neq q$, Schwartz functions $\varsigma_l\in\mathcal{S}_{\psi_0}(V_1(\mathbb{Q}_l)),\vartheta_l\in\mathcal{S}_{\psi_0}(V_2(\mathbb{Q}_l))$. Then for any $\varsigma\in\mathcal{S}_{\psi_0}(V_1(\mathbb{Q}_q)),\vartheta\in\mathcal{S}_{\psi_0}(V_2(\mathbb{Q}_q))$, set

$$I(\varsigma,\vartheta)=\int_{\mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{SL}_2(\mathbb{A})} h_\chi(\sigma)t_{\psi_0}(\varsigma\otimes\varsigma^q,\sigma,\mu)t_{\psi_0}(\vartheta\otimes\vartheta^q,\sigma,\eta)d^{(1)}\sigma, \tag{5.1}$$

where $\varsigma^q=\otimes_{l\neq q}\varsigma_l,\vartheta^q=\otimes_{l\neq q}\vartheta_l$. Suppose $\delta_q\in B_q^\times$ is chosen such that $\varphi_q^\delta(\cdot):=\varphi_q(\delta_q^{-1}\cdot\delta_q)$ is a scalar multiple of φ_q. Let $i_q^\delta: K\otimes\mathbb{Q}_q\hookrightarrow B_q$ be given by $i_q^\delta(x)=\delta_q x\delta_q^{-1}$ and set $W=i_q^\delta(K\otimes\mathbb{Q}_q)$. Also let $f:B\to B$ denote the isomorphism given by conjugation by δ, i.e. $f(x)=\delta x\delta^{-1}$. Then f induces isomorphisms of quadratic spaces $f: V_{1,q}\simeq W$ and $f: V_{2,q}\simeq W^\perp$. Now, for $\varsigma\in\mathcal{S}_{\psi'}(W),\vartheta\in\mathcal{S}_{\psi'}(W^\perp)$, set $\varsigma^\delta=f^*(\varsigma),\vartheta^\delta=f^*(\vartheta)$ and $J(\varsigma,\vartheta)=I(\varsigma^\delta,\vartheta^\delta)$.

We now need to compute the theta lift of η to $\mathrm{SL}_2(\mathbb{A})$. However it is more useful to compute the theta lift of η to $\mathrm{GL}_2(\mathbb{A})$ using the extension of the theta correspondence to similitude groups (as in [12]). We have then for $\sigma\in\mathrm{GL}_2(\mathbb{A})$,

$$\begin{aligned}
t_{\psi_0}(\vartheta^\delta\otimes\vartheta^q,\sigma,\eta) &= \int_{K^{(1)}\backslash K^{(1)}_{\mathbb{A}}}\sum_{x\in V_2} r_{\psi_0}(\sigma,h\tilde{h})(\vartheta^\delta\otimes\vartheta^q)(x)\eta(h\tilde{h})d_1^\times h\\
&= \int_{K^{(1)}\backslash K^{(1)}_{\mathbb{A}}}\sum_{x\in V_2} r_{\psi_0}(\sigma^q,h^q\tilde{h}^q)\vartheta^q(x)r_{\psi_0}(\sigma_q,h_q\tilde{h}_q)\vartheta^\delta(x)\eta(h\tilde{h})d_1^\times h\\
&= \int_{K^{(1)}\backslash K^{(1)}_{\mathbb{A}}}\sum_{x\in V_2} r_{\psi_0}(\sigma^q,h^q\tilde{h}^q)\vartheta^q(x)r_{\psi_0}(\sigma_q,h_q\tilde{h}_q)\vartheta(x^\delta)\eta(h\tilde{h})d_1^\times h,
\end{aligned} \tag{5.2}$$

for $\tilde{h} \in K_{\mathbb{A}}^{\times}$ with $\mathrm{Nm}(\tilde{h}) = \det(\sigma)$, where the measure $d_1^{\times}h$ is defined as in [22], p.925. Likewise,

$$t_{\psi_0}(\varsigma^{\delta} \otimes \varsigma^{q}, \sigma, \mu) = \int_{\{\pm 1\}\backslash\{\pm 1\}_{\mathbb{A}}} \sum_{x \in V_1} r_{\psi_0}(\sigma^q, h^q\tilde{h}^q)\vartheta^q(x) r_{\psi_0}(\sigma_q, h_q\tilde{h}_q)\varsigma(x^{\delta})\mu(h\tilde{h}) d_1^{\times}h$$

(5.3)

for $\tilde{h} \in \mathbb{Q}_{\mathbb{A}}^{\times}$ with $\tilde{h}^2 = \det(\sigma)$. For convenience of notation, set $t_1(\varsigma, \sigma) = t_{\psi_0}(\varsigma^{\delta} \otimes \varsigma^{q}, \sigma, \mu)$ and $t_2(\vartheta, \sigma) = t_{\psi_0}(\vartheta^{\delta} \otimes \vartheta^{q}, \sigma, \eta)$.

Suppose $\varphi_i \in \mathcal{S}_{\psi_0}(V_i(\mathbb{A}))$, $\varphi_1 = \otimes_q \varsigma_q^{\delta}$, $\varphi_2 = \otimes_q \vartheta_q^{\delta}$, with $\delta = (\delta_q)$. Then for $g \in \mathrm{GL}_2(\mathbb{A})$, $\det(g) \in \mathrm{Nm}(K_{\mathbb{A}}^{\times})$, $\nu_0 = -|\nu|$,

$$t_{\psi_0}(\varphi_2, \sigma, \eta) = \sum_{\substack{\xi \in \mathbb{Q}^{\times} \\ \xi\nu_0 \,\mathrm{Nm}(j)^{-1} \in \mathrm{Nm}(K_{\mathbb{A}}^{\times})}} W_{\eta}^{\psi}\left(\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix} g\right),$$

where, choosing $\tilde{h} = (\tilde{h}_q)$ such that $\mathrm{Nm}(\tilde{h}) = \mathrm{Nm}(j)^{-1}\nu_0 \det(g)$,

$$\begin{aligned} W_{\eta}^{\psi}(g) &= \int_{K_{\mathbb{A}}^{(1)}} r_{\psi_0}(\mathbf{a}(\mathrm{Nm}(j)^{-1}\nu_0)g, h\tilde{h})\varphi_2(j)\eta(h\tilde{h}) d_1^{\times}h = \prod_q W_{\eta,q}^{\psi}(g_q), \\ W_{\eta,q}^{\psi}(g_q) &= \int_{K_q^{(1)}} r_{\psi_0}(\mathbf{a}(\mathrm{Nm}(j)^{-1}\nu_0)g_q, h\tilde{h}_q)\vartheta_q^{\delta}(j)\eta(h\tilde{h}_q) d_1^{\times}h. \end{aligned}$$

Suppose $f_q(j) = \alpha_q j_q$. Since $r_{\psi_0}(\mathbf{a}(\mathrm{Nm}(\alpha_q)), \alpha_q^i)\vartheta_q(\cdot) = |\alpha_q|^{1/2}\vartheta_q(\alpha_q \cdot)$,

$$\begin{aligned} W_{\eta,q}^{\psi}(g_q) &= |\alpha_q|^{-1/2}\eta(\alpha_q^i)^{-1}\Theta_{\eta}(g_q), \\ \Theta_{\eta}(g_q) &= \int_{K_q^{(1)}} r_{\psi_0}(\mathbf{a}(\mathrm{Nm}(j_q)^{-1}\nu_0)g_q, h\tilde{h}_q)\vartheta_q(j_q)\eta(h\tilde{h}_q) d_1^{\times}h, \end{aligned}$$

where now $\mathrm{Nm}(\tilde{h}_q) = \mathrm{Nm}(j_q)^{-1}\nu_0 \det(g_q)$, and $\Theta_{\eta}(g_q) = 0$ if $\mathrm{Nm}(j_q)^{-1}\nu_0 \det(g_q) \notin \mathrm{Nm}(K_q^{\times})$.

On the other hand, the theta lift $t_{\psi_0}(\varphi_1, \sigma, \mu)$ could possibly be an Eisenstein series. Suppose $K = \mathbb{Q}(\sqrt{-d})$ with d square-free and set $v_0 = \sqrt{-d}$. Then setting $\tilde{\psi} = \psi_0^d$ one easily computes the Fourier development of $t_{\psi_0}(\varphi_1, \sigma, \mu)$ (for $\sigma \in \widetilde{\mathrm{S}}_{\mathbb{A}}$) to be given by

$$t(\psi_0, \varphi_1, \sigma, \mu) = C_0(\sigma) + \sum_{\xi \in \mathbb{Q}^{>0}} W_{\mu}^{\tilde{\psi}}(\mathbf{d}(\xi)\sigma),$$

where

$$C_0(\sigma) = \begin{cases} 0, \text{ if } \mu \text{ is not a square.} \\ r_{\psi_0}(\sigma)\varphi_1(0) = \prod_q r_{\psi_0}(\sigma_q)\varsigma(0) \text{ if } \mu \text{ is a square.} \end{cases}$$

and

$$W_{\mu}^{\tilde{\psi}}(\sigma) = \int_{\{\pm 1\}_{\mathbb{A}}} r_{\psi_0}(\sigma, h)\varphi_1(v_0)\mu(h) d_1^{\times}h = \prod_q \Theta_{\mu}(\sigma_q),$$

$$\Theta_{\mu}(\sigma_q) = \int_{\{\pm 1\}} r_{\psi_0}(\sigma_q, h)\varsigma_q(v_0)\mu_q(h) d_1^{\times}h = \frac{1}{2}[r_{\psi_0}(\sigma_q)\varsigma_q(v_0) + \mu_q(-1)r_{\psi_0}(\sigma_q)\varsigma_q(-v_0)],$$

$$\Theta_0(\sigma_q) = r_{\psi_0}(\sigma_q)\varsigma(0).$$

Let ς_q^μ denote the $\mu_q(-1)$ component of ς_q i.e. $\varsigma_q^\mu(\sigma_q) = \frac{1}{2}[\varsigma_q(\sigma_q) + \mu_q(-1)\varsigma_q(-\sigma_q)]$ and set $\varsigma^\mu = \otimes \varsigma_q^\mu$. Then

$$\begin{aligned} C_0(\sigma) &= r_{\psi_0}(\sigma)\varsigma^\mu(0) \quad W_\mu^{\tilde{\psi}}(\sigma) = r_{\psi_0}(\sigma)\varsigma^\mu(v_0). \\ t(\psi_0,\varphi_1,\sigma,\mu) &= r_{\psi_0}(\sigma)\varsigma^\mu(0) + \sum_{\xi\in\mathbb{Q}^{>0}} r_{\psi_0}(\mathbf{d}(\xi)\sigma)\varsigma^\mu(v_0) = \sum_{\xi\in\mathbb{Q}^{\geq 0}} r_{\psi_0}(\sigma)\varsigma^\mu(\xi v_0). \end{aligned}$$

5.2. **Local analysis of the triple integral.** Let π_η denote the automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ corresponding to the character η. Let $\tilde{\Omega}$ be the set of primes dividing $N\nu$ at which π_η is supercuspidal and $\tilde{\Omega}'$ the set of primes dividing $\gcd(\nu,d)$. We will see later that π_η must be a ramified principal series representation at $q \in \tilde{\Omega}'$, hence $\tilde{\Omega}$ and $\tilde{\Omega}'$ are mutually exclusive sets. Denote by Σ (resp. Σ') the set of positive square-free integers all whose prime factors lie in $\tilde{\Omega}$ (resp. $\tilde{\Omega}'$.) In what follows, t will denote any element of Σ and χ_t is as usual the quadratic character $\left(\frac{t}{\cdot}\right)$. Also we use the symbol $\tilde{W}^\psi$ to denote an anti-newform in the $\overline{\psi}$-Whittaker model of π_η i.e. one that transforms by a character of a rather than that of d for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}_q)$. Further, let $A_q(s) = D_q(s-k,\theta_\eta,\theta_{\eta_\rho})$ (defined as in [27]), $B_q(s) = L_q(\eta(\eta^\rho)^{-1},s)$ and set

$$C_q(s) = A_q(s)B_q(s)^{-1}\zeta_{K,q}(s)^{-1}\zeta_{\mathbb{Q},q}(2s),$$

so that

$$D_q(s+k,\theta_\eta,\theta_{\eta,\rho}) = C_q(s)\cdot\frac{L_q(\eta(\eta^\rho)^{-1},s)\zeta_{K,q}(s)}{\zeta_{\mathbb{Q},q}(2s)}.$$

For each q, we also define an integer c_q that is set to be equal to 1 except when explicitly listed below. In what follows, we denote by η_K the quadratic character associated to the quadratic extension $K/\mathbb{Q}$. Further, for the rest of this section, $\mathcal{F}$ will denote the Fourier transform taken with respect to the character ψ_0.

5.2.1. *Case A:* $(q,2N\nu)=1$. **Subcase (i):** K is split at q. Then $K_q \simeq \mathbb{Q}_q\times\mathbb{Q}_q$, $B_q \simeq M_2(\mathbb{Q}_q)$. Set $\mathfrak{r} = \mathbb{Z}_q\times\mathbb{Z}_q$. We may pick $\delta_q \in \mathrm{GL}_2(\mathbb{Z}_q)$ such that $i_q^\delta(a,b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. Let $j_q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then $\overline{\varphi_q} = \varphi_q = \varsigma\otimes\vartheta$, where $\varsigma = \mathbb{I}_{\mathfrak{r}^0}$, $\vartheta = \mathbb{I}_{\mathfrak{r}j_q}$.

It is easy to see that Θ_η, Θ_μ and Θ_0 are right invariant by $\overline{\mathbf{n}}(x), \underline{\mathbf{n}}(y)$ for $v_q(x)\geq 0, v_q(y)\geq 0$. Suppose $\eta = (\lambda_1,\lambda_2)$. Then λ_1/λ_2 is unramified. Set $\lambda = \lambda_1|_{\mathbb{Z}_q^\times} = \lambda_2|_{\mathbb{Z}_q^\times}$, and $\alpha = \lambda_1(\pi), \beta = \lambda_2(\pi)$, for π a uniformiser in $\mathbb{Z}_q$. Also note that $\mu_q(-1)=1$. Then

$$\begin{aligned} \Theta_\eta\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} &= |a|^{1/2}\lambda(\nu_0 a/\pi^n)\frac{\beta^{n+1}-\alpha^{n+1}}{\beta-\alpha}\mathbb{I}_{\mathbb{Z}_q}(a), \qquad \text{if } v_q(a)=n; \\ \Theta_\mu(\mathbf{d}(a)) &= |a|^{1/2}\mu_{\psi_0}(a)\chi_{d,q}(a)\mathbb{I}_{\mathbb{Z}_q}(a), \quad \Theta_0(\mathbf{d}(a)) = |a|^{1/2}\mu_{\psi_0}(a)\chi_{d,q}(a). \end{aligned}$$

If λ_1 and λ_2 are unramified, so that λ is trivial and μ_q is unramified,

$$\Theta_\eta = \tilde{W}_\eta^\psi = \tilde{W}_{\eta\otimes\chi_t}^\psi\otimes\chi_t$$

for any $t\in\Sigma$. By a familiar computation (see [27]), $C_q(s)=1$.

Subcase (ii): K is inert at q. Then $K_q = \mathbb{Q}_q(v)$, where $v^2 = u$ is a non-square unit in $\mathbb{Z}_q$. Set $\mathfrak{r} = \mathbb{Z}_q + \mathbb{Z}_q v$. We may pick $\delta_q \in \mathrm{GL}_2(\mathbb{Z}_q)$ such that $i_q^\delta(v) = \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix}$. Let $j_q = \begin{pmatrix} 0 & -1 \\ u & 0 \end{pmatrix}$. Then $\overline{\varphi_q} = \varphi_q = \varsigma \otimes \vartheta$, where $\varsigma = \mathbb{I}_{\mathfrak{r}^0}$, $\vartheta = \mathbb{I}_{\mathfrak{r}j_q}$.

Since any unit in $K_q^{(1)}$ is of the form $\kappa/\bar{\kappa}$ for some unit κ, we see that $\eta_q|_{K_q^{(1)}}$ is trivial, whence η factors as $\lambda \circ \mathrm{Nm}$ and $\mu_q(-1) = 1$. Again, Θ_η, Θ_μ and Θ_0 are right invariant by $\overline{\mathbf{n}}(x), \underline{\mathbf{n}}(y), x, y \in \mathbb{Z}_q$ and

$$\begin{aligned} \Theta_\eta \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} &= \frac{1}{2}(1 + \eta_{K,q}(a))|a|^{1/2}\lambda(\nu_0 ua)\mathbb{I}_{\mathbb{Z}_q}(a); \\ \Theta_\mu(\mathbf{d}(a)) &= \mu_{\psi_0}(a)\chi_{d,q}(a)|a|^{1/2}\mathbb{I}_{\mathbb{Z}_q}(a), \quad \Theta_0(\mathbf{d}(a)) = |a|^{1/2}\mu_{\psi_0}(a)\chi_{d,q}(a), \end{aligned}$$

where $\tilde{h}$ is any element of K_q with $\mathrm{Nm}(\tilde{h}) = \nu_0 ua$. If λ is chosen to be unramified (so that μ_q is also unramified,)

$$\Theta_\eta = \tilde{W}_\eta^\psi = \tilde{W}^\psi_{\eta\otimes\chi_t} \otimes \chi_t$$

for any $t \in \Sigma$. Again, $C_q(s) = 1$.

Subcase (iii): K is ramified at q. Then $K_q = \mathbb{Q}_q(v)$, where $v^2 = \pi$ is a uniformizer at q. (Without loss, we may take $v = v_0$.) Set $\mathfrak{r} = \mathbb{Z}_q + \mathbb{Z}_q v$. We may pick $\delta_q \in \mathrm{GL}_2(\mathbb{Z}_q)$ such that $i_q^\delta(v) = \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}$. Let $j_q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $\overline{\varphi_q} = \varphi_q = \sum_{i=0}^{q-1} \varsigma_i \otimes \vartheta_i$, where $\varsigma_i = \mathbb{I}_{(\frac{i}{\pi}+\mathbb{Z}_q)v}$ and $\vartheta_i = \mathbb{I}_{(\mathbb{Z}_q+(\frac{i}{\pi}+\mathbb{Z}_q)v)j_q}$. Set $J_{ij} = J(\varsigma_i, \vartheta_j)$. For $y \in \mathbb{Q}_q$ denote by n_y the element $\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}_q)$. Since $h_\chi(\sigma n_1) = h_\chi(\sigma)$, $r_{\psi_0}(n_1)\varsigma_i = \psi_0(i^2/\pi)\varsigma_i$ and $r_{\psi_0}(n_1)\vartheta_j = \psi_0(-j^2/\pi)\vartheta_j$, we see that $J_{ij} = 0$ if $i^2 \neq j^2$. For $a \in \{1, \ldots, q-1\}$ let $d_a = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_q)$. Since $h_\chi(\sigma d_a) = h_\chi(\sigma)$, $r_{\psi_0}(d_a)\varsigma_i = \varsigma_{ai}$ and $r_{\psi_0}(d_a)\vartheta_j = \vartheta_{aj}$ we get $J_{ij} = J_{(ai)(aj)}$, hence $\sum_i J_{ii} = J_{00} + (q-1)J_{11}$. Finally, let $\beta = (\beta_l) \in \mathbb{Q}_\mathbb{A}^\times$ be the element given by $\beta_q = -1$, $\beta_l = 1$ if $l \neq q$. Making the change of variables $h \mapsto h\beta$ in (5.3), one gets $J_{ij} = \mu_q(-1)J_{(-i)j}$.

We now make the following observation. A unit $z = x + yv \in \mathfrak{r}, x, y \in \mathbb{Z}_q, y \neq 0$ with norm 1 such that $v_q(x+1) \le v_q(y)$ is always of the form $\kappa/\bar{\kappa}$ for some unit $\kappa \in \mathfrak{r}$. In particular, for such units z,

$$\eta_q(z) = \eta_q(\kappa/\bar{\kappa}) = \eta'(\kappa)\chi_q\chi_{\nu,q}(\mathrm{Nm}(\kappa)) = 1.$$

If $x \not\equiv -1 \mod q$ and $y \neq 0$, this shows that $\eta_q(z) = 1$ and by continuity the same is true without the assumption $y \neq 0$. If $q > 3$ (as we may always arrange to be the case by picking K appropriately), this forces $\eta_q(z) = 1$ even if $x \equiv -1 \mod q$. Thus η_q and μ_q must be unramified, hence $\mu_q(-1) = \eta_q(-1) = 1$. Let $\varsigma' = \sum_i \varsigma_i = \mathbb{I}_{\frac{1}{q}\mathbb{Z}_q v}$ and $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_q)$. Since $h_\chi(\sigma w) = h_\chi(\sigma)$, $\mathcal{F}(\varsigma') = q^{1/2}\varsigma_0$ and $\mathcal{F}(\vartheta_j) = q^{-1/2}\psi_0(\langle -\frac{j}{\pi}v, \cdot\rangle)\mathbb{I}_{\frac{1}{v}\mathfrak{r}j_q}$,

$$\sum_i J_{ij} = J(\varsigma', \vartheta_j) = J(\mathcal{F}(\varsigma'), \mathcal{F}(\vartheta_j)) = \sum_i \psi_0(\langle -\frac{j}{\pi}v, \frac{i}{\pi}v\rangle) J(\varsigma_0, \vartheta_i) = J_{00}.$$

Since $J_{ii} = J_{(-i)i}$, we have $2J_{ii} = J_{00}$ for $i \neq 0$. Thus $J = J_{00} + (q-1)(\frac{1}{2}J_{00}) = \frac{1}{2}(q+1)J_{00} = \frac{1}{2}(q+1)J(\varsigma, \vartheta)$ for $\varsigma = \varsigma_0, \vartheta = \vartheta_0$. Suppose $\eta_q = \lambda \circ \mathrm{Nm}$ with λ unramified, so that $\pi_\eta \simeq \pi(\eta_K\lambda, \lambda)$. Then one checks that $\Theta_\eta, \Theta_\mu, \Theta_0$ are all invariant by $\overline{\mathbf{n}}(x), \underline{\mathbf{n}}(y), v_q(x) \geq 0, v_q(y) \geq 1$ and

$$\begin{aligned}\Theta_\eta(\mathbf{d}(a)) &= |a|^{1/2}\eta_{K,q}(-\nu_0 a)\lambda(-\nu_0 a)\mathbb{I}_{\mathbb{Z}_q}(a);\\ \Theta_\mu(\mathbf{d}(a)) &= |a|^{1/2}\mu_{\psi_0}(a)\chi_{d,q}(a)\mathbb{I}_{\mathbb{Z}_q}(a), \qquad \Theta_0(\mathbf{d}(a)) = |a|^{1/2}\mu_{\psi_0}(a)\chi_{d,q}(a).\end{aligned}$$

so that $\Theta_\eta = \eta_{K,q}(-\nu_0)\tilde{W}^\psi_\eta = \eta_{K,q}(-\nu_0)\tilde{W}^\psi_{\eta\otimes\chi_t}\otimes\chi_t$ for any $t \in \Sigma$. Also, $A_q(s) = (1-q^{-s})^{-1}$, $B_q(s) = (1-q^{-s})^{-1}$, $\zeta_{K_q}(s) = (1-q^{-s})^{-1}$ and $\zeta_{\mathbb{Q},q}(2s) = (1-q^{-2s})^{-1}$. Thus $C_q(s) = (1+q^{-s})^{-1}$. Set $c_q = (q+1)$.

5.2.2. *Case B: $q \mid \nu$, $(q, 2N) = 1$.* **Subcase (i):** K is split at q. Then $K_q \simeq \mathbb{Q}_q \times \mathbb{Q}_q$, $B_q \simeq M_2(\mathbb{Q}_q)$. It could happen that $q = p$, in which case we pick the first factor to correspond to the completion at $\mathfrak{p}$ and the second to $\overline{\mathfrak{p}}$ where $\mathfrak{p}$ is the prime induced by λ on K. Suppose $\eta_q = (\lambda_1, \lambda_2)$. Set $\mathfrak{r} = \mathbb{Z}_q \times \mathbb{Z}_q$. We may pick $\delta_q \in \mathrm{GL}_2(\mathbb{Z}_q)$ such that $i_q^\delta(a,b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. Let $j_q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $v = (1,-1) \in \mathbb{Q}_q \times \mathbb{Q}_q$. Also for $i, j, k \in \{0, 1, \ldots, q-1\}$, set $\varsigma_i = \mathbb{I}_{(q\mathbb{Z}_q + i)v}, \vartheta_{jk} = \mathbb{I}_{(q\mathbb{Z}_q+j, q\mathbb{Z}_q+k)j_q}$. One checks easily that

$$\overline{\varphi_q} = \varphi_q = \sum_{j=1}^{q-1} \varrho(j)\varsigma_0 \otimes \vartheta_{j0} + \sum_{k=1}^{q-1} \varrho(k)\varsigma_0 \otimes \vartheta_{0k} + \sum_{\substack{i,j,k=1\\ i^2 \equiv jk \mod q}}^{q-1} \varrho(j)\varsigma_i \otimes \vartheta_{jk},$$

and further, we may replace $\varrho(j)$ in the last term by $\varrho(k)$. Set $J_{ijk} = J(\varsigma_i, \vartheta_{jk})$. Note that $J_{ijk} = 0$ if $i^2 \not\equiv jk \mod q$ (since making the change of variables $\sigma \mapsto \sigma n_1$ in the integral defining J_{ijk} multiplies the integral by $\psi_0(i^2 - jk)$, which is not 1 unless $i^2 \equiv jk \mod q$.) Let $c = (-1, 1) \in \mathbb{Q}_q \times \mathbb{Q}_q$. Then

$$\begin{aligned}\eta_q(-1) &= \eta_q(c/c^\rho) = \eta_q'(c) \cdot \chi_q\chi_{\nu,q}(Nm(c)) = \chi_{\nu,q}(-1);\\ \mu_q(-1) &= \eta_q^{-1}(-1) \cdot \chi_q\chi_{\nu,q}(-1) = 1.\end{aligned}$$

Hence $J_{ijk} = \mu_q(-1)J_{(-i)jk} = J_{(-i)jk}$. Also set $\varsigma = \sum_i \varsigma_i = \mathbb{I}_{\mathfrak{r}^0}$. Now, since $h_\chi(\sigma w) = h_\chi(\sigma)$, $\mathcal{F}(\varsigma) = q^{1/2}\varsigma_0$, $\mathcal{F}(\vartheta_{jk})((a,c)j_q) = q^{-1}\psi_0(-cj)\psi_0(-ak)\mathbb{I}_{\mathfrak{r}}(a,c)$,

$$\begin{aligned}\sum_i J_{ijk} &= J(\varsigma, \vartheta_{jk}) = \gamma_{\psi_0}^2\gamma_{\psi_0^{-1}}J(\mathcal{F}(\varsigma), \mathcal{F}(\vartheta_{jk})) = q^{1/2}\gamma_{\psi_0}^2\gamma_{\psi_0^{-1}}J(\varsigma_0, \mathcal{F}(\vartheta_{jk}))\\ &= q^{-1/2}\gamma_{\psi_0}^2\gamma_{\psi_0^{-1}}\sum_{j',k'}\psi_0(-jk')\psi_0(-kj')J(\varsigma_0, \vartheta_{j'k'})\\ &= q^{-1/2}\gamma_{\psi_0}^2\gamma_{\psi_0^{-1}}[J_{000} + \sum_{j'\neq 0}\psi_0(-kj')J_{0j'0} + \sum_{k'\neq 0}\psi(-jk')J_{00k'}].\end{aligned}$$

Thus

$$\begin{aligned}\sum_{\substack{i,j,k=1\\ i^2\equiv jk \mod q}}^{q-1} \varrho(j)J_{ijk} &= \sum_{i,j,k=1}^{q-1} \varrho(j)J_{ijk} \\ &= q^{-1/2}\gamma_{\psi_0}^2\gamma_{\psi_0^{-1}} \sum_{j,k=1}^{q-1} \varrho(j)[J_{000} + \sum_{j'\neq 0} \psi_0(-kj')J_{0j'0} + \sum_{k'\neq 0}\psi(-jk')J_{00k'}] \\ &= q^{-1/2}\gamma_{\psi_0}^2\gamma_{\psi_0^{-1}}(q-1)\varrho(-1)G(\varrho,\psi_0)\sum_{k'\neq 0}\varrho(k')J_{00k'} \\ &= (q-1)\sum_{k\neq 0}\varrho(k)J_{00k},\end{aligned}$$

and by symmetry, this last term also equals $(q-1)\sum_{j\neq 0}\varrho(j)J_{0j0}$. Thus $J=(q+1)J(\varsigma_0,\vartheta)$ where $\vartheta((a,b)j_q)=\varrho(a)\mathbb{I}_{q\mathbb{Z}_q}(a)\mathbb{I}_{\mathbb{Z}_q^\times}(b)=\chi_{\nu,q}(a)\mathbb{I}_{q\mathbb{Z}_q}(a)\mathbb{I}_{\mathbb{Z}_q^\times}(b)$. One may check that $\Theta_\eta,\Theta_\mu,\Theta_0$ are invariant by $\overline{\mathbf{n}}(x),\underline{\mathbf{n}}(y),v_q(x)\geq 0,v_q(y)\geq 1$ and

$$\begin{aligned}\Theta_\eta(\mathbf{d}(a)) &= \lambda_2(\nu_0 a)|\nu_0 a|^{1/2}\mathbb{I}_{\mathbb{Z}_q}(a);\\ \Theta_\mu(\mathbf{d}(a)) &= |a|^{1/2}\mu_{\psi_0}(a)\chi_{d,q}(a)\mathbb{I}_{q\mathbb{Z}_q}(a), \qquad \Theta_0(\mathbf{d}(a))=|a|^{1/2}\mu_{\psi_0}(a)\chi_{d,q}(a).\end{aligned}$$

Note that for $x\in\mathbb{Z}_q^\times$,

$$\begin{aligned}\lambda_1\lambda_2^{-1}(x) &= \eta_q'(1,x)\chi_q\chi_{\nu,q}(x)=\chi_{\nu,q}(x)=\varrho(x);\\ \mu_q(x) &= \eta^{-1}(x)\chi_q\chi_{\nu,q}(x)=(\lambda_1\lambda_2)^{-1}(x)\chi_{\nu,q}(x).\end{aligned}$$

Choosing λ_2 to be ramified and λ_1 unramified, we see that $\lambda_2\chi_{\nu,q}$ and μ_q are unramified, and

$$\Theta_\eta=\lambda_2(\nu_0)|\nu_0|^{1/2}\tilde{W}_\eta^\psi=\lambda_2(\nu_0)|\nu_0|^{1/2}(\tilde{W}_{\eta\chi_t}^\psi\otimes\chi_t)$$

for any $t\in\Sigma$. In this case, $A_q(s)=(1-q^{-s})^{-1}$, $B_q(s)=1$, $\zeta_{K,q}(s)=(1-q^{-s})^{-2}$, $\zeta_{\mathbb{Q},q}(2s)=(1-q^{-2s})^{-1}$. Thus $C_q(s)=(1+q^{-s})^{-1}$. Since η has weight $(-k/2,k/2)$, $v_p(\lambda_2(\nu_0))=v_{\mathfrak{p}}(\lambda_2(\nu_0))=k/2$. Set $c_q=(q+1)q^{\frac{k-1}{2}}$.

Subcase (ii): K is inert at q. Then $K_q=\mathbb{Q}_q(v)$, where $v^2=u$ is a non-square unit in $\mathbb{Z}_q$. Set $\mathfrak{r}=\mathbb{Z}_q+\mathbb{Z}_q v$. We may pick $\delta_q\in\mathrm{GL}_2(\mathbb{Z}_q)$ such that $i_q^\delta(v)=\begin{pmatrix}0&1\\u&0\end{pmatrix}$. Let $j_q=\begin{pmatrix}-1&0\\0&1\end{pmatrix}$. For $i,j,k\in\{0,1,\ldots,q-1\}$, set $\varsigma_i=\mathbb{I}_{(q\mathbb{Z}_q+i)v}$, $\vartheta_{jk}=\mathbb{I}_{(q\mathbb{Z}_q+j+(q\mathbb{Z}_q+k)v)j_q}$. Then one checks that

$$\begin{aligned}\overline{\varphi_q} &= \sum_{i\neq 0}\varrho(-2i)\varsigma_i\otimes\vartheta_{0i}+\sum_{i\neq 0}\varrho(2iu)\varsigma_i\otimes\vartheta_{0(-i)}+\sum_{\substack{i,j,k;i\neq\pm k\\ j^2\equiv(k^2-i^2)u}}\varrho(-(i+k))\varsigma_i\otimes\vartheta_{jk}.\\ &= \sum_{i\neq 0}\varrho(2iu)\varsigma_i\otimes\vartheta_{0(-i)}+\sum_{\substack{i,j,k;i\neq -k\\ j^2\equiv(k^2-i^2)u}}\varrho(-(i+k))\varsigma_i\otimes\vartheta_{jk}\end{aligned}$$

As usual, set $J_{ijk} = J(\varsigma_i, \vartheta_{jk})$. Now note that

$$\begin{aligned} \eta_q(-1) &= \eta_q(v/v^\rho) = \eta_q'(v)\chi_q\chi_{\nu,q}(-u) = -\chi_{\nu,q}(-1); \\ \mu_q(-1) &= \eta_q^{-1}(-1)\chi_q\chi_{\nu,q}(-1) = -1. \end{aligned}$$

so that $J_{ijk} = \mu_q(-1)J_{(-i)jk} = -J_{(-i)jk}$. Let $\varsigma^k = \sum_{i\neq -k} \varrho(-(i+k))\varsigma_i$. Since $\mathcal{F}(\varsigma_i)(xv) = q^{-1/2}\psi_0(2ixu)\mathbb{I}_{\mathbb{Z}_q}(x)$, one has

$$\begin{aligned} \mathcal{F}(\varsigma^k)(xv) &= q^{-1/2}\sum_{i\neq -k} \varrho(-(i+k))\psi_0(2ixu)\mathbb{I}_{\mathbb{Z}_q}(x) \\ &= q^{-1/2}\psi_0(2kxu)\sum_{i\neq 0}\varrho(-i)\psi_0(2xiu)\mathbb{I}_{\mathbb{Z}_q}(x) \\ &= q^{-1/2}\psi_0(2kxu)\varrho(2xu)\varrho(-1)G(\varrho,\psi_0)\mathbb{I}_{\mathbb{Z}_q}(x). \end{aligned}$$

Further,

$$\mathcal{F}(\vartheta_{jk})((y+zv)j_q) = q^{-1}\psi_0(-2yj+2zuk)\mathbb{I}_{\mathbb{Z}_q}(y)\mathbb{I}_{\mathbb{Z}_q}(z).$$

Thus

$$\begin{aligned} \sum_{i\neq -k}\varrho(-(i+k))J_{ijk} &= J(\varsigma^k,\vartheta_{jk}) = \gamma_{\psi_0}^2\gamma_{\psi_0^{-1}}J(\mathcal{F}(\varsigma^k),\mathcal{F}(\vartheta_{jk})) \\ &= q^{-3/2}\gamma_{\psi_0}^2\gamma_{\psi_0^{-1}}\varrho(-1)G(\varrho,\psi_0)\sum_{x,y,z}\varrho(2xu)\psi_0(2kxu-2yj+2zku)\}J_{xyz}, \end{aligned}$$

and

$$\begin{aligned} \sum_{\substack{i,j,k;i\neq -k \\ j^2\equiv(k^2-i^2)u}}\varrho(-(i+k))J_{ijk} &= \sum_{\substack{i,j,k \\ i\neq -k}}\varrho(-(i+k))J_{ijk} \\ &= q^{1/2}\gamma_{\psi_0}^2\gamma_{\psi_0^{-1}}\varrho(-1)G(\varrho,\psi_0)\sum_i\varrho(2iu)J_{i0(-i)} \\ &= q\sum_i\varrho(2iu)J_{i0(-i)} = -q\sum_i\varrho(2i)J_{i0(-i)}. \end{aligned}$$

Since $J_{i0i} = \mu_q(-1)J_{(-i)0i} = -J_{(-i)0i}$, one has $J = (q+1)\varrho(-2)\sum_i\varrho(i)J(\varsigma_i,\vartheta_{0i})$. Set $\varsigma = \sum_{i\neq 0}\mu_q(i)\varsigma_i, \vartheta = \sum_{i\neq 0}\eta_q(i)\vartheta_{0i}$. Noting that $\mu_q\eta_q(i) = \chi_q\chi_{\nu,q}(i) = \varrho(i)$, we see that

$$J(\varsigma,\vartheta) = \sum_{\substack{i\neq 0 \\ j\neq 0}}\mu_q(i)\eta_q(j)J_{i0j} = \sum_{i\neq 0}\{\mu_q(i)\eta_q(i)J_{i0i} + \mu_q(-i)\eta_q(i)J_{(-i)0i}\} = 2\sum_{i\neq 0}\varrho(i)J_{i0i}.$$

Thus $J = \frac{1}{2}(q+1)\varrho(-2)J(\varsigma,\vartheta)$. Now note that for x any unit in $\mathfrak{r}$,

$$\eta_q(x/x^\rho) = \eta_q'(x)\chi_q\chi_{\nu,q}(\mathrm{Nm}(x)) = \chi_{\nu,q}(\mathrm{Nm}(x)).$$

Since the norm map is surjective onto the units of $\mathbb{Z}_q^\times$, $\eta_q(\eta_q^\rho)^{-1}$ is not the trivial character. Thus η_q does not factor through the norm, whence $\pi_{\eta,q}$ must be supercuspidal.

Set $\Theta'_\eta(g) = \Theta_\eta(g\omega_q)$, $\Theta'_\mu(g) = \Theta_\mu(g\omega_q)$, $\Theta'_0(g) = \Theta_0(g\omega_q)$, where $\omega_q = \begin{pmatrix} \nu_0^{-1} & 0 \\ 0 & 1 \end{pmatrix}$. Then $\Theta'_\eta, \Theta'_\mu, \Theta'_0$ are invariant by $\overline{\mathbf{n}}(x), \underline{\mathbf{n}}(y)$, for $v_q(x) \geq 0, v_q(y) \geq 2$ and

$$\begin{aligned}
\Theta'_\eta \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} &= \frac{1}{2}(1 - \varrho(a))\eta(av^{-1})\mathbb{I}_{\mathbb{Z}_q^\times}(a); \\
\Theta'_\mu(\mathbf{d}(a)\omega_q^{-1}) &= |a|^{1/2}\mu_{\psi_0}(a)\chi_{d,q}\mu_q(a)\mathbb{I}_{\mathbb{Z}_q^\times}(a), \qquad \Theta'_0(\mathbf{d}(a)\omega_q^{-1}) = 0.
\end{aligned}$$

Choose η such that π_η has conductor q^2. Then for any $t \in \Sigma$ with $q \mid t$, $\pi_{\eta\otimes\chi_t}$ has conductor q^2 as well and for any $t_1, t_2 \in \Sigma$ with $q \nmid t_1, q \mid t_2$,

$$\Theta'_\eta = \eta(v^{-1})\{\tilde{W}^\psi_{\eta\otimes\chi_{t_1}} \otimes \chi_{t_1} - \tilde{W}^\psi_{\eta\otimes\chi_{t_2}} \otimes \chi_{t_2}\}.$$

Also, $A_q(s) = 1$, $B_q(s) = 1$, $\zeta_{K,q}(s) = \zeta_{\mathbb{Q},q}(2s) = (1 - q^{-2s})^{-1}$. Hence $C_q(s) = 1$. Set $c_q = q + 1$.

Subcase (iii): K is ramified at q. Then $K_q = \mathbb{Q}_q(v)$, where $v^2 = \pi$ is a uniformizer at q. Set $\mathfrak{r} = \mathbb{Z}_q + \mathbb{Z}_q v$. We may pick $\delta_q \in \mathrm{GL}_2(\mathbb{Z}_q)$ such that $i_q^\delta(v) = \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}$. Let $j_q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. For $r, i, j, k, l \in \{0, 1, \ldots, q-1\}$, set $\varsigma_{rj} = \mathbb{I}_{(\frac{r}{\pi}+j+q\mathbb{Z}_q)v}$, $\vartheta_{ikl} = \mathbb{I}_{(l+q\mathbb{Z}_q+(\frac{i}{\pi}+k+q\mathbb{Z}_q)v)j_q}$. Then one checks that

$$\varrho(-1)\overline{\varphi_q} = \sum_{\substack{i,j \\ i\neq 0}} \varrho(-2i)\varsigma_{ij} \otimes \vartheta_{ij0} + \sum_{\substack{j,k \\ j\neq k}} \varrho(j-k)\varsigma_{0j} \otimes \vartheta_{0k0} + \sum_{\substack{i,j,k,l \\ l\neq 0, l^2\equiv 2i(k-j)}} \varrho(j-k)\varsigma_{ij} \otimes \vartheta_{ikl}.$$

(5.4)

Set $J_{rjikl} = J(\varsigma_{rj}, \vartheta_{ikl})$. As usual, we have $J_{rjikl} = \mu_q(-1)J_{(-r)(-j)ikl}$. It is easy to see that if $J_{rjikl} \neq 0$ then either $r = i$ and $l^2 \equiv 2i(k-j)$ or $r = -i$ and $l^2 \equiv 2i(k+j)$. Now fix $i \neq 0, l \neq 0$ for the moment. Let t be such that $l^2 \equiv 2it$. Then

$$\varrho(-t)\sum_{j,k} J_{ijikl} = \varrho(-t)\sum_{\substack{j,k \\ l^2\equiv 2i(k-j)}} J_{ijikl} = \sum_{\substack{j,k \\ l^2\equiv 2i(k-j)}} \varrho(j-k)J_{ijikl}.$$

Set $\varsigma_r = \mathbb{I}_{(\frac{r}{\pi}+\mathbb{Z}_q)v}$, $\vartheta_{il} = \mathbb{I}_{(l+q\mathbb{Z}_q+(\frac{i}{\pi}+\mathbb{Z}_q)v)j_q}$ and $J_{ril} = J(\varsigma_r, \vartheta_{il})$. Thus the contribution of the last term in (5.4) to the integral $\varrho(-1)J$ is

$$\sum_{\substack{i,j,k,l \\ l\neq 0, l^2\equiv 2i(k-j)}} \varrho(j-k)J_{ijikl} = \sum_{i\neq 0, l\neq 0} \varrho(-2i)J_{iil}.$$

Set $\vartheta_i = \mathbb{I}_{(\mathbb{Z}_q+(\frac{i}{\pi}+\mathbb{Z}_q)v)j_q}$ and $J_{ri} = J(\varsigma_r, \vartheta_i)$. Note that if $i \neq 0$, $J_{ijik0} = 0$ for $j \neq k$. Hence the contribution of the first term of (5.4) to $\varrho(-1)J$ equals

$$\sum_{i\neq 0} \varrho(-2i)\sum_{j,k} J_{ijik0} = \sum_{i\neq 0} \varrho(-2i)J_{ii0},$$

whence the first and last terms of (5.4) together contribute $\sum_{i\neq 0} \varrho(-2i)J_{ii0} + \sum_{i\neq 0} \varrho(-2i) \sum_{l\neq 0} J_{iil} = \sum_{i\neq 0} \varrho(-2i)J_{ii}$ to the integral $\varrho(-1)J$.

The contribution of the middle term of (5.4) is somewhat tricky to compute. First we begin by computing the Fourier transforms of ς_{0j} and ϑ_{0k0}. One checks that

$$\mathcal{F}(\varsigma_{0j}) \;=\; q^{-1}\sum_r \psi_0(2jr)\varsigma_r \qquad \mathcal{F}(\vartheta_{0k0}) = q^{-3/2}\sum_i \psi_0(-2ik)\vartheta_i.$$

Thus

$$\begin{aligned}\sum_{j\neq k}\varrho(j-k)J(\varsigma_{0j},\vartheta_{0k0}) &= \sum_{j\neq k}\gamma_{\psi_0}^2\gamma_{\psi_0^{-1}}\varrho(j-k)J(\mathcal{F}(\varsigma_{0j}),\mathcal{F}(\vartheta_{0k0}))\\ &= q^{-5/2}\gamma_{\psi_0}^2\gamma_{\psi_0^{-1}}\sum_{j\neq k}\sum_{r,i}\varrho(j-k)\psi_0(2jr)\psi_0(-2ik)J(\varsigma_r,\vartheta_i)\\ &= q^{-5/2}\gamma_{\psi_0}^2\gamma_{\psi_0^{-1}}\sum_{s\neq 0}\sum_{r,i,k}\varrho(s)\psi_0(2(k+s)r)\psi_0(-2ik)J(\varsigma_r,\vartheta_i)\\ &= q^{-3/2}\gamma_{\psi_0}^2\gamma_{\psi_0^{-1}}\sum_{s\neq 0}\sum_{i}\varrho(s)\psi_0(2si)J(\varsigma_i,\vartheta_i)\\ &= q^{-3/2}\gamma_{\psi_0}^2\gamma_{\psi_0^{-1}}G(\varrho,\psi_0)\sum_{i\neq 0}\varrho(2i)J_{ii} = q^{-1}\varrho(-2i)J_{ii}.\end{aligned}$$

Thus $\varrho(-1)J = (1+\frac{1}{q})\sum_{i\neq 0}\varrho(-2i)J_{ii}$. Now setting $\varsigma = \sum_{i\neq 0}\mu(i)\varsigma_i, \vartheta = \sum_{i\neq 0}\eta(i)\vartheta_i$, one sees that $J = \frac{q+1}{2q}\varrho(2)J(\varsigma,\vartheta)$. Set $\Theta'_\eta(g) = \Theta_\eta(g\omega_q)$, $\Theta'_\mu(g) = \Theta_\mu(g\omega'_q)$, $\Theta'_0(g) = \Theta_0(g\omega'_q)$, where $\omega_q = \begin{pmatrix}\pi^{-2} & 0\\ 0 & 1\end{pmatrix}$, $\omega'_q = \begin{pmatrix}\pi^{-1} & 0\\ 0 & \pi\end{pmatrix}$. Then $\Theta'_\eta, \Theta'_\mu, \Theta'_0$ are invariant by $\overline{\mathbf{n}}(x), \underline{\mathbf{n}}(y)$, for $v_q(x)\geq 0, v_q(y)\geq 2$ and

$$\begin{aligned}\Theta'_\eta\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} &= \frac{1}{2}\eta_q(-\nu_0 a v^{-1})(1\pm\varrho(a))|a|^{1/2}\mathbb{I}_{\mathbb{Z}_q^\times}(a);\\ \Theta'_\mu(\mathbf{d}(a)) &= \mu_{\psi_0}(a)\chi_{d,q}\mu_q(a)|a|^{1/2}\mathbb{I}_{\mathbb{Z}_q^\times}(a), \qquad \Theta'_0(\mathbf{d}(a)) = 0.\end{aligned}$$

where the $\pm$ sign holds according as $(\nu_0,-\pi) = \pm 1$. Arguing exactly as in the case $q\mid d, d\nmid\nu$, we see that η must be unramified and factor as $\eta = \lambda\circ\mathrm{Nm}$ for some unramified character λ. Thus $\pi_\eta \simeq \pi(\lambda\eta_{K,q},\lambda)$ has conductor q.

Let $\breve{W}_\eta^\psi(g) = W_\eta^\psi\left(g\begin{pmatrix}q^{-1} & 0\\ 0 & 1\end{pmatrix}\right)$, $\breve{\tilde{W}}_\eta^\psi = \tilde{W}_\eta^\psi\left(g\begin{pmatrix}q^{-1} & 0\\ 0 & 1\end{pmatrix}\right)$. Note that

$$\begin{aligned}W_\eta^\psi\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} &= \lambda(a)|a|^{1/2}\mathbb{I}_{\mathbb{Z}_q}(a) \qquad \tilde{W}_\eta^\psi\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} = \eta_{K,q}(a)\lambda(a)|a|^{1/2}\mathbb{I}_{\mathbb{Z}_q}(a);\\ \breve{W}_\eta^\psi\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} &= \lambda(aq^{-1})|aq^{-1}|^{1/2}\mathbb{I}_{\mathbb{Z}_q}(aq^{-1}) = (\lambda(q)^{-1}q^{1/2})\lambda(a)|a|^{1/2}\mathbb{I}_{q\mathbb{Z}_q}(a);\\ \breve{\tilde{W}}_\eta^\psi\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} &= \eta_{K,q}(aq^{-1})\lambda(aq^{-1})|aq^{-1}|^{1/2}\mathbb{I}_{\mathbb{Z}_q}(aq^{-1})\\ &= (\eta_{K,q}(q)q^{1/2}\lambda(q)^{-1})\eta_{K,q}(a)\lambda(a)|a|^{1/2}\mathbb{I}_{q\mathbb{Z}_q}(a).\end{aligned}$$

Now setting

$$\begin{aligned}W_\eta^{\psi,+}(g) &= \tilde{W}_\eta^\psi(g) - q^{-1/2}(\lambda\eta_{K,q})(q)\breve{\tilde{W}}_\eta^\psi(g);\\ W_\eta^{\psi,-}(g) &= W_\eta^\psi(g) - q^{-1/2}\lambda(q)\breve{W}_\eta^\psi(g).\end{aligned}$$

we see that

$$\Theta'_\eta = \frac{1}{2}\eta_q(-\nu_0 v)\{W_\eta^{\psi,-} \pm W_\eta^{\psi,+}\} = \frac{1}{2}\eta_q(-\nu_0 v)\{W_{\eta\otimes\chi_t}^{\psi,-} \pm W_{\eta\otimes\chi_t}^{\psi,+}\}$$

for any $t \in \Sigma$. Also, $A_q(s) = (1-q^{-s})^{-1}$, $B_q(s) = (1-q^{-s})^{-1}$, $\zeta_{K,q}(s) = (1-q^{-s})^{-1}$ and $\zeta_{\mathbb{Q},q}(2s) = (1-q^{-2s})^{-1}$. Thus $C_q(s) = (1+q^{-s})^{-1}$. Set $c_q = q+1$.

5.2.3. $q \mid N^+$. In this case, K is split, so we fix an isomorphism $K \otimes \mathbb{Q}_q \simeq \mathbb{Q}_q \times \mathbb{Q}_q$. Set $\mathfrak{r} = \mathbb{Z}_q \times \mathbb{Z}_q$.

Subcase (i): $q \nmid \nu, \chi$ is unramified at q. We may pick $\delta_q \in N_{\mathrm{GL}_2(\mathbb{Q}_q)}(\mathcal{O}_{\chi,q}^\times)$ such that $i_q^\delta(a,b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. Let $j_q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $v = (1,-1) \in \mathbb{Q}_q \times \mathbb{Q}_q$. Then $\overline{\varphi_q} = \varsigma \otimes \vartheta$ where $\varsigma = \mathbb{I}_{\mathfrak{r}^0}$ and $\vartheta = \mathbb{I}_{(\mathbb{Z}_q \times q\mathbb{Z}_q)j_q}$. Set $\Theta'_\eta(g) = \Theta'_\eta(g\omega_q)$, where $\omega_q = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$. Set $\lambda = \lambda_1/\lambda_2$, where $\eta_q = (\lambda_1, \lambda_2)$. Then λ is unramified, $\mu_q(-1) = 1$, $\Theta'_\eta, \Theta_\mu, \Theta_0$ are invariant by $\overline{\mathbf{n}}(x), \underline{\mathbf{n}}(y), x, y \in \mathbb{Z}_q$, and

$$\begin{aligned} \Theta'_\eta \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} &= |aq|^{1/2}\lambda_1(\nu_0 aq)\frac{\lambda^{-1}(aq)-1}{\lambda^{-1}(q)-1}\mathbb{I}_{\mathbb{Z}_q}(a); \\ \Theta_\mu(\mathbf{d}(a)) &= |a|^{1/2}\mu_{\psi_0}(a)\chi_{d,q}(a)\mathbb{I}_{\mathbb{Z}_q}(a), \qquad \Theta_0(\mathbf{d}(a)) = |a|^{1/2}\mu_{\psi_0}(a)\chi_{d,q}(a). \end{aligned}$$

If we pick λ_1 and λ_2 to be unramified,

$$\Theta'_\eta \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = |q|^{1/2}\lambda_1(q)|a|^{1/2}\frac{\lambda_1(aq)-\lambda_2(aq)}{\lambda_1(q)-\lambda_2(q)},$$

so that $\Theta'_\eta = |q|^{1/2}\lambda_1(q)\tilde{W}_\eta^\psi$. One checks easily that $C_q(s) = 1$.

Subcase (ii): $q \nmid \nu, \chi$ is ramified at q. We may pick $\delta_q \in \mathcal{O}_{\chi,q}^\times$ such that either $i_q^\delta(a,b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ or $i_q^\delta(a,b) = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}$. Let $j_q = \begin{pmatrix} 0 & -q \\ q^{-1} & 0 \end{pmatrix}$ and $v = (1,-1) \in \mathbb{Q}_q \times \mathbb{Q}_q$. Then $\overline{\varphi_q} = \varsigma \otimes \vartheta$ where $\varsigma = \mathbb{I}_{\mathfrak{r}^0}$ and $\vartheta((a,b)j_q) = \chi_q(a)\mathbb{I}_{\mathbb{Z}_q^\times}(a)\mathbb{I}_{q\mathbb{Z}_q}(c)$ or $\vartheta((a,b)j_q) = \chi_q(c)\mathbb{I}_{\mathbb{Z}_q^\times}(c)\mathbb{I}_{q\mathbb{Z}_q}(a)$. We assume we are in the former case, since the latter case is exactly similar. Set $\Theta'_\eta(g) = \Theta_\eta(g\omega_q)$, where $\omega_q = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$. Note that $\eta'_q(a,b) = \chi_q^{-2}(b)$ if a,b are units, $\eta_q(-1) = \eta'_q(-1,1)\chi_q\chi_{\nu,q}(-1) = \chi_q(-1)$ and $\mu_q(-1) = 1$. Now one checks that Θ_μ, Θ_0 are invariant by $\overline{\mathbf{n}}(x), \underline{\mathbf{n}}(y), v_q(x) \geq 0, v_q(y) \geq 0$, Θ'_η is invariant by $\overline{\mathbf{n}}(x), \underline{\mathbf{n}}(y), v_q(x) \geq 0, v_q(y) \geq 1$ and

$$\begin{aligned} \Theta'_\eta \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} &= q^{-1/2}|a|^{1/2}\lambda_1(\nu_0 aq)\mathbb{I}_{\mathbb{Z}_q}(a); \\ \Theta_\mu(\mathbf{d}(a)) &= |a|^{1/2}\mu_{\psi_0}(a)\chi_{d,q}(a)\mathbb{I}_{q\mathbb{Z}_q}(a), \qquad \Theta_0(\mathbf{d}(a)) = |a|^{1/2}\mu_{\psi_0}(a)\chi_{d,q}(a). \end{aligned}$$

For any $a \in \mathbb{Z}_q^\times$, $\lambda_1\lambda_2^{-1}(a) = \eta_q(a,a^{-1}) = \eta'_q(a,1)\chi_q\chi_{\nu,q}(a) = \chi_q(a)$. Thus we may pick η such that λ_2 is unramified and λ_1 is ramified with conductor q. Then $\pi_{\eta,q} \simeq \pi(\lambda_1,\lambda_2)$ has conductor q and $\Theta'_\eta = \lambda_1(\nu_0 q)\tilde{W}_\eta^\psi$. (If on the other hand, $\vartheta((a,b)j_q) = \chi^{-1}(c)\mathbb{I}_{\mathbb{Z}_q^\times}(c)\mathbb{I}_{q\mathbb{Z}_q}(a)$, one gets $\Theta'_\eta = \lambda_2(\nu_0 q)\tilde{W}_\eta^\psi$.) In this case, $A_q(s) = (1-q^{-s})^{-1}$, $B_q(s) = 1$, $\zeta_{K,q}(s) = (1-q^{-s})^{-2}$, $\zeta_{\mathbb{Q}}(2s) = (1-q^{-2s})^{-1}$ and $C_q(s) = (1+q^{-s})^{-1}$.

Subcase (iii): $q \mid \nu$. In this case, χ has been chosen to be unramified at q. We may pick $\delta_q \in \mathcal{O}_{\chi,q}^\times$ such that either $i_q^\delta(a,b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ or $i_q^\delta(a,b) = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}$. Let $j_q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $v = (1,-1) \in \mathbb{Q}_q \times \mathbb{Q}_q$. Then $\overline{\varphi_q} = \varphi_q = \varsigma \otimes \vartheta$ where $\varsigma = \mathbb{I}_{q\mathfrak{r}^0}$ and $\vartheta((a,b)j_q) = \chi_{\nu,q}(a)\mathbb{I}_{\mathbb{Z}_q^\times}(a)\mathbb{I}_{q\mathbb{Z}_q}(c)$ or $\vartheta((a,b)j_q) = \chi_{\nu,q}(c)\mathbb{I}_{\mathbb{Z}_q^\times}(c)\mathbb{I}_{q\mathbb{Z}_q}(a)$. Without loss we may assume we are in the former case. In this case, η'_q is unramified, $\eta_q(-1) = \chi_{\nu,q}(-1)$ and $\mu_q(-1) = 1$. Arguing as in the previous case, $\lambda_1\lambda_2^{-1}(a) = \chi_{\nu,q}(a) = \varrho(a)$ for $a \in \mathbb{Z}_q^\times$, so we may assume that λ_2 is unramified and λ_1 is ramified, but $\lambda_1\chi_{\nu,q}$ is unramified. One may check that $\Theta_\eta, \Theta_\mu, \Theta_0$ are invariant by $\overline{\mathbf{n}}(x), \underline{\mathbf{n}}(y), v_q(x) \geq 0, v_q(y) \geq 1$, and

$$\begin{aligned} \Theta_\eta(\mathbf{d}(a)) &= q^{-1/2}|\nu_0 a|^{1/2}\lambda_1(\nu_0 a)\mathbb{I}_{\mathbb{Z}_q}(a); \\ \Theta_\mu(\mathbf{d}(a)) &= |a|^{1/2}\mu_{\psi_0}(a)\chi_{d,q}(a)\mathbb{I}_{q\mathbb{Z}_q}(a), \qquad \Theta_0(\mathbf{d}(a)) = |a|^{1/2}\mu_{\psi_0}(a)\chi_{d,q}(a). \end{aligned}$$

We see that $\Theta_\eta = \lambda_1(\nu_0)|\nu_0|^{1/2}\tilde{W}_\eta^\psi$. $A_q(s) = (1-q^{-s})^{-1}$, $B_q(s) = 1$, $\zeta_{K,q}(s) = (1-q^{-s})^{-2}$, $\zeta_{\mathbb{Q}}(2s) = (1-q^{-2s})^{-1}$ and $C_q(s) = (1+q^{-s})^{-1}$.

5.2.4. $q \mid N^-$. In this case, K is inert at q; we use the notation of Sec. 3.2 in what follows. We pick an isomorphism $K_q \simeq L_q$ and identify K_q and L_q via this isomorphism. Set $\mathfrak{r} = \mathbb{Z}_q + \mathbb{Z}_q\omega$.

Subcase (i): $q \nmid \nu$, χ is unramified at q. We may pick $\delta_q \in B_q^\times$ such that $i_q^\delta(a) = a$. Clearly, $\varphi_q^\delta = \varphi_q$, since B_q has a unique maximal order. Also, $\overline{\varphi_q} = \varsigma \otimes \vartheta$, where $\varsigma = \mathbb{I}_{\mathfrak{r}^0}$ and $\vartheta = \mathbb{I}_{\mathfrak{r}u}$ and we may set $j_q = u$. In this case, η_q and μ_q are unramified, hence $\eta_q = \lambda \circ \mathrm{Nm}$ for an unramified character λ. Let $\Theta'_\eta(g) = \Theta_\eta(g\omega_q)$ with $\omega_q = \begin{pmatrix} \mathrm{Nm}(\omega) & 0 \\ 0 & 1 \end{pmatrix}$. Then $\Theta'_\eta, \Theta_\mu, \Theta_0$ are invariant by $\overline{\mathbf{n}}(x), \underline{\mathbf{n}}(y), x, y \in \mathbb{Z}_q$ and

$$\begin{aligned} \Theta'_\eta\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} &= \frac{1}{2}(1+\eta_{K,q}(\nu_0 a))|a|^{1/2}\lambda(\nu_0 a)\mathbb{I}_{\mathbb{Z}_q}(a) = \frac{1}{2}(1+\eta_{K,q}(a))|a|^{1/2}\lambda(a)\mathbb{I}_{\mathbb{Z}_q}(a); \\ \Theta_\mu(\mathbf{d}(a)) &= |a|^{1/2}\mu_{\psi_0}(a)\chi_{d,q}(a)\mathbb{I}_{\mathbb{Z}_q}(a), \qquad \Theta_0(\mathbf{d}(a)) = |a|^{1/2}\mu_{\psi_0}(a)\chi_{d,q}(a). \end{aligned}$$

Hence $\Theta'_\eta = \tilde{W}_\eta^\psi$. Also, $C_q(s) = 1$.

Subcase (ii): $q \nmid \nu, \chi$ is ramified at q. We may pick $\delta_q \in B_q^\times$ such that $i_q^\delta(a) = a$. It is easy to check that $\varphi_q^\delta = \varphi_q$. Also, $\overline{\varphi_q} = \varsigma \otimes \vartheta$, where $\varsigma(av) = \chi_q(a)\mathbb{I}_{\mathbb{Z}_q^\times}(a)$ and $\vartheta = \mathbb{I}_{\mathfrak{r}u}$. Then $\eta'_q(a) = \chi_q^{-1}(\mathrm{Nm}(a))$ for a any unit in $\mathfrak{r}$, $\eta_q|_{K^{(1)}}$ is trivial and $\mu_q(-1) = \chi_q(-1)$. Thus $\eta_q = \lambda \circ \mathrm{Nm}$ for some unramified character λ. Set $\Theta'_\eta(g) = \Theta_\eta\left(g\begin{pmatrix} \mathrm{Nm}(\omega) & 0 \\ 0 & 1 \end{pmatrix}\right)$. Then Θ'_η is invariant by $\overline{\mathbf{n}}(x), \underline{\mathbf{n}}(y), v_q(x) \geq 0, v_q(y) \geq 0$, Θ_μ, Θ_0 are invariant by $\overline{\mathbf{n}}(x), \underline{\mathbf{n}}(y), v_q(x) \geq 0, v_q(y) \geq 2$ and

$$\begin{aligned} \Theta'_\eta\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} &= \frac{1}{2}(1+\eta_{K,q}(a))|a|^{1/2}\mathbb{I}_{\mathbb{Z}_q}(a); \\ \Theta_\mu(\mathbf{d}(a)) &= |a|^{1/2}\mu_{\psi_0}(a)\chi_{d,q}(a)\mathbb{I}_{\mathbb{Z}_q^\times}(a), \qquad \Theta_0(\mathbf{d}(a)) = 0. \end{aligned}$$

As in the previous case, $\Theta'_\eta = \tilde{W}_\eta^\psi$. Again, $C_q(s) = 1$.

Subcase (iii): $q \mid \nu$. In this case, χ has been chosen to be ramified at q, indeed $\chi_q(-1) = -1$. We may pick $\delta_q \in B_q^\times$ such that $i_q^\delta(a) = a$. It is easy to check that $\varphi_q^\delta = \varphi_q$ and $\eta'_q(a) = \chi_q^{-1}(\mathrm{Nm}(a))$ for $a \in \mathfrak{r}^\times$. Also, $\overline{\varphi_q} = \varsigma \otimes \vartheta$, where $\varsigma = \mathbb{I}_{q\mathfrak{r}^0}$ and ϑ is given by the following formula: $\vartheta(bu) = 0$ unless $N(b) \in (\mathbb{Z}_q^\times)^2$. In that case, write $b = c\frac{\bar{e}}{e}$ for some $c \in \mathbb{Z}_q^\times, e \in \mathfrak{r}^\times$. Then $\vartheta(b) = \chi_{\nu,q}\chi_q(c) \cdot \chi_{\nu,q}(N(e))$. Note that for $x \in \mathfrak{r}^\times$,

$$\eta_q(x/x^\rho) = \eta'_q(x)\chi_q\chi_{\nu,q}(\mathrm{Nm}\, x) = \chi_{\nu,q}(\mathrm{Nm}\, x).$$

In particular, η does not factor through the norm, hence π_η is supercuspidal. Setting $x = v$, one gets $\eta_q(-1) = \chi_{\nu,q}(\mathrm{Nm}\,\omega) = -\chi_{\nu,q}(-1) = \chi_q\chi_{\nu,q}(-1)$. Hence we may assume that $\mu_q = \eta_q^{-1}|_{\mathbb{Q}_q^\times} \cdot \chi_q\chi_{\nu,q}$ is unramified. One checks that Θ_η is invariant by $\overline{\mathbf{n}}(x), \underline{\mathbf{n}}(y), v_q(x) \geq 0, v_q(y) \geq 2$, and

$$\Theta_\eta \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = |a|^{1/2}\chi_{\nu,q}(\epsilon a) \int_{K_q^{(1)}} \vartheta(\epsilon a\tilde{h}^{-1}h^{-1})\eta_q(\tilde{h}h)d^\times h,$$

for any $\tilde{h} \in K_q^\times$ with $\mathrm{Nm}(\tilde{h}) = \epsilon a$. Now $\vartheta(\epsilon a\tilde{h}^{-1}h^{-1}) = 0$ unless $\epsilon a \in (\mathbb{Z}_q^\times)^2$. Suppose $\epsilon a = b^2$. Pick $\tilde{h} = b$, so that $\epsilon a\tilde{h}^{-1}h^{-1} = bh^{-1}$. Let us write $h = x/x^\rho$ for some $x \in \mathfrak{r}^\times$. Then

$$\begin{aligned} \eta_q(\tilde{h}h) &= \eta_q(b\frac{x}{x^\rho}) = \chi_q\chi_{\nu,q}(b) \cdot \chi_{\nu,q}(\mathrm{Nm}(x)), \\ \vartheta(\epsilon a\tilde{h}^{-1}h^{-1}) &= \vartheta(bh^{-1}) = \vartheta(b\frac{x^\rho}{x}) = \chi_{\nu,q}\chi_q(b) \cdot \chi_{\nu,q}(\mathrm{Nm}(x)), \end{aligned}$$

whence from (5.5) above, we see that

$$\Theta_\eta \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}|a|^{1/2}\chi_{\nu,q}\chi_q(\epsilon a)(1 + \varrho(\epsilon a))\mathbb{I}_{\mathbb{Z}_q^\times}(a).$$

Thus for $t_1, t_2 \in \Sigma$, with $q \nmid t_1, q \mid t_2$, $\Theta_\eta = \frac{1}{2}\chi_{\nu,q}\chi_q(\epsilon)\{\tilde{W}^\psi_{\eta\otimes\chi_{t_1}} \otimes \chi_{t_1} \pm \tilde{W}^\psi_{\eta\otimes\chi_{t_2}} \otimes \chi_{t_2}\}$ where the $\pm$ sign appears according as $\varrho(\epsilon) = \pm 1$. Also Θ_μ, Θ_0 are invariant by $\overline{\mathbf{n}}(x), \underline{\mathbf{n}}(y), v_q(x) \geq 0, v_q(y) \geq 1$ and

$$\Theta_\mu(\mathbf{d}(a)) = |a|^{1/2}\mu_{\psi_0}(a)\chi_{d,q}(a)\mathbb{I}_{q\mathbb{Z}_q}(a), \qquad \Theta_0(\mathbf{d}(a)) = |a|^{1/2}\mu_{\psi_0}(a)\chi_{d,q}(a).$$

In this case, $A_q(s) = 1$, $B_q(s) = (1 - q^{-s})^{-1}$, $\zeta_{K,q}(s) = \zeta_{\mathbb{Q}}(2s)$ and $C_q(s) = (1 - q^{-s})$.

5.2.5. $q = 2$. We assume that K is split at 2; the other cases can be handled similarly. Pick $\delta_q, i_q^\delta, j_q$ as in Case (A), subcase (i). Then $\varphi_q = \varsigma \otimes \vartheta$, where $\varsigma = \mathbb{I}_{\mathfrak{r}^0}$, $\vartheta = \mathbb{I}_{2\mathfrak{r}j_q}$. Since η', χ_2 and $\chi_{\nu,2}$ are unramified, we may pick η and μ to be unramified. Let $\Theta'_\eta(g) = \Theta_\eta(g \cdot \begin{pmatrix} 2^{-2} & 0 \\ 0 & 1 \end{pmatrix})$. One checks that $\Theta'_\eta = (\lambda_1\lambda_2)(2) \cdot \tilde{W}^\psi_\eta$. Further, Θ_μ, Θ_0 are invariant by $\overline{\mathbf{n}}(x), \underline{\mathbf{n}}(y), v_2(x) \geq 0, v_2(y) \geq 2$ and

$$\Theta_\mu(\mathbf{d}(a)) = |a|^{1/2}\mu_{\psi_0}(a)\chi_{d,2}(a)\mathbb{I}_{\mathbb{Z}_q}(a), \qquad \Theta_0(\mathbf{d}(a)) = |a|^{1/2}\mu_{\psi_0}(a)\chi_{d,2}(a).$$

5.2.6. $q = \infty$. Let $j_\infty = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $M_2(\mathbb{R}) = \mathbb{C} + \mathbb{C}j_\infty$ and $\varphi = (-2i)^k|\nu|^{-1/2} \cdot \varphi'_{1,\infty} \otimes \varphi'_{2,\infty}$ where $\varphi'_{1,\infty}(x) = e^{-2\pi|x|^2/|\nu|}$, $\varphi'_{2,\infty}(yj_\infty) = y^k e^{-2\pi|y|^2/|\nu|}$. Here we think of $\mathbb{C} \hookrightarrow \mathrm{GL}_2(\mathbb{R})$ via $x = a + bi \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. Suppose $\alpha^{-1}\tilde{j}\alpha = y_0 j_\infty$ for $y_0 \in \mathbb{C}$. Then

$\varphi(\alpha^{-1}(x+yj)\alpha) = (-2i)^k\varphi_{1,\infty}(x)\varphi_{2,\infty}(yj)$ where $\varphi_{1,\infty}(x) = e^{-2\pi|x|^2/|\nu|}$ and $\varphi_{2,\infty}(yj) = |\nu|^{-1/2}y_0^{-k}y^k e^{-2\pi|y|^2/|\nu|}$. One checks easily that

$$\begin{aligned}\Theta_\eta\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} &= y_0^{-k}|\nu|^{\frac{k}{2}}|a|^{\frac{k+1}{2}}e^{-2\pi a}\mathbb{I}_{\mathbb{R}^+}(a);\\ \Theta_\mu(\mathbf{d}(a)) &= |a|^{1/2}\mu_{\psi_0}(a)e^{-2\pi a^2/|\nu|}, \quad \Theta_0(\mathbf{d}(a)) = |a|^{1/2}\mu_{\psi_0}(a).\end{aligned}$$

Set $c_\infty = |\nu|^{k/2}$. Also notice that $\overline{y_0} = -\Im(z)J(\tilde{j},z)j(\alpha,i)^2$ (see the discussion on p. 940 of [22]). Hence

$$(-\overline{y_0})^{-k}\{J(\tilde{j},z)\Im(z)\}^k j(\alpha,i)^{2k} = 1.$$

5.3. **Statement of the main theorem and proof of rationality.** We begin by summarizing the calculations of the previous section in more classical language. For each prime q define integers l_q, r_q, m_q, n_q, s_q as below.

(i) If $q \nmid 2N\nu d$, $l_q = r_q = m_q = n_q = s_q = 0$.

(ii) If $q \nmid 2N, q \mid d, q \nmid \nu$,

$l_q = 0, r_q = 1, m_q = n_q = 1, s_q = 0$.

(iii) If $q \nmid 2N, q \mid \nu$,

$l_q = 0, r_q = 1, m_q = n_q = 1, s_q = 0$, if K is split at q,
$l_q = 1, r_q = 0, m_q = n_q = 2, s_q = 0$, if K is inert at q,
$l_q = 2, r_q = 0, m_q = n_q = 2, s_q = 0$, if K is ramified at q.

(iv) If $q \mid N^+$,

$l_q = r_q = 0, m_q = 1, n_q = 0, s_q = 1$, if $q \nmid \nu$ and $\chi_{0,q}$ is unramified,
$l_q = r_q = 0, m_q = 2, n_q = 1, s_q = 1$, if $q \nmid \nu$ and $\chi_{0,q}$ is ramified,
$l_q = 0, r_q = 1, m_q = n_q = 1, s_q = 0$, if $q \mid \nu$.

(v) If $q \mid N^-$,

$l_q = r_q = 0, m_q = 1, n_q = 0, s_q = 0$, if $q \nmid \nu$ and $\chi_{0,q}$ is unramified,
$l_q = r_q = 0, m_q = 2, n_q = 0, s_q = 0$, if $q \nmid \nu$ and $\chi_{0,q}$ is ramified,
$l_q = 0, r_q = 1, m_q = n_q = 2, s_q = 0$, if $q \mid \nu$.

(vi) If $q = 2$, $l_q = r_q = 0$, $m_q = 2, n_q = 0, s_q = 2$.

Set $l = \prod_q q^{l_q}, r = \prod_q q^{r_q}, m = \prod_q q^{m_q}$, $n = \prod_q q^{n_q}$, $s = \prod_q q^{s_q}$. Let κ be the Grossencharacter of weight $(k,0)$ defined by $\kappa = \overline{\tilde{\eta}}$ and set $\kappa_t = \kappa\cdot(\chi_t\circ \mathrm{Nm})$ for $t \in \Sigma$. It is easy to check that $\mathfrak{c}_{\kappa_t} = \mathfrak{c}_\kappa = \mathfrak{c}_{\tilde{\eta}}$ for all $t \in \Sigma$ where $\mathfrak{c}_{\kappa_t}$ (resp. $\mathfrak{c}_{\tilde{\eta}}$) denotes the conductor of κ_t (resp. of $\tilde{\eta}$.) Let

$$\theta_\mu(z) = \sum_{j\in\mathbb{Z}^{\geq 0}}\mu(j)e^{2\pi i j^2 z}, \qquad \theta_{\kappa_t}(z) = \sum_{\substack{\mathfrak{a}\in\mathcal{O}_K\\(\mathfrak{a},\mathfrak{c}_\kappa)=1}}\kappa_t(\mathfrak{a})e^{2\pi i N(\mathfrak{a})z}$$

and denote by $\tilde{\theta}_{\kappa_t}$ the modular form obtained by dropping the Euler factor at q for $q \in \Sigma'$ in the Euler product expansion of θ_{κ_t}. When $t = 1$, we simply write θ_κ or $\tilde{\theta}_\kappa$. Let $s' = \prod_{q\in\Sigma'} q$. Note that $\theta_{\kappa_t} \in S_{k+1}(\Gamma_0(n/s'), \eta|_{\mathbb{Q}^\times}^{-1}\eta_K)$ while $\tilde{\theta}_{\kappa_t} \in S_{k+1}(\Gamma_0(n), \eta|_{\mathbb{Q}^\times}^{-1}\eta_K)$. Let V_q denote the Atkin-Lehner operator usually denoted by the symbol W_{q^2}, and for $t' \in \Sigma'$, set $V_{t'} = \prod_{q|t'} V_q$. Then the computations of the previous section express $L_{\eta'}$ explicitly as a linear combination of the Petersson inner products $\langle h_\chi(lz)\theta_\mu(rz), V_{t'}\tilde{\theta}_{\kappa t}(sz)\rangle$ for $t \in \Sigma$ and $t' \in \Sigma'$. For a vector $\mathfrak{b} = [l, r, s]$, set

$$I^{\mathfrak{b},t'}_{f,\chi}(\mu, \tilde{\boldsymbol{\eta}}) = \langle V^*_{t'}\{h_\chi(lz)\theta_\mu(rz)\}, \tilde{\theta}_{\kappa t}(sz)\rangle,$$

where $\tilde{\boldsymbol{\eta}}$ is the algebraic Hecke character corresponding to $\tilde{\eta}$. We have then more precisely,

$$(2\pi i)^k\{\pi\Im(z)J(\tilde{j}, z)\}^k L_{\eta'} = (2\pi i)^k\pi^{k+1}\sum_{t\in\Sigma}\sum_{t'\in\Sigma'} c^{\mathfrak{b},t,t'}_{f,\chi}(\mu, \tilde{\boldsymbol{\eta}}) I^{\mathfrak{b},t'}_{f,\chi}(\mu\chi_t, \tilde{\boldsymbol{\eta}}\cdot\chi_t\circ \mathrm{Nm}), \tag{5.5}$$

with explicit coefficients $c^{\mathfrak{b},t,t'}_{f,\chi}(\mu, \tilde{\boldsymbol{\eta}}) \in K(f, \chi, \tilde{\boldsymbol{\eta}})$ that are p-adic integers and satisfy

$$\left(i^k|\nu|^{1/2}c^{\mathfrak{b},t,t'}_{f,\chi}(\mu, \tilde{\boldsymbol{\eta}})\right)^\sigma = i^k|\nu|^{1/2}c^{\mathfrak{b},t,t'}_{f^\sigma,\chi^\sigma}(\mu^\sigma, \tilde{\boldsymbol{\eta}}^\sigma) \tag{5.6}$$

for any $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. Recall now that Ω is the CM period associated to K, that is well defined up to a p-adic unit, and $\Omega(\hat{\boldsymbol{\eta}})$ is the CM period associated to the pair $(K, \hat{\boldsymbol{\eta}})$, that is well defined up to an element of $\mathbb{Q}(\hat{\boldsymbol{\eta}})^\times$. Also $\Omega(\hat{\boldsymbol{\eta}}) = (2\pi i)^{2k}p(\hat{\boldsymbol{\eta}}, 1)$ where $p(\hat{\boldsymbol{\eta}}, 1)$ is the period that occurs in [10].

Theorem 5.2. *(a) For all $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$,*

$$\left(\frac{\pi^{2k+1}i^k\sqrt{-d}\cdot\mathfrak{g}(\chi\chi_\nu)\cdot I^{l,s}_{f,\chi}(\mu, \tilde{\boldsymbol{\eta}})}{\Omega(\hat{\boldsymbol{\eta}})}\right)^\sigma = \frac{\pi^{2k+1}i^k\sqrt{-d}\cdot\mathfrak{g}(\chi^\sigma\chi_\nu)\cdot I^{l,s}_{f^\sigma,\chi^\sigma}(\mu^\sigma, \tilde{\boldsymbol{\eta}}^\sigma)}{\Omega(\hat{\boldsymbol{\eta}}^\sigma)}.$$

(b) Suppose that p is split in K, $p \nmid h_K$, $p > 2k + 1$ and $p \nmid \tilde{N}$. Then, for all t, t', the ratio

$$\frac{\pi^{2k+1}c^{\mathfrak{b},t,t'}_{f,\chi}(\mu, \tilde{\boldsymbol{\eta}})\cdot I^{\mathfrak{b},t'}_{f,\chi}(\mu\chi_t, \tilde{\boldsymbol{\eta}}\cdot\chi_t\circ \mathrm{Nm})}{\Omega^{2k}}$$

is a λ-adic integer.

Proof: The p-integrality of part (b) may be proved along the lines of Thm. 4.15 of [22], using Rubin's theorem on the main conjecture of Iwasawa theory for imaginary quadratic fields with some modifications to account for the more complicated situation of the present article. We defer the details to the next section.

The reciprocity law of part (a) may be obtained as follows. By [27] Lemmas 3, 4 (and their proofs), for all $\sigma \in Aut(\mathbb{C}/K)$,

$$\left(\frac{I^{l,s}_{f,\chi}(\mu, \tilde{\boldsymbol{\eta}})}{\langle\theta_\kappa, \theta_\kappa\rangle}\right)^\sigma = \frac{I^{l,s}_{f^\sigma,\chi^\sigma}(\mu^\sigma, \tilde{\boldsymbol{\eta}}^\sigma)}{\langle\theta_{\kappa^\sigma}, \theta_{\kappa^\sigma}\rangle}.$$

Also, by equation (2.5) of [27], $\sqrt{d}\pi^{k+2}\langle\theta_\kappa, \theta_\kappa\rangle/L(1, \kappa^{-1}\kappa^\rho) \in \mathbb{Q}^\times$. But $L(1, \kappa^{-1}\kappa^\rho) = L(k + 1, \kappa^{-1}\kappa^\rho\mathrm{N}^k) = L(k + 1, \check{\boldsymbol{\eta}}\cdot(\chi\chi_\nu)\circ \mathrm{Nm})$ where $\check{\boldsymbol{\eta}} = (\hat{\boldsymbol{\eta}}^\rho)^{-1}$. By [27], Thm. 1,

$$\left(\frac{\mathfrak{g}(\chi\chi_\nu)L(k + 1, \check{\boldsymbol{\eta}}\cdot(\chi\chi_\nu)\circ \mathrm{Nm})}{L(k + 1, \check{\boldsymbol{\eta}})}\right)^\sigma = \left(\frac{\mathfrak{g}(\chi^\sigma\chi_\nu)L(k + 1, \check{\boldsymbol{\eta}}^\sigma\cdot(\chi^\sigma\chi_\nu)\circ \mathrm{Nm})}{L(k + 1, \check{\boldsymbol{\eta}}^\sigma)}\right).$$

Finally, by the main theorem of Blasius's article on Deligne's conjecture for Hecke L-functions of K ([2]) reinterpreted as in [12], Appendix (see also the correction in [11], p.82)

$$\left(\frac{L(k + 1, \check{\boldsymbol{\eta}}}{(2\pi i)^{k+1}p(\hat{\boldsymbol{\eta}}, 1)}\right)^\sigma = \frac{L(k + 1, \check{\boldsymbol{\eta}}^\sigma)}{(2\pi i)^{k+1}p(\hat{\boldsymbol{\eta}}^\sigma, 1)},$$

from which the required reciprocity law follows. ■

Corollary 5.3. *(a)* $i^{k+\tau}\mathfrak{g}(\chi)\beta \in \mathbb{Q}(f,\chi)$. *(b)* $v_\lambda(\beta) \geq 0$.

Proof: Part (a) follows from part (a) of the theorem, the rationality criterion in Prop. 5.1 (a) and equations (5.5) and (5.6), using that $\mathfrak{g}(\chi\chi_\nu)/\mathfrak{g}(\chi)\mathfrak{g}(\chi_\nu) \in \mathbb{Q}(\chi)$ and $\mathfrak{g}(\chi_\nu)|\nu|^{-1/2}i^\tau \in \mathbb{Q}^\times$. Part (b) follows from part (b) of the theorem and the integrality criterion Prop. 5.1 (b), since there exist infinitely many Heegner points with p split in K and $p \nmid h_K$. ([22], Lemma 5.1.) ∎

Let us then set $\boldsymbol{\beta} = i^{k+\tau}\mathfrak{g}(\chi)\beta$. The following reciprocity law for $\boldsymbol{\beta}$ is now immediate:

Corollary 5.4. *For any* $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$,

$$(\boldsymbol{\beta}(g,\chi))^\sigma = \boldsymbol{\beta}(g^\sigma,\chi^\sigma). \tag{5.7}$$

5.4. **Integrality of the Shimura lift.** We indicate in this section the modifications to the arguments in [22] needed to prove part (b) of Thm. 5.2. Since $\mathfrak{b}$ is fixed and the p-adic valuation of $c_{f,\chi}^{\mathfrak{b},t,t'}(\mu,\tilde{\boldsymbol{\eta}})$ is independent of t and t', in what follows we omit the superscripts $\mathfrak{b}, t, t'$ and simply write $c_{f,\chi}(\mu,\tilde{\boldsymbol{\eta}})$. Also, since the pair $(\mu\tilde{\chi}, \tilde{\boldsymbol{\eta}}\cdot\tilde{\chi}\circ \mathrm{Nm})$ is again of the form $(\mu,\tilde{\boldsymbol{\eta}})$, we may assume without loss that $t=1$. Let $S = S_{k+1}(m, \eta|_{\mathbb{Q}^\times}^{-1}\eta_K)$. By Thm. A.1 of the Appendix, $V_{t'}^*\{h_\chi(lz)\theta_\mu(rz)\}$ is a p-integral modular form in S. It suffices then to prove the following theorem.

Theorem 5.5. *Suppose that p is split in K, $p \nmid h_K$, $p > 2k+1$ and $p \nmid \tilde{N}$. Let g be any p-integral form in S. Then*

$$c_{f,\chi}(\mu,\tilde{\boldsymbol{\eta}}) \cdot \frac{\pi^{2k+1}\langle g(z), \tilde{\theta}_\kappa(sz)\rangle}{\Omega^{2k}}$$

is a p-adic integer.

Let T_0 be the set of primes q (dividing $2N^+$) such that $n_q = 0$ but $s_q > 0$ and let T be the set of primes q in T_0 such that $a_q(\theta_\kappa)^2 \equiv q^{k-1}(q+1)^2 \mod p$. For $q \in T$, let α_q, β_q be the parameters associated to θ_κ at q, ordered such that $\alpha_q/\beta_q \equiv q \mod p$. Denote by $\tilde{\mathbb{T}}$ the subalgebra of $\mathrm{End}_{\mathbb{C}}(S)$ generated by the Hecke operators T_q for $q \nmid m$ and the U_q for primes $q \in T$. If $V \subset S$ denotes the oldspace corresponding to θ_κ, then V is $\tilde{\mathbb{T}}$-invariant and the action of $\tilde{\mathbb{T}}$ on V is diagonalizable. Let $\mathcal{P}$ denote the set of eigencharacters of $\tilde{\mathbb{T}}$ that appear in its action on V. For every $i \in \mathcal{P}$, the corresponding eigenspace $V_i \subset V$ is one dimensional. Let $T_i \subset T$ be such that the action of U_q on V_i is by α_q for $q \in T_i$ and by β_q (or 0, if $2 \in T$ and $q = 2$) for $q \in T \setminus T_i$. For g any p-integral form in S, we may expand g as

$$g = \sum_{i \in P} \delta_i g_i + g'$$

where each $g_i \in V_i$ is a p-unit and g' is orthogonal to the oldspace of θ_κ.

Let F' be a number field that contains all the Hecke eigenvalues of all eigenforms in S, $\mathcal{O}$ the ring of integers of F' and $\tilde{\pi}$ any prime of F' over p. We shall prove in fact the following theorem from which Thm. 5.5 follows immediately.

Theorem 5.6. *Suppose p satisfies the assumptions of the previous theorem. Then, for all $i \in \mathcal{P}$,*

$$\delta_i \cdot c_{f,\chi}(\mu,\tilde{\boldsymbol{\eta}}) \cdot \frac{\pi^{2k+1}\langle g_i(z), \tilde{\theta}_\kappa(sz)\rangle}{\Omega^{2k}}$$

is a p-adic integer.

Let us now take and fix an $i \in \mathcal{P}$. The following lemma is the analog of Lemma 4.2 of [22].

Lemma 5.7.

$$v_{\tilde{\pi}}\left(c_{f,\chi}\frac{\pi^{2k+1}\langle g_i(z), \tilde{\theta}_\kappa(sz)\rangle}{\Omega^{2k}}\right) \geq \sum_{q|\nu, q\nmid N} f_q + \sum_{q\in T_i} v_{\tilde{\pi}}\left(\frac{\alpha_q}{\beta_q} - q\right) + v_{\tilde{\pi}}\left(\frac{\pi^{k-1}\cdot h_K \cdot L(1, \kappa^{-1}\kappa^\rho)}{\Omega^{2k}}\right)$$

where $f_q = (k+\frac{1}{2})v_{\tilde{\pi}}(q)$ (resp. $f_q = v_{\tilde{\pi}}(q+1)$, resp. $f_q = v_{\tilde{\pi}}(q-1)$) if q is split (resp. inert, resp. ramified) in K.

Remark 5.8. The assumption that $p \nmid h_K$ made earlier in the article will be essential later in this section. However some of the initial propositions do not require this, hence we do not make this assumption in the beginning but introduce it later when needed. Also we write $c_{f,\chi}$ instead of $c_{f,\chi}(\mu, \tilde{\boldsymbol{\eta}})$ for simplicity of notation.

Proof: Let P be defined by

$$P = \prod_{q|2N^+, q\notin T} q^{m_q - n_q} \cdot \prod_{q|N^-, q\nmid\nu} q^{m_q-n_q} \cdot \prod_{q\in\tilde{\Omega}'} q.$$

Let $\theta_i \in V_i$ be the $\tilde{\mathbb{T}}$-eigenform normalized to have its first Fourier coefficient equal to 1 and let u_q denote the eigenvalue of U_q acting on θ_i i.e. the L-series associated to θ_i is obtained by obtained by dropping the factors $(1-\alpha_q q^{-s})$ (resp. $(1-\beta_q q^{-s})$, resp. $(1-\alpha_q q^{-s})(1-\beta_q q^{-s})$) for $q \in T_i$ (resp. for $q \in T \setminus T_i$, $u_q = \alpha_q$, resp. $q \in T\setminus T_i$, $u_q = 0$) from the L-series for θ_κ. Then the collection $\{\theta_i(d'z); d' \mid P\}$ is a basis for V_i over $\mathbb{C}$ and one checks easily that g_i is a p-integral linear combination of the elements of this basis. For $d' \mid P$, one finds using Lemma 3 of [27] (and its proof) that

$$\langle\theta_i(d'z), \tilde{\theta}_\kappa(sz)\rangle = \prod_q R_q \cdot \langle\theta_\kappa, \theta_\kappa\rangle, \tag{5.8}$$

where $R_q = 1$ except in the cases listed below:

(i) If $q \mid N^+, q\nmid\nu, q \notin T$,

$$R_q = \frac{a_q(\theta_\kappa)}{q^k(q+1)}, \text{ if } q\nmid d', \quad R_q = q^{-(k+1)}, \text{ if } q \mid d'.$$

(ii) If $q \mid N^+, q \in T$,

$$R_q = \frac{q\beta_q - \alpha_q}{q^{k+1}(q+1)}, \text{ if } q \in T_i, \quad R_q = \frac{q\alpha_q - \beta_q}{q^{k+1}(q+1)}, \text{ if } q \in T\setminus T_i.$$

(iii) If $q \mid N^-, q\nmid\nu$, $\chi_{0,q}$ unramified,

$$R_q = 1, \text{ if } q\nmid d', \quad R_q = 0 \text{ if } q \mid d'.$$

(iv) If $q \mid N^-, q \nmid \nu$, $\chi_{0,q}$ ramified,

$$R_q = 1, \text{ if } q\nmid d', \quad R_q = 0 \text{ if } v_q(d') = 1, \quad R_q = q^{-(k+2)} \text{ if } v_q(d') = 2.$$

(v) If $q \in \tilde{\Omega}'$,

$$R_q = \frac{q-1}{q}, \text{ if } q\nmid d', \quad R_q = 0, \text{ if } q \mid d'.$$

(vi) If $q = 2 \notin T$,

$$R_q = \begin{cases} \dfrac{a_{q^2}(\theta_\kappa) - \varepsilon_{\theta_\kappa}(q)q^{k-1}}{q^{2k-1}(q+1)}, \text{ if } q \nmid d'; \\ \dfrac{a_q(\theta_\kappa)}{q^{2k-1}(q+1)} \text{ if } v_q(d') = 1; \\ q^{-2k} \text{ if } v_q(d') = 2. \end{cases}$$

where $\varepsilon_{\theta_\kappa}$ is the central character of θ_κ. On the other hand, if $q = 2 \in T$,

$$R_q = \begin{cases} \dfrac{\beta_q(q\beta_q - \alpha_q)}{q^{2k}(q+1)}, \text{ if } q \in T_i; \\ \dfrac{\alpha_q(q\alpha_q - \beta_q)}{q^{2k}(q+1)}, \text{ if } q \in T \setminus T_i \text{ and } u_q = \alpha_q; \\ 0, \text{ if } q \in T \setminus T_i \text{ and } u_q = 0. \end{cases}$$

Further

$$\frac{(4\pi)^{k+1}}{k!}\langle \theta_\kappa, \theta_\kappa \rangle = \mathrm{Res}_{s=k+1} D(s, \theta_{\bar{\kappa}}, \theta_\kappa) = \prod_q C_q(1) \cdot \frac{L(1, \kappa^{-1}\kappa^\rho)L(1, \eta_K)}{\zeta_{\mathbb{Q}}(2)}. \tag{5.9}$$

Recall that we have defined for each q (including $q = \infty$) an algebraic integer c_q such that $\sum_q v_{\tilde{\pi}}(c_q) = v_{\tilde{\pi}}(c_{f,\chi})$. Since $p \nmid q(q+1)$ for $q \mid N$, $p \nmid d$ and $L(1, \eta_K) = 2\pi h_K/w\sqrt{d}$, combining (5.8) and (5.9), we get

$$v_{\tilde{\pi}}\left(c_{f,\chi}\frac{\pi^{2k+1}\langle g_i(z), \tilde{\theta}_\kappa(sz)\rangle}{\Omega^{2k}}\right) \geq \sum_{q<\infty} v_{\tilde{\pi}}(c_q C_q(1)) + v_{\tilde{\pi}}(c_\infty) + \sum_{q\in\Sigma'} v_{\tilde{\pi}}(q-1) + \sum_{q\in T_i} v_{\tilde{\pi}}\left(\frac{\alpha_q}{\beta_q} - q\right) + v_{\tilde{\pi}}\left(\frac{\pi^{k-1}h_K L(1, \kappa^{-1}\kappa^\rho)}{\Omega^{2k}}\right).$$

One checks immediately that for finite q, $v_{\tilde{\pi}}(c_q C_q(1)) = 0$ unless $q \mid \nu, q \nmid N$ and q is unramified in K, in which case it equals $\frac{k+1}{2}v_{\tilde{\pi}}(q)$ or $v_{\tilde{\pi}}(q+1)$ according as q is split or inert in K. On the other hand, $v_{\tilde{\pi}}(c_\infty) = \frac{k}{2}v_{\tilde{\pi}}(\nu)$, whence we get the equality of the lemma, noting that if $q \mid \nu$ and $q = p$, q must be split in K. ∎

Note that in the case $v_{\tilde{\pi}}(\delta_i) \geq 0$, Thm. 5.6 follows immediately from the above lemma since all the terms on the right in the statement of the lemma are nonnegative. Therefore we may assume that $v_{\tilde{\pi}}(\delta_i) = -e_i$ with $e_i > 0$. Now write

$$S = V_i \oplus \oplus_{j\neq i} V_j \oplus W$$

with W the orthogonal complement to $\oplus_j V_j$ (the oldspace of θ_κ). Further suppose $W = W_1 \oplus W_2$ where W_1 is the subspace of W spanned by all the oldspaces corresponding to newforms in S that are theta functions associated to Grossencharacters of K and are congruent to θ_κ modulo λ. Thus $g' = g'_1 + g'_2$ for a uniquely determined $g'_1 \in W_1, g'_2 \in W_2$.

We will now need to study in more detail the space W_1. We have the following proposition.

Proposition 5.9. *Let κ' be a Grossencharacter of K of type $(k, 0)$ such that $\theta_{\kappa'} \in S$ and $\theta_{\kappa'}$ is congruent modulo λ to θ_κ. Then $\kappa' = \kappa \cdot \varepsilon$ for a finite order character ε of $K_{\mathbb{A}}^\times$ that satisfies* *(i)* $\varepsilon|_{\mathbb{Q}_{\mathbb{A}}^\times} = 1$*, and*

(ii) ε *is unramified outside the set of primes* $\tilde{\Omega}'' := \{q \mid \nu, q \nmid N, \left(\frac{d}{q}\right) = -1\}$.

Proof: We begin with a modification of the argument in the proof of Prop. 2.2 of [14]. Let κ' be a Grossencharacter of K of type $(k, 0)$ such that $\theta_{\kappa'}$ is congruent modulo λ to θ_κ. Thus the mod λ representations of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ associated to $\theta_{\kappa'}$ and θ_κ must be equal. Restricting to $Gal(\overline{\mathbb{Q}}/K)$ one must have $\tilde{\kappa}_\lambda \oplus \tilde{\kappa}_\lambda^\rho = \tilde{\kappa'_\lambda} \oplus \tilde{\kappa'_\lambda}^\rho$.

We claim that, with our assumptions, $\tilde{\kappa}_\lambda \neq \tilde{\kappa'_\lambda}^\rho$. Indeed, if $p \nmid \nu$, both κ and κ' are unramified at p, and the same argument as in [14] shows that $\tilde{\kappa}_\lambda \neq \tilde{\kappa'_\lambda}^\rho$ provided $p > k+1$. If on the other hand $p \mid \nu$, κ^2 is unramified at p, whence κ'^2 must also be unramified at p. Since κ^2 has weight $(2k, 0)$, the argument cited above then shows that $(\tilde{\kappa}_\lambda)^2 \neq (\tilde{\kappa'_\lambda}^\rho)^2$ (and hence $\tilde{\kappa}_\lambda \neq \tilde{\kappa'_\lambda}^\rho$) provided $p > 2k+1$.

Thus we must have $\tilde{\kappa}_\lambda = \tilde{\kappa'_\lambda}$. Let $\varepsilon = \kappa\kappa'^{-1}$ so that $\varepsilon_\lambda = \kappa_\lambda^{-1}\kappa'_\lambda$ and $\tilde{\varepsilon}_\lambda = 1$. Since $\theta_{\kappa'} \in S$, it must have the same central character as θ_κ. Thus ε is a finite order character with $\varepsilon|_{\mathbb{Q}_{\mathbb{A}}^\times} = 1$.

We now show that ε must be unramified outside the set of primes of K that lie over $\{q \mid \nu, \left(\frac{d}{q}\right) = -1\}$. To start with, it is clear that ε must be unramified outside the primes above m. If $q \mid N^+$, $q = \mathfrak{q}\bar{\mathfrak{q}}$ in K, the condition $\varepsilon|_{\mathbb{Q}_{\mathbb{A}}^\times} = 1$ forces $\mathfrak{f}_{\varepsilon,\mathfrak{q}} = \mathfrak{f}_{\varepsilon,\bar{\mathfrak{q}}}$. Since $v_q(m_i) \leq 2$, one sees that ε is at worst tamely ramified at $\mathfrak{q}$ and $\bar{\mathfrak{q}}$. But $\tilde{\varepsilon}_\lambda = 1$ and $p \nmid q-1$ by assumption, hence ε must in fact be unramified at $\mathfrak{q}$ and $\bar{\mathfrak{q}}$. Similarly, if $q \mid N^-$, so that q is inert in K, ε must be at worst tamely ramified and hence unramified at q since $p \nmid q^2 - 1$. If $q \mid d$ and $q \nmid \nu$, $v_q(m_i) \leq 1$, hence κ_i and ε must be unramified at such q. If $q \mid \nu$ and $q = \mathfrak{q}\bar{\mathfrak{q}}$ is split in K, identifying $K_q \simeq \mathbb{Q}_q \times \mathbb{Q}_q$ one has $\kappa_q = (\kappa_{q,1}, \kappa_{q,2})$ where $\kappa_{q,1}\chi_{\nu,q}$ and $\kappa_{q,2}$ are unramified. As before, the condition $\varepsilon|_{\mathbb{Q}_{\mathbb{A}}^\times} = 1$ forces $\mathfrak{f}_{\varepsilon,\mathfrak{q}} = \mathfrak{f}_{\varepsilon,\bar{\mathfrak{q}}}$. Since $v_q(m_i) \leq 1$, if ε were ramified at $\mathfrak{q}$ and $\bar{\mathfrak{q}}$, $\varepsilon_{\mathfrak{q}}\chi_{\nu,q}$ and $\varepsilon_{\bar{\mathfrak{q}}}\chi_{\nu,q}$ would both have to be unramified. However the condition $\tilde{\varepsilon}_\lambda = 1$ now forces ε to be unramified at $\mathfrak{q}$ and $\bar{\mathfrak{q}}$ since $p \neq 2$ and $\chi_{\nu,q}|_{\mathbb{Z}_q^\times}$ is a nontrivial quadratic character. Finally, if $q \mid (\nu, d)$, $v_q(m_i) \leq 2$, hence ε is at worst tamely ramified at q. However the condition $\varepsilon|_{\mathbb{Q}_{\mathbb{A}}^\times} = 1$ forces ε to be unramified at q. This completes the proof of the proposition. ■

Recall from the statement of the proposition that $\tilde{\Omega}''$ has been defined to be the set of primes $q \mid \nu, q \nmid N$ such that q is inert in K. Let κ' and ε be as in the proposition and let $q \in \tilde{\Omega}''$. Since $v_q(m_i) \leq 2$, ε must be a tamely ramified or unramified character with $\varepsilon_q|_{\mathbb{Z}_q^\times = 1}$. Let $U_q = \mathcal{O}_{K_q}^\times$. Then $\varepsilon_q|_{U_q}$ factors through the quotient U_q/U'_q where U'_q is the subgroup $\mathbb{Z}_q^\times(1 + q\mathcal{O}_{K_q})$ of index $q+1$. Set $U' = \prod_{q \notin \tilde{\Omega}''} U_q \times \prod_{q \in \tilde{\Omega}''} U'_q$ so that ε factors through the abelian extension K' of K corresponding the open subgroup $K^\times U' K_\infty^\times$ of $K_{\mathbb{A}}^\times$. We may thus think of ε as being a character of G' where G' is the p-part of the Galois group $\mathrm{Gal}(K'/K) \simeq K_{\mathbb{A}}^\times / K^\times U' K_\infty^\times$ (thought of as a quotient of $\mathrm{Gal}(K'/K)$). In this way one obtains a bijection between the set of κ' with $\theta_{\kappa'}$ congruent to θ_κ modulo λ and the nontrivial characters ε of the group G'. Notice that $v_{\tilde{\pi}}(|G'|) = v_{\tilde{\pi}}(h_K) + \sum_{q \in \Omega''} v_{\tilde{\pi}}(q+1)$. Also note that for any such character ε, $\varepsilon|_{\mathbb{Q}_{\mathbb{A}}^\times} = 1$ (thinking of ε as a character of $K_{\mathbb{A}}^\times$). In particular for any prime $q = \mathfrak{q}\bar{\mathfrak{q}}$ split in K at which ε is unramified, $\varepsilon(\mathfrak{q})\varepsilon(\bar{\mathfrak{q}}) = 1$.

Suppose that $G' \cong C_1 \times C_2 \times \ldots C_v$ with C_l being the cyclic factors of G' and $|C_l| = p^{a_l}$. For $l = 1, \ldots, v$, let ξ_l be a generator of C_l and ε_l be a generator of the character group of C_l. Also, we now pick for each l, $l = 1, \ldots, v$, a prime q_l such that

(i) q_l is split in K, $q_l = \mathfrak{q}_l\bar{\mathfrak{q}}_l$ and $\mathfrak{q}_l$, $\bar{\mathfrak{q}}_l$ are unramified in K'.

(ii) $\mathrm{Frob}_{\mathfrak{q}_l}$ corresponds to the element $(1, \ldots, \xi_l, \ldots, 1)$ i.e. the element of G' that projects to 1 on the factor C_j for $j \neq l$ and that projects to ξ_l on the factor C_l.

(iii) $q_l \nmid pN$ and $(\overline{\eta' \cdot \chi_0 \circ \mathbb{N}})^2(\mathfrak{q}_l) \not\equiv 1 \mod \tilde{\pi}$.

Since $(\eta' \cdot \chi_0 \circ \mathbb{N})^2$ is a Hecke character of type $(-2k, 2k)$ with conductor only divisible by the primes above N (recall $p > 2k+1$), and since ε has conductor divisible only by the primes in Ω'', a simple application of Chebotcharev's theorem allows us to pick primes q_l satisfying the properties above. Now define a Hecke operator Δ by

$$\Delta = \prod_{l=1}^{v} \prod_{j=1}^{p^{a_l}-1} (T_{q_l} - \kappa(\mathfrak{q}_l)\varepsilon_l^j(\mathfrak{q}_l) - \kappa(\bar{\mathfrak{q}})\varepsilon_l^j(\bar{\mathfrak{q}}_l)).$$

Since

$$g = \delta_i g_i + \sum_{j \neq i} \delta_j g_j + g_1' + g_2'$$

we see that $g_i \equiv H \mod \tilde{\pi}^{e_i}$ where H is given by

$$H = -\delta_i^{-1}(\sum_{j \neq i} \delta_j g_j + g_1' + g_2').$$

Notice that H is in fact p-integral since $H = g_i - \delta_i^{-1} g$ and that $H \in \oplus_{j \neq i} V_j \oplus W$. Applying the integral Hecke operator Δ to the equation $g_i \equiv H \mod \tilde{\pi}^{e_i}$, we see that

$$\Delta g_i \equiv \Delta H \mod \tilde{\pi}^{e_i}.$$

We now state and prove two lemmas about Δg_i and ΔH.

Lemma 5.10. $\Delta H \in \tilde{W} := \oplus_{j \neq i} V_j \oplus W_2$.

Lemma 5.11. $\Delta g_i = \tilde{\alpha} g_i$ *with* $\tilde{\alpha} \in F'$ *satisfying* $v_{\tilde{\pi}}(\tilde{\alpha}) = v_{\tilde{\pi}}(|G'|)$.

We first prove Lemma 5.10. It suffices to show that Δ annihilates any newform $\theta_{\kappa'}$ which is congruent to θ_κ mod λ. Write $\kappa' = \kappa \cdot \varepsilon$ and suppose that $\varepsilon = \prod_{l=1}^{v} \varepsilon_l^{b_l}$ for $0 \leq b_l \leq a_l$. Since ε is not the trivial character we may pick j such that $b_j \neq 0$. The Hecke operator $T_{q_j} - \kappa(\mathfrak{q}_j)\varepsilon_j^{b_j}(\mathfrak{q}_j) - \kappa(\bar{\mathfrak{q}}_j)\varepsilon_j^{b_j}(\bar{\mathfrak{q}}_j)$ occurs as a factor of Δ. On the other hand this Hecke operator acts on $\theta_{\kappa'}$ with eigenvalue

$$\begin{aligned} & \kappa(\mathfrak{q}_j)\prod_{l=1}^{v}\varepsilon_l^{b_l}(\mathfrak{q}_j) + \kappa(\bar{\mathfrak{q}}_j)\prod_{l=1}^{v}\varepsilon_l^{b_l}(\bar{\mathfrak{q}}_j) - \kappa(\mathfrak{q}_j)\varepsilon_j^{b_j}(\mathfrak{q}_j) - \kappa(\bar{\mathfrak{q}}_j)\varepsilon_j^{b_j}(\bar{\mathfrak{q}}_j) \\ = \; & \kappa(\mathfrak{q}_j)\varepsilon_j^{b_j}(\mathfrak{q}_j)\{\prod_{l \neq j}\varepsilon_l^{b_l}(\mathfrak{q}_j) - 1\} + \kappa(\bar{\mathfrak{q}}_j)\varepsilon_l^{b_l}(\bar{\mathfrak{q}}_j)\{\prod_{l \neq j}\varepsilon_l^{b_l}(\bar{\mathfrak{q}}_j) - 1\} \\ = \; & 0 \end{aligned}$$

since $\varepsilon_l(\mathfrak{q}_j) = \varepsilon_l(\bar{\mathfrak{q}}_j) = 1$ for $l \neq j$. Thus $\Delta\theta_{\kappa'} = 0$ as well, as was required to be shown.

Now we prove Lemma 5.11. Clearly $\Delta g_i = \tilde{\alpha} g_i$, where

$$\tilde{\alpha} = \prod_{l=1}^{v} \prod_{j=1}^{p^{a_l}-1} (\kappa(\mathfrak{q}_l) + \kappa(\overline{\mathfrak{q}}_l) - \kappa(\mathfrak{q}_l)\varepsilon_l^j(\mathfrak{q}_l) - \kappa(\overline{\mathfrak{q}})\varepsilon_l^j(\overline{\mathfrak{q}}_l)).$$

Here $\varepsilon_l(\mathfrak{q}_l) = \zeta_l$, with ζ_l a primitive p^{a_l}th root of unity. Let $\beta_l = \kappa(\mathfrak{q}_l) + \kappa(\overline{\mathfrak{q}}_l) - \kappa(\mathfrak{q}_l)\varepsilon_l^j(\mathfrak{q}_l) - \kappa(\overline{\mathfrak{q}})\varepsilon_l^j(\overline{\mathfrak{q}}_l)$. Then

$$v_{\tilde{\pi}}(\beta_l) = v_{\tilde{\pi}}\left(\kappa(\mathfrak{q}_l)(1-\zeta_l^j) + \kappa(\overline{\mathfrak{q}}_l)(1-\zeta_l^{-j})\right) = v_{\tilde{\pi}}(1-\zeta_l^j) + v_{\tilde{\pi}}\left(\zeta_l^j\kappa(\mathfrak{q}_l) - \kappa(\overline{\mathfrak{q}}_l)\right).$$

We claim that $v_{\tilde{\pi}}\left(\zeta_l^j\kappa(\mathfrak{q}_l) - \kappa(\overline{\mathfrak{q}}_l)\right) = 0$. Suppose to the contrary that $\zeta_l^j\kappa(\mathfrak{q}_l) - \kappa(\overline{\mathfrak{q}}_l) \equiv 0 \mod \tilde{\pi}$. Then $\kappa(\overline{\mathfrak{q}}_l) \equiv \zeta_l^j\kappa(\mathfrak{q}_l) \equiv \kappa(\mathfrak{q}_l) \mod \tilde{\pi}$. Since $\kappa(\kappa^\rho)^{-1} = \overline{\tilde{\eta}(\tilde{\eta}^\rho)^{-1}} = \overline{\eta' \cdot (\chi_0\chi_\nu \circ \mathbb{N})}$, we get $\overline{\eta' \cdot (\chi_0\chi_\nu \circ \mathbb{N})}(\mathfrak{q}_l) \equiv 1 \mod \tilde{\pi}$, hence $\overline{(\eta' \cdot \chi_0 \circ \mathbb{N})}^2(\mathfrak{q}_l) \equiv 1 \mod \tilde{\pi}$. However we have chosen q_l to expressly avoid this congruence, hence the claim above is verified. Thus

$$v_{\tilde{\pi}}(\tilde{\alpha}) = \sum_{l=1}^{v} \sum_{j=1}^{p^{a_l}-1} v_{\tilde{\pi}}(1-\zeta_l^j) = \sum_{l=1}^{v} v_{\tilde{\pi}}(p^{a_l}) = v_{\tilde{\pi}}(|G'|),$$

which proves Lemma 5.11.

Now consider the congruence $\tilde{\alpha} g_i \equiv \Delta H \mod \tilde{\pi}^{e_i}$. If $v_{\tilde{\pi}}(\tilde{\alpha}) \geq v_{\tilde{\pi}}(e_i)$, Thm. 5.6 follows again from Lemma 5.7 since $v_{\tilde{\pi}}(\tilde{\alpha}) = v_{\tilde{\pi}}(|G'|) = v_{\tilde{\pi}}(h_K) + \sum_{q \in \tilde{\Omega}''} v_{\tilde{\pi}}(q+1) = v_{\tilde{\pi}}(h_K) + \sum_{q \in \tilde{\Omega}''} f_q$. Thus we may assume that $v_{\tilde{\pi}}(\tilde{\alpha}) < v_{\tilde{\pi}}(e_i)$. In this case, $g_i \equiv \tilde{\alpha}^{-1}\Delta H \mod \tilde{\pi}^{e_i - v_{\tilde{\pi}}(\tilde{\alpha})}$ and $\tilde{\alpha}^{-1}\Delta H$ is a p-integral form in $\tilde{W}$. Set $e = e_i - v_{\tilde{\pi}}(\tilde{\alpha})$. Let $\mathbb{T}'$ be the subalgebra of $End_{\mathbb{C}}(\tilde{W})$ generated by the image of $\tilde{\mathbb{T}}$ and let $\mathbb{T} = \mathbb{T}' \otimes \mathcal{O}$. Define $I = Ann_{\mathbb{T}}(\tilde{\alpha}^{-1}\Delta H \mod \tilde{\pi}^e)$. Then $\mathbb{T}/I \simeq \mathcal{O}/\tilde{\pi}^e$ and the elements $T' - \lambda_{\theta_i}(T') \in I$ for all $T' \in \mathbb{T}'$.

Let $[\tilde{W}]$ be a set of representatives for the eigenspaces of $\tilde{\mathbb{T}}$ contained in $\tilde{W}$ and F be the ring $\prod_{h' \in [\tilde{W}]} F'$ (where by $h' \in [\tilde{W}]$ we mean h' is any normalized eigenform of $\tilde{\mathbb{T}}$ contained in $\tilde{W}$, i.e. with first Fourier coefficient equal to 1.) Then $\mathbb{T}$ is naturally a subring of F via the embedding given by the various characters of $\mathbb{T}'$ and $\mathbb{T} \otimes_{\mathcal{O}} F' = F$. Let $V = F \oplus F$ and $L = \prod_{h' \in [\tilde{W}]} \mathcal{O}^2 \subset V$. Then L is a sublattice of V that is stable under the action of $\mathbb{T}$. Below we write $K'_{h'}$ for the appropriate copy of the field F' in F (and $\mathcal{O}'_{h'}$ for the appropriate copy of $\mathcal{O}$) so that $F = \prod_{h' \in [\tilde{W}]} K'_{h'}$.

Let β be the maximal ideal of $\tilde{\mathbb{T}}$ containing I and let L_β denote the completion of L at β. The natural map $L \mapsto L_\beta$ factors through $\prod_{h' \in [\tilde{W}]} (\mathcal{O}'_{h',\lambda})^2$. As in [22], Lemma 4.5 and Lemma 4.6 (note our slightly different notation), one has

Lemma 5.12. *(i) If $(\mathcal{O}'_{h',\lambda})^2$ is not in the kernel of the map $L \to L_\beta$, then h' is congruent to θ_i mod λ i.e. the characters of $\tilde{\mathbb{T}}$ corresponding to h' and θ_i are congruent mod λ.*

(ii) If h' is an eigenform in W_2 corresponding to a theta lift from K, then $(\mathcal{O}'_{h',\lambda})^2$ is in the kernel of this map.

(iii) The terms $(\mathcal{O}'_{\theta_j,\lambda})^2$, $j \neq i$ are in the kernel of this map.

Lemma 5.13. *Let $[\mathcal{W}]$ denote the set of forms h in $[\tilde{W}]$ such that one of the eigenforms h' of $\tilde{\mathbb{T}}$ corresponding to h is congruent to θ_i mod λ. Then, for each such h exactly one of the eigenforms corresponding to h can be congruent to θ_i mod λ. Denoting this eigenform by h', one has $L_\beta \simeq (\prod_{h \in [\mathcal{W}]} \mathcal{O}'_{h',\lambda})^2$ and $\mathbb{T}_\beta \otimes_{\mathcal{O}} F' \simeq \prod_{h \in [\mathcal{W}]} K'_{h',\lambda}$.*

Since $V_\beta = L_\beta \otimes_{\mathcal{O}} F' \simeq (\prod_{h\in[\mathcal{W}]} K'_{h',\lambda})^2 \simeq \prod_{h\in[\mathcal{W}]}(K'_{h',\lambda})^2$, and $K_{h,\lambda}$ is contained in $K'_{h',\lambda}$, V_{β_2} is naturally a representation space for $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, the action on the component $V_{h,\lambda} = (K'_{h',\lambda})^2$ being via $\rho_{h,\lambda}$. The Galois action preserves L_β and thus L_β is a $\mathbb{T}_\beta[Gal(\overline{\mathbb{Q}}/\mathbb{Q})]$ module with commuting actions of the Galois group and the Hecke algebra. We shall only be concerned with its structure as a $\mathbb{T}_\beta[Gal(\overline{\mathbb{Q}}/K)]$ module.

Let κ_λ and κ_λ^ρ denote the λ-adic characters associated to κ and κ^ρ respectively, and denote by $\tilde{\kappa}_\lambda$ and $\tilde{\kappa}_\lambda^\rho$ their reductions mod λ. An application of the Brauer-Nesbitt theorem gives

Lemma 5.14. *Let $\mathcal{L}$ be a compact sub-bimodule of V_β. Suppose that U is an irreducible subquotient (as $\mathbb{T}_\beta[Gal(\overline{\mathbb{Q}}/K)]$ module) of $\mathcal{L}/\pi^r\mathcal{L}$ for some r. Then U has one of the following two types.*
(i) $U \simeq \mathbb{T}_\beta/\beta\mathbb{T}_\beta \simeq \mathcal{O}_\pi/\pi\mathcal{O}_\pi$ with $Gal(\overline{\mathbb{Q}}/K)$ acting via $\tilde{\kappa}_\lambda$.
(ii) $U \simeq \mathbb{T}_\beta/\beta\mathbb{T}_\beta \simeq \mathcal{O}_\pi/\pi\mathcal{O}_\pi$ with $Gal(\overline{\mathbb{Q}}/K)$ acting via $\tilde{\kappa}_\lambda^\rho$.

We say that U is of type κ or κ^ρ respectively in these two cases. Note that these types are distinct since $\tilde{\kappa}_\lambda \neq \tilde{\kappa}_\lambda^\rho$. Indeed since $p > 2k+1$, $\tilde{\kappa}_\lambda$ is ramified at $\mathfrak{p}$ and unramified at $\overline{\mathfrak{p}}$ while $\tilde{\kappa}_\lambda^\rho$ is unramified at $\mathfrak{p}$ and ramified at $\overline{\mathfrak{p}}$.

By the method of [22] (p. 947-950) one constructs a compact sub-bimodule $\mathcal{L}$ of V_β such that $\mathcal{L}/I\mathcal{L}$ sits in an exact sequence of bimodules

$$0 \to C \to \mathcal{L}/I\mathcal{L} \to M \to 0 \tag{5.10}$$

such that M is a free module of rank one over $\mathbb{T}_\beta/I$, $C \simeq \mathcal{L}^0/I\mathcal{L}^0$ for a faithful $\mathbb{T}_\beta$ module $\mathcal{L}^0$ and the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ on C (resp. M) is given by $\tilde{\kappa}_\lambda$ (resp. $\tilde{\kappa}_\lambda^\rho$). Let $\mathfrak{g}$ be the conductor of $\eta' \cdot \chi \circ \mathbb{N}$, $K_{\mathfrak{g}}$ denote the ray class field of K modulo $\mathfrak{g}$ and set $K_0 = K_{\mathfrak{g}}(\sqrt{\nu})$. Let K_∞ be the unique $\mathbb{Z}_p^2$ extension of K_0 abelian over K (so that κ_λ factors through $\mathrm{Gal}(K_\infty/K)$) and L' the splitting field over K_∞ of the representation $\mathcal{L}/I\mathcal{L}$. Denote by G' the Galois group $\mathrm{Gal}(L'/K_\infty)$. We define a pairing

$$G' \times M \to C, \qquad \langle \sigma, m\rangle \mapsto \sigma\tilde{m} - \kappa_\lambda(g)\tilde{m}, \tag{5.11}$$

where $\tilde{m}$ is any lift of m to $\mathcal{L}/I\mathcal{L}$. The following lemma may be proved in exactly the same way as Lemma 4.12 of [22].

Lemma 5.15. *The extension L'/K_∞ is unramified outside the primes lying above $\Xi \cup \mathfrak{p}$ where Ξ is the following set of primes in K.*

$$\Xi = \{2\} \cup \{\mathfrak{q}; q \mid \nu\} \cup \{\mathfrak{q};\ q \in T_i\} \cup \{\mathfrak{q}, \overline{\mathfrak{q}};\ q \mid N^+, n_q > 0\} \cup \{q;\ q \mid N^-, n_q > 0\}.$$

We view the pairing (5.11) as one of $Gal(\overline{\mathbb{Q}}/K)$ modules where $Gal(\overline{\mathbb{Q}}/K)$ acts on G' in the usual way (via conjugation). Then we obtain a Galois equivariant injection

$$G' \hookrightarrow \mathrm{Hom}_R(M, C). \tag{5.12}$$

Let R_κ be the ring generated over $\mathbb{Z}_p$ by the values of $\kappa_\lambda = \chi_\lambda(\chi_\lambda^\rho)^{-1}$. The image of G' under (5.12) is easily seen to be stable under R_κ, and this gives G' the structure of an R_κ module. We thus get a map $\phi : G' \otimes_{R_\kappa} \mathcal{O}_\pi \to \mathrm{Hom}_{\mathcal{O}_\pi}(M, C) = C$.

Lemma 5.16. *The map ϕ is surjective. Also* $\mathrm{Fitt}_{\mathcal{O}_\pi}(G' \otimes_{R_\kappa} \mathcal{O}_\pi) \subseteq \pi^e$.

Proof: See [22], Lemma 4.13. ■

We now assume that $p \nmid h_K$. Thus $p \nmid [K_0 : K]$ as well. An application of the main conjecture as in [22], Sec. 4.3 yields

Proposition 5.17. *Let $\epsilon = \kappa(\kappa^\rho)^{-1}\mathbb{N}$, and let γ be given by*

$$\gamma = G(\epsilon)\left(1 - \frac{\epsilon(\overline{\mathfrak{p}})}{p}\right)(1 - \epsilon^{-1}(\overline{\mathfrak{p}}))\frac{\pi^{k-1}L_{\mathfrak{g}\,\cup\,\Xi}(1, \kappa^{-1}\kappa^\rho)}{\Omega^{2k}},$$

where $G(\epsilon)$ is the modified Gauss sum defined in [6] *Thm 4.14. Then*

$$\mathrm{Fitt}_{R_\kappa}(G') \supseteq (\gamma).$$

One can check easily that $1 - \frac{\epsilon(\overline{\mathfrak{p}})}{p}$ and $1 - \epsilon^{-1}(\overline{\mathfrak{p}})$ are λ-units and $v_{\tilde{\pi}}(G(\epsilon)) = (k + \frac{1}{2})v_{\tilde{\pi}}(|\nu|)$. (For the computation of $v_{\tilde{\pi}}(G(\epsilon))$ the reader may also refer to II Sec. 6.3 of [6] or the remarks in Sec. 7.6 of [7].) Further one checks immediately that for $\mathfrak{q} \in \mathfrak{g} \cup \Xi$, the Euler factor at $\mathfrak{q}$ of ϵ^{-1} evaluated at 0 is a p-unit except possibly when $\mathfrak{q}^2 = (q), q \mid (\nu, d)$ or $\mathfrak{q} \in T_i$. In these case, the inverse of the Euler factors evaluated at 0 have p-adic valuation equal to that of $q - 1$ and $\frac{\alpha_q}{\beta_q} - q$ respectively. Since $f_q = (k + \frac{1}{2})v_{\tilde{\pi}}(q)$ for $q \mid \nu$, $\left(\frac{d}{q}\right) = 1$, $f_q = v_{\tilde{\pi}}(q-1)$ for $q \mid (\nu, d)$, and $(\gamma) \subseteq (\pi^e)$ from the previous proposition and lemma, we get

$$e \leq \sum_{q \mid \nu, q \nmid N, \left(\frac{d}{q}\right) \neq -1} f_q + \sum_{q \in T_i} v_{\tilde{\pi}}\left(\frac{\alpha_q}{\beta_q} - q\right) + v_{\tilde{\pi}}\left(\frac{\pi^{k-1} \cdot L(1, \kappa^{-1}\kappa^\rho)}{\Omega^{2k}}\right).$$

Since $e = e_i - v_{\tilde{\pi}}(\tilde{\alpha})$ where $v_{\tilde{\pi}}(\alpha) = \sum_{q \in \Omega''} f_q + v_{\tilde{\pi}}(h_K)$, we get finally

Theorem 5.18. *Assume $p \nmid h_K$. Then*

$$e_i \leq \sum_{q \mid \nu, q \nmid N} f_q + \sum_{q \in T_i} v_{\tilde{\pi}}\left(\frac{\alpha_q}{\beta_q} - q\right) + v_{\tilde{\pi}}\left(\frac{\pi^{k-1} \cdot h_K \cdot L(1, \kappa^{-1}\kappa^\rho)}{\Omega^{2k}}\right).$$

Combining the theorem above with Lemma 5.7 completes the proofs of Thms. 5.6 and 5.5.

Remark 5.19. We have assumed that $p \nmid h_K$ since we need $p \nmid [K_0 : K]$ in order to apply Rubin's theorem [25]. However we have stated the above theorem including the term h_K since the statement above is presumably true even without the assumption $p \nmid h_K$.

6. Applications

6.1. **A plethora of formulae.** Recall the following notation and results from the previous chapters:

$$
\begin{aligned}
f &\in S_{2k}(\Gamma_0(N)), g \in S_{2k}(\Gamma), g := \mathrm{JL}(f) \\
\nu &\quad \text{an odd fundamental quadratic discriminant}, \chi_\nu = \left(\frac{\nu}{\cdot}\right), \psi' := \psi^{1/|\nu|} \\
\boldsymbol{\chi} &\quad \text{a finite order character}, N' := c_{\boldsymbol{\chi}} \mid 4N, M := \gcd(4, N'N) \\
F_0 &= \text{a number field over which } B \text{ splits.} \\
\tilde{F}_0 &= \mathbb{Q} \text{ if } k = 1, \tilde{F}_0 = F_0 \text{ if } k \geq 2. \\
F &= \text{a number field containing } F_0 \text{ and all the eigenvalues of } f. \\
F(\chi) &= \text{the field generated over } F \text{ by the values of } \chi. \\
\mathbb{Q}(f,\chi) &= \text{the field generated over } \mathbb{Q} \text{ by the eigenvalues of } f \text{ and the values of } \chi. \\
s_{g_\chi} &:= \text{a newform in } \pi' \otimes \chi, \text{ well defined up to a } \lambda\text{-unit in } \mathbb{Q}(f,\chi). \\
h_\chi &\in S_{k+\frac{1}{2}}(M, \boldsymbol{\chi}, f_\chi), t := t_{h_\chi} \in \tilde{A}_{k+\frac{1}{2}}(M, \chi_0, f_\chi), \text{ both well defined up to} \\
&\qquad \text{a } \lambda\text{-unit in } \mathbb{Q}(f,\chi). \\
s &:= s_{g_\chi} \otimes \chi^{-1}\chi_\nu \circ \mathrm{Nm} \in \pi' \otimes \chi_\nu. \\
\varphi &\in V(\mathbb{A}), t' := t(\psi', \varphi, s), s' := T(\psi', \varphi, t).
\end{aligned}
$$

We have shown that

$$
\begin{aligned}
t' &= \alpha' u_+(g_\chi) t = \alpha u_\epsilon(g) t, \text{ i.e. } t' = t_{\alpha u_\epsilon(g) h_\chi}. \\
s' &= \beta s, \text{ with } \boldsymbol{\alpha} := \alpha/\mathfrak{g}(\chi) \in F(\chi), v_\lambda(\alpha), v_\lambda(\boldsymbol{\alpha}) \geq 0, \\
&\qquad \boldsymbol{\beta} := i^{k+\tau}\mathfrak{g}(\chi)\beta \in \mathbb{Q}(f,\chi), v_\lambda(\beta) \geq 0.
\end{aligned}
$$

We now write down several formulae that explain the relations between the objects and quantities mentioned above. All the constants below are completely explicit, but for ease of notation we suppress their exact values.

1. See-Saw duality

$$
\begin{aligned}
\langle t, t' \rangle &= \langle s', s \rangle. \\
\Rightarrow \bar{\alpha}\overline{u_\epsilon(g)}\langle h_\chi, h_\chi \rangle &= \beta \langle g_\chi, g_\chi \rangle.
\end{aligned} \tag{6.1}
$$

2. The formula from Prop. 4.1 for the Fourier coefficients of t': for ξ satisfying the conditions

(a) If $q \mid N$, $q \nmid \nu$, $\left(\frac{\xi_0}{q}\right) \neq -w_q$;

(b) If $q \mid N$, $q \mid \nu$, $\left(\frac{\xi_0}{q}\right) = -w_q$;

(c) $\xi_0 \equiv 0, 1 \mod 4$;

$$
|\alpha u_\epsilon(g) a_\xi(h_\chi)|^2 = C(f,\chi,\nu)\pi^{-2k}|\nu\xi|^{k-\frac{1}{2}} L(\frac{1}{2}, \pi_f \otimes \chi_\nu) L(\frac{1}{2}, \pi_f \otimes \chi_{\xi_0}) \cdot \frac{\langle g_\chi, g_\chi \rangle}{\langle f_\chi, f_\chi \rangle} \tag{6.2}
$$

for an explicit nonzero constant $C(f,\chi,\nu) \in \mathbb{Q}^\times$.

3. A formula of Baruch and Mao [1] for the Fourier coefficients of h_χ: for the ξ satisfying conditions (a),(b),(c) above,

$$\frac{|a_\xi(h_\chi)|^2}{\langle h_\chi, h_\chi\rangle} = C'(f,\chi)\frac{\pi^{-k}|\xi|^{k-\frac{1}{2}}L(\frac{1}{2},\pi\otimes\chi_{\xi_0})}{\langle f,f\rangle} \tag{6.3}$$

for an explicit nonzero constant $C(f,\chi) \in \mathbb{Q}^\times$.

4. Taking the ratio of (6.2) to (6.3), we get

$$|\alpha u_\epsilon(g)|^2\langle h_\chi, h_\chi\rangle = \frac{C(f,\chi,\nu)}{C'(f,\chi)}\cdot \pi^{-k}|\nu|^{k-\frac{1}{2}}L(\frac{1}{2},\pi\otimes\chi_\nu)\frac{\langle f,f\rangle}{\langle f_\chi, f_\chi\rangle}\cdot\langle g_\chi, g_\chi\rangle. \tag{6.4}$$

Set $C''(f,\chi,\nu) := \frac{C(f,\chi,\nu)}{C'(f,\chi)}\cdot\frac{\langle f,f\rangle}{\langle f_\chi,f_\chi\rangle}$. Now substituting (6.1) in (6.4) yields the fundamental formula

Theorem 6.1.

$$\alpha\beta u_\epsilon(g) = C''(f,\chi,\nu)\cdot\pi^{-k}|\nu|^{k-\frac{1}{2}}L(\frac{1}{2},\pi\otimes\chi_\nu). \tag{6.5}$$

5. As a bonus, multiplying both sides of (6.1) by $\bar\beta$ gives

$$\begin{aligned}\beta\bar\beta\langle g_\chi, g_\chi\rangle &= \bar\alpha\bar\beta\bar u_\epsilon(g)\langle h_\chi,h_\chi\rangle = \alpha\beta u_\epsilon(g)\langle h_\chi,h_\chi\rangle.\\ \text{i.e. } \langle s',s'\rangle = \langle\beta g,\beta g\rangle &= C''(f,\chi,\nu)\cdot\pi^{-k}|\nu|^{k-\frac{1}{2}}L(\frac{1}{2},\pi\otimes\chi_\nu)\langle h_\chi,h_\chi\rangle. \qquad (6.6)\end{aligned}$$

This is nothing but the explicit version of the Rallis inner product formula.

6.2. **Period ratios of modular forms. Proofs of Thms. 1.1 and 1.2:**

We begin by making use of the main formula (6.5). In the notation of the introduction, we have

$$\boldsymbol{\alpha}\boldsymbol{\beta}u_\epsilon(g) = C''(f,\chi,\nu)A(f,\nu)u_\epsilon(f)$$

since $\boldsymbol{\alpha} = \alpha/\mathfrak{g}(\chi)$, $\boldsymbol{\beta} = i^{k+\tau}\mathfrak{g}(\chi)\beta$ and $\mathfrak{g}(\chi_\nu) = i^\tau|\nu|^{1/2}$. Under the assumption $p\nmid\tilde N$, one checks easily that $C''(f,\chi,\nu)$ is a p-unit in $\mathbb{Q}$. Since $\boldsymbol{\alpha}\in F(\chi), \boldsymbol{\beta}\in\mathbb{Q}(f,\chi)$ and $A(f,\nu)\in\mathbb{Q}(f)$, we have $u_\epsilon(f)/u_\epsilon(g)\in F(\chi)$. Setting $\chi = 1$ (and making an appropriate compatible choice of ν), we obtain the reciprocity law of Thm. 1.1 by combining (4.5), (5.7) and Thm 1, (iii) of [31]. Further, we have shown that $v_\lambda(\boldsymbol{\alpha})\ge 0, v_\lambda(\boldsymbol{\beta})\ge 0$. Thus, if $A(f,\nu)$ is a p-unit, we get $v_\lambda(u_\epsilon(f)/u_\epsilon(g))\ge 0$. This completes the proof of Thm. 1.2 of the introduction. ■

6.3. **Isogenies between new-quotients of Jacobians of Shimura curves.** We show now, if N is odd and square-free, that $J_0(N)^{\text{new}}$ and $Jac(X)^{new}$ are isogenous $/\mathbb{Q}$ without using Faltings' isogeny theorem. Indeed it suffices to prove the following

Theorem 6.2. *Let A_f and A_g denote the abelian variety quotients of $J_0(N)$ and $Jac(X)$ corresponding to newforms f and g that are Jacquet-Langlands transfers of each other. Then A_f and A_g are isogenous over $\mathbb{Q}$.*

Proof: Let $V_f = \oplus\mathbb{C}f^\sigma \subset S_2(\Gamma_0(N))^{\text{new}}$, $V_g = \oplus\mathbb{C}g^\sigma\subset S_2(\Gamma)^{\text{new}}$, where σ runs over the embeddings of $\mathbb{Q}(f)$ in $\mathbb{C}$. Then we have canonical identifications of V_f, V_g with the cotangent space at the identity of A_f, A_g respectively. Further, if f, g are chosen to be $\mathbb{Q}(f)$-rational, then the $\mathbb{Q}$-subspaces $V_{f,0} := \{\sum_\sigma a^\sigma f^\sigma : a\in\mathbb{Q}(f)\}$, $V_{g,0} := \{\sum_\sigma b^\sigma g^\sigma : b\in\mathbb{Q}(f)\}$

are identified with the natural $\mathbb{Q}$-structures on V_f, V_g respectively coming from the $\mathbb{Q}$-structures of A_f, A_g. Let $\xi_f : V_f^\vee \to A_f, \xi_g : V_g^\vee \to A_g$ denote the canonical exponential uniformizations and L_f, L_g the kernels of ξ_f, ξ_g respectively.

Define a $\mathbb{C}$-linear isomorphism $\varphi : V_g \to V_f$ by $\varphi(g^\sigma) = f^\sigma$. Clearly φ restricts to a $\mathbb{Q}$-linear isomorphism of $V_{f,0}$ onto $V_{g,0}$. Now consider the dual map $\varphi^\vee : V_f^\vee \to V_g^\vee$. We claim that $\varphi^\vee$ maps $L_f \otimes \mathbb{Q}$ isomorphically onto $L_g \otimes \mathbb{Q}$. To prove this note first that $H^1(X_0(N), \mathbb{C}) \simeq H^1_p(\Gamma_0(N), \mathbb{C})$ is spanned by the classes $\xi_\pm(f')$ (for varying $f' \in S_2(\Gamma_0(N))$. (Here we use the notation of Sec. 4.3, except we write $\xi_\pm$ for $\xi_\pm(f', K_{f'})$). Since $J_0(N) \twoheadrightarrow A_f$, $H^1(A_f, \mathbb{C}) \subseteq H^1(X_0(N), \mathbb{C})$. Further the $\mathbb{Q}$-subspace $H^1(A, \mathbb{Q})$ is given by

$$H^1(A_f, \mathbb{Q}) = \{\sum_\sigma (a^\sigma \xi_+(f^\sigma) + b^\sigma \xi_-(f^\sigma)) : a, b \in K(f)\}.$$

Likewise $H^1(X, \mathbb{C}) \simeq H^1_p(\Gamma, \mathbb{C})$ is spanned by the classes $\xi_\pm(g')$ for varying g' and

$$H^1(A_g, \mathbb{Q}) = \{\sum_\sigma (a^\sigma \xi_+(g^\sigma) + b^\sigma \xi_-(g^\sigma)) : a, b \in K(f)\}.$$

Now $L_f \otimes \mathbb{Q} \simeq H_1(A_f, \mathbb{Q})$, $L_g \otimes \mathbb{Q} \simeq H_1(A_g, \mathbb{Q})$. Let $\{\xi^*_\pm(f^\sigma)\}$ (resp. $\{\xi^*_\pm(g^\sigma)\}$) denote the basis of $H_1(A_f, \mathbb{Q})$ (resp. of $H_1(A_g, \mathbb{Q})$) that is dual to the basis $\{\xi_\pm(f^\sigma)\}$ (resp. $\{\xi_\pm(g^\sigma)\}$). It is easy to see that

$$\varphi^\vee(\xi^*_\pm(f^\sigma)) = \frac{u_\pm(f^\sigma)}{u_\pm(g^\sigma)} \xi^*_+(g^\sigma).$$

The rationality result Thm. 1.1 implies then that $\varphi^\vee$ carries $L_f \otimes \mathbb{Q}$ isomorphically onto $L_g \otimes \mathbb{Q}$ and hence L_f into a lattice commensurable with L_g. Thus $n\varphi^\vee$ for n a sufficiently large integer, induces an isogeny from A_f to A_g, that must be defined over some number field. Since φ restricts to a $\mathbb{Q}$-linear isomorphism of $V_{f,0} = H^0(A_{f,\mathbb{Q}}, \Omega^1)$ onto $V_{g,0} = H^0(A_{g,\mathbb{Q}}, \Omega^1)$, this isogeny must in fact be defined over $\mathbb{Q}$. ■

Appendix A. An integrality property for the Atkin-Lehner operator by Brian Conrad

Let Q and Q' be relatively prime positive integers and let $N = QQ'$. For $k \geq 1$ let $w_{Q,k}$ denote the usual Atkin-Lehner involution on the space $\mathrm{M}_k(\Gamma_0(N))$ of weight-k classical modular forms on $\Gamma_0(N)$, defined by

$$f \mapsto f|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for any $a, b, c, d \in \mathbb{Z}$ such that $N|c$, $Q|a$, $Q|d$, and $ad - bc = Q$. For $f \in \mathrm{M}_k(\Gamma_0(N))$ such that the q-expansion $f_\infty(q) \in \mathbb{C}[\![q]\!]$ at the cusp ∞ has all coefficients in a number field $K \subseteq \mathbb{C}$, it is an easy consequence of the algebraic theory of modular forms (as in [16, §1]) that the q-expansion $(w_{Q,k}(f))_\infty(q)$ also has all coefficients in K. We aim to prove a stronger integrality property:

Theorem A.1. *Fix a prime $p \nmid Q$ and a prime $\mathfrak{p}$ of K over p. If $f \in \mathrm{M}(\Gamma_0(N))$ satisfies $f_\infty(q) \in \mathscr{O}_{K,\mathfrak{p}}[\![q]\!]$ then likewise $w_{Q,k}(f)$ has $\mathfrak{p}$-integral q-expansion coefficients at ∞. More* *generally, if* $R \subseteq \mathbb{C}$ *is any* $\mathbb{Z}[1/Q]$*-subalgebra and if* f *has all* q*-expansion coefficients at* ∞ *lying in* R *then the same holds for* $w_{Q,k}(f)$*.*

To prove this theorem we wish to use an integral model for a modular curve by interpreting f as a section of a line bundle and identifying $w_{Q,k}$ as the pullback operation on its global sections induced by line bundle map covering a self-map of such an integral model. The most natural way to do this is to work with the moduli stack $\mathscr{X}_0(N)$ over $\operatorname{Spec}\mathbb{Z}$ that classifies generalized elliptic curves equipped with a $\Gamma_0(N)$-structure (i.e., ample finite locally free subgroups of the smooth locus that have order N and are cyclic in the sense of Drinfeld); working over $\operatorname{Spec}\mathbb{Z}_{(p)}$ for a prime $p \nmid Q$ is all that we really require. This stack is generally only an Artin stack (especially when working over $\mathbb{Z}_{(p)}$ with $p^2|N$, which is certainly a case of much interest). In [4] the basic theory of such stacks was systematically developed by building on the work [5] of Deligne and Rapoport over $\mathbb{Z}[1/N]$ and the work [17] of Katz and Mazur over $\mathbb{Z}$ away from the cusps, and for example it is shown there (see [4, Thm. 1.2.1]) that $\mathscr{X}_0(N)$ is a normal (even regular) Artin stack that is proper and flat over $\mathbb{Z}$ with geometrically connected fibers of pure dimension 1.

Remark A.2. For the purposes of proving Theorem A.1 it will turn out to only be necessary to work with certain open substacks of $\mathscr{X}_0(N)$ that are Deligne–Mumford stacks. In fact, by working systematically with enough auxiliary prime-to-p level structure to force stacks to be schemes it is possible to prove Theorem A.1 for normal R without leaving the category of schemes. (The role of normality is to make it harmless to check the result after adjoining roots of unity to R so that the Tate curve over $R[\![q]\!]$ admits enough auxiliary level structure.) However, it is certainly more natural to work directly with stacks, and to avoid unnecessary normality hypotheses on R it seems to be unavoidable to use stacks. For these reasons, we have decided to work directly on $\mathscr{X}_0(N)$ rather than try to avoid it.

Since R is a flat $\mathbb{Z}[1/Q]$-algebra we have $R = \cap_{p \nmid Q} R_{(p)}$ with the intersection taken inside of $R_{\mathbb{Q}} = R \otimes \mathbb{Q}$. It therefore suffices to prove Theorem A.1 for each $R_{(p)}$, so from now on we may and do assume that $R \subseteq \mathbb{C}$ is a $\mathbb{Z}_{(p)}$-subalgebra for a fixed choice of prime $p \nmid Q$. We let $\mathscr{U} \subseteq \mathscr{X}_0(N)_{\mathbb{Z}_{(p)}}$ be the open substack that has full generic fiber and (irreducible) closed fiber classifying level-structures with multiplicative p-part. The idea for proving Theorem A.1 is rather simple: identify the space of classical modular forms having p-adically integral q-expansion at ∞ with the space of $\mathscr{U}$-sections of the line bundle of weight-k modular forms over $\mathscr{X}_0(N)$, and then invoke the fact that for any line bundle on a $\mathbb{Z}_{(p)}$-flat normal Artin stack (such as $\mathscr{U}$) any section over the generic fiber extends (uniquely) to a global section if it extends over some open locus meeting every irreducible component of the closed fiber (as then it is "defined in codimension 1"). To make this idea work we use a geometric Atkin-Lehner self-map w_Q of both $\mathscr{U}$ and the universal generalized elliptic curve over $\mathscr{U}$, and the construction of this map rests on the fact that $p \nmid Q$ and $\mathscr{U}_{\mathbb{F}_p}$ classifies precisely the level structures in characteristic p with multiplicative p-part. The relevant technical problems were either solved in [4] or will be settled by adapting arguments given there.

As a first step, we shall translate our given setup into purely algebro-geometric language. The underlying set of the classical analytic modular curve $X_0(N)$ is identified with the set of isomorphism classes of objects in the category $\mathscr{X}_0(N)(\mathbb{C})$, and in this way the cusp ∞ arises from the object in $\mathscr{X}_0(N)(\operatorname{Spec}\mathbb{Z})$ given by the standard Néron 1-gon C_1 over $\operatorname{Spec}\mathbb{Z}$ equipped with the cyclic subgroup $\mu_N \subseteq \mathbf{G}_m = C_1^{\mathrm{sm}}$. This object over $\operatorname{Spec}\mathbb{Z}$ canonically lifts to a morphism $\operatorname{Spec}\mathbb{Z}[\![q]\!] \to \mathscr{X}_0(N)$ given by the Tate curve $\underline{\mathrm{Tate}}$ equipped with $\Gamma_0(N)$-structure $\mu_N \subseteq \underline{\mathrm{Tate}}^{\mathrm{sm}}[N]$. We refer the reader to [4, §2.5] for a review of the basic facts from the algebraic and formal theories of the Tate curve, including the existence and uniqueness of an isomorphism of formal $\mathbb{Z}[\![q]\!]$-group schemes $\underline{\mathrm{Tate}}_0^\wedge \simeq \widehat{\mathbf{G}}_m$ (formal completion along the identity) lifting the evident isomorphism modulo q. Since global sections of the relative dualizing sheaf of a generalized elliptic curve are canonically identified (via restriction) with invariant relative 1-forms over the smooth locus (as each of these spaces of sections is compatibly identified with the space of sections of the cotangent space along the identity section), the relative dualizing sheaf of $\underline{\mathrm{Tate}} \to \mathbb{Z}[\![q]\!]$ admits a unique trivializing section whose pullback to $\underline{\mathrm{Tate}}_0^\wedge$ goes over to the invariant 1-form $\mathrm{d}t/t$ on the formal multiplicative group; this trivializing section is also denoted $\mathrm{d}t/t$. Let us now briefly recall how the Tate curve underlies the algebraic theory of q-expansions, and the relation of this algebraic theory with the analytic theory of q-expansions. If $E \to S$ is a generalized elliptic curve then we write $\omega_{E/S}$ to denote the pushforward of its relative dualizing sheaf; this is a line bundle on S whose formation commutes with any base change on S [5, II, 1.6], so we get an invertible sheaf $\omega = \omega_{\mathscr{E}/\mathscr{X}_0(N)}$ on $\mathscr{X}_0(N)$. For any ring A we write ω_A to denote $\omega_{\mathscr{E}_A/\mathscr{X}_0(N)_A}$ (with $\mathscr{E}_A \to \mathscr{X}_0(N)_A$ denoting the scalar extension of $\mathscr{E} \to \mathscr{X}_0(N)$ by $\mathbb{Z} \to A$), so there is a canonical A-linear map

$$\mathrm{H}^0(\mathscr{X}_0(N)_A, \omega_A^{\otimes k}) \to \mathrm{H}^0(\mathrm{Spec}\, A[\![q]\!], \omega^{\otimes k}_{\underline{\mathrm{Tate}}_A/A[\![q]\!]}) = A[\![q]\!]$$

using the basis $(\mathrm{d}t/t)^{\otimes k}$ (with $\underline{\mathrm{Tate}}_A$ denoting the scalar extension on $\underline{\mathrm{Tate}}$ by $\mathbb{Z}[\![q]\!] \to A[\![q]\!]$). This map is called the *algebraic q-expansion at* ∞ over A. In the special case $A = \mathbb{C}$, descent theory and GAGA provide a canonical $\mathbb{C}$-linear isomorphism

$$\mathrm{H}^0(\mathscr{X}_0(N)_{\mathbb{C}}, \omega_{\mathbb{C}}^{\otimes k}) \simeq \mathrm{M}_k(\Gamma_0(N))$$

that identifies the analytic q-expansion at ∞ and the algebraic q-expansion at ∞ over $\mathbb{C}$; this is proved as in [5, IV, §4] (which treats $\Gamma(N)$). Since the natural map $M \otimes_B B[\![q]\!] \to M[\![q]\!]$ is injective for any module M over any noetherian ring B (such as $B = \mathbb{Z}$), the image of the q-expansion map over a ring A lies in $A \otimes_{\mathbb{Z}} \mathbb{Z}[\![q]\!]$.

By descent theory, the q-expansion principle as in [16, 1.6.2] ensures that for any $\mathbb{Z}[1/N]$-algebra $A \subseteq \mathbb{C}$ the A-submodule of classical modular forms in $\mathrm{M}_k(\Gamma_0(N))$ with q-expansion in $A[\![q]\!]$ coincides with

$$\mathrm{H}^0(\mathscr{X}_0(N)_A, \omega_A^{\otimes k}) \subseteq \mathrm{H}^0(\mathscr{X}_0(N)_{\mathbb{C}}, \omega_{\mathbb{C}}^{\otimes k}).$$

However, this fails for more general subrings of $\mathbb{C}$ in which N is not necessarily a unit because fibers of $\mathscr{X}_0(N)$ in characteristic dividing N are reducible. This is why we will need to make fuller use of the structure of $\mathscr{X}_0(N)$ near ∞ in characteristic p in order to prove Theorem A.1.

We now construct the analytic operator $w_{Q,k}$ algebraically over $\mathbb{Z}_{(p)}$ for an arbitrary prime p (allowing $p|Q$). Using primary decomposition for cyclic subgroups in the sense of Drinfeld, for any scheme S the objects in the category $\mathscr{X}_0(N)(S)$ may be described as triples $(E; C_Q, C_{Q'})$ where $E \to S$ is a generalized elliptic curve, C_Q and $C_{Q'}$ are finite locally free cyclic subgroups of the smooth locus E^{sm} whose respective orders are Q and Q', and the relative effective Cartier divisor $C_Q + C_{Q'}$ on E is S-ample. Letting $\mathscr{Y}_0(N) \subseteq \mathscr{X}_0(N)$ be the open substack classifying such triples $(E; C_Q, C_{Q'})$ for which E is an elliptic curve, we can define a morphism $w_Q^0 : \mathscr{Y}_0(N) \to \mathscr{Y}_0(N)$ by the functorial recipe

$$(E; C_Q, C_{Q'}) \rightsquigarrow (E/C_Q; E[Q]/C_Q, (C_Q + C_{Q'})/C_Q).$$

This is an involution in the sense that there is a canonical isomorphism of 1-morphisms $w_Q^0 \circ w_Q^0 \simeq \mathrm{id}_{\mathscr{Y}_0(N)}$ via the canonical isomorphism $E/E[Q] \simeq E$ induced by multiplication by Q on E. The quotient process defining w_Q^0 makes no sense over $\mathscr{X}_0(N)$ because for generalized elliptic curves there is no reasonable general theory of quotients for the action by a finite locally free subgroup scheme of the smooth locus when there are non-smooth fibers, but there is a unique way (up to unique isomorphism) to extend this construction over the open substack $\mathscr{V} \subseteq \mathscr{X}_0(N)_{\mathbb{Z}_{(p)}}$ complementary to the closed substack of cusps in characteristic p whose level structure has p-part that is neither étale nor multiplicative. (If $\mathrm{ord}_p(N) \leq 1$ then $\mathscr{V} = \mathscr{X}_0(N)_{\mathbb{Z}_{(p)}}$.) The following lemma makes this precise.

Lemma A.3. *Let $(\mathscr{E}; \mathscr{C}_Q, \mathscr{C}_{Q'}) \to \mathscr{X}_0(N)$ be the universal object, and let $(\mathscr{E}^0; \mathscr{C}_Q^0, \mathscr{C}_{Q'}^0) \to \mathscr{Y}_0(N)$ denote its restriction away from the closed substack of cusps. The open substack $\mathscr{V} \subseteq \mathscr{X}_0(N)_{\mathbb{Z}_{(p)}}$ defined as above is Deligne–Mumford and up to unique isomorphism there is a unique generalized elliptic curve $\mathscr{E}'$ over $\mathscr{V}$ equipped with a $\Gamma_0(N)$-structure restricting to*

$$(\mathscr{E}^0/\mathscr{C}_Q^0; \mathscr{E}^0[Q]/\mathscr{C}_Q^0, (\mathscr{C}_Q^0 + \mathscr{C}_{Q'}^0)/\mathscr{C}_Q^0)$$

over $\mathscr{Y}_0(N)_{\mathbb{Z}_{(p)}}$.

Proof: By [4, Thm. 3.2.7], $\mathscr{V}$ lies in an open substack of $\mathscr{X}_0(N)_{\mathbb{Z}_{(p)}}$ that is Deligne–Mumford. Thus, $\mathscr{V}$ is Deligne–Mumford. Since $\mathscr{Y}_0(N)_{\mathbb{Z}_{(p)}} \subseteq \mathscr{V}$ is the complement of a relative effective Cartier divisor (as this even holds for $\mathscr{Y}_0(N)$ viewed inside of $\mathscr{X}_0(N)$, by [4, Thm. 4.1.1(1)]), the uniqueness up to unique isomorphism follows by descent after applying the uniqueness criterion for extending generalized elliptic curves equipped with ample Drinfeld level structures in [4, Cor. 3.2.3] (applied over a smooth scheme covering $\mathscr{V}$). For existence, one argues exactly as in the deformation-theoretic arguments in [4, §4.4] where it is proved that the pth Hecke correspondence T_p on moduli stacks is defined over $\mathbb{Z}$ (including the cusps). The main points in adapting this argument to work for our problem over the Deligne–Mumford stack $\mathscr{V}$ are that (i) the property of p-torsion at cusps that makes the analysis of T_p work in [4, §4.4] is that such torsion is either multiplicative or étale on fibers (this is the main reason that we work over $\mathscr{V}$ rather than $\mathscr{X}_0(N)_{\mathbb{Z}_{(p)}}$) and (ii) if G is a multiplicative or étale cyclic subgroup of order p^n ($n \geq 1$) in an elliptic curve E over an $\mathbb{F}_p$-scheme then $E[p^n]/G$ is étale or multiplicative respectively. ■

Since the Deligne–Mumford stack $\mathscr{V}$ is normal, by [4, Lemma 4.4.5] the morphism $w^0_{Q/\mathbb{Z}_{(p)}}$ has at most one extension (up to unique isomorphism) to a morphism $w_Q : \mathscr{V} \to \mathscr{V}$, and moreover such a morphism does exist via the generalized elliptic curve with $\Gamma_0(N)$-structure over $\mathscr{V}$ provided by Lemma A.3 (the key point is that it suffices to solve the extension problem on deformation rings at geometric points, again by [4, Lemma 4.4.5]). The resulting isomorphism $w_Q^*(\mathscr{E}) \simeq \mathscr{E}'$ respecting $\Gamma_0(N)$-structures over $\mathscr{V}$ defines (by pullback) a map of line bundles $\omega_{\mathscr{E}/\mathscr{V}} \to \omega_{\mathscr{E}'/\mathscr{V}}$. Fix a $\mathbb{Z}_{(p)}$-algebra A, so passing to kth tensor powers for any $k \geq 1$ and using extension of scalars thereby defines an A-linear map

$$\mathrm{H}^0(\mathscr{V}_A, \omega_A^{\otimes k}) \to \mathrm{H}^0(\mathscr{V}_A, \omega^{\otimes k}_{\mathscr{E}'_A/\mathscr{V}_A}). \tag{A.1}$$

We want to compose this with another map to obtain an endomorphism of $\mathrm{H}^0(\mathscr{V}_A, \omega_A^{\otimes k})$, at least if $\mathbb{Z}_{(p)} \to A$ is flat.

Consider the canonical isogeny of elliptic curves $\phi_Y : \mathscr{E}^0 \to \mathscr{E}^0/\mathscr{C}_Q^0$ over $\mathscr{Y}_0(N)$. Since $\mathscr{X}_0(N)_{\mathbb{Q}}$ is a regular 1-dimensional Deligne–Mumford stack we can use descent theory and

Néron models over étale scheme covers of this stack to uniquely extend the isogeny $\phi_{Y/\mathbb{Q}}$ over $\mathscr{Y}_0(N)_{\mathbb{Q}}$ to a homomorphism ϕ_X over $\mathscr{X}_0(N)_{\mathbb{Q}} = \mathscr{V}_{\mathbb{Q}}$ from the relative identity component of $(\mathscr{E}_{\mathbb{Q}})^{\mathrm{sm}}$ to the relative identity component of $(\mathscr{E}'_{\mathbb{Q}})^{\mathrm{sm}}$. But global sections of the relative dualizing sheaf of a generalized elliptic curve are canonically identified with global sections of the relative cotangent space along the identity section, so we can use the cotangent space map induced by ϕ_X to define a (necessarily unique) map of line bundles $\omega_{\mathscr{E}'_{\mathbb{Q}}/\mathscr{V}_{\mathbb{Q}}} \to \omega_{\mathbb{Q}}$ over $\mathscr{X}_0(N)_{\mathbb{Q}}$. This can be glued to the canonical pullback map over $\mathscr{Y}_0(N)_{\mathbb{Z}_{(p)}}$ induced by ϕ_Y to define a map of line bundles from $\omega_{\mathscr{E}'/\mathscr{V}}$ to $\omega_{\mathbb{Z}_{(p)}}$ over the open substack $\mathscr{V}' \subseteq \mathscr{X}_0(N)_{\mathbb{Z}_{(p)}}$ complementary to the cusps in characteristic p. (This open complement is contained in $\mathscr{V}$.) Passing to kth tensor powers and composing with (A.1) after extending scalars to A and forming global sections defines an A-linear map

$$\mathrm{H}^0(\mathscr{V}_A, \omega_A^{\otimes k}) \to \mathrm{H}^0(\mathscr{V}'_A, \omega_A^{\otimes k}).$$

If $\mathbb{Z}_{(p)} \to A$ is flat then I claim that the target of this map coincides with the module of $\mathscr{V}_A$-sections of $\omega_A^{\otimes k}$. By the compatibility of cohomology and flat base change it suffices to treat the case $A = \mathbb{Z}_{(p)}$. Since $\mathscr{V}$ is a $\mathbb{Z}_{(p)}$-flat normal Deligne–Mumford stack and the open substack $\mathscr{V}'$ contains the entire generic fiber and is dense in the closed fiber, we get the desired equality of modules of sections.

To summarize, for any prime p and any flat $\mathbb{Z}_{(p)}$-algebra A we have defined an A-linear endomorphism of $\mathrm{H}^0(\mathscr{V}_A, \omega_A^{\otimes k})$. Moreover, if $p \nmid Q$ then since Q-isogenies of elliptic curves induce isomorphisms on p-power torsion, the exact same method works with $\mathscr{V}$ replaced by the open substack $\mathscr{U}$ whose closed fiber consists of the geometric points of $\mathscr{X}_0(N)_{\mathbb{F}_p}$ whose level structure has p-part that is multiplicative. In particular, for $p \nmid Q$ we have constructed an A-linear endomorphism

$$w_{Q,k/A} : \mathrm{H}^0(\mathscr{U}_A, \omega_A^{\otimes k}) \to \mathrm{H}^0(\mathscr{U}_A, \omega_A^{\otimes k}).$$

(Obviously via restriction this is compatible with the endomorphism that we have just constructed on sections over $\mathscr{V}_A$.) Note that as a special case of working over either $\mathscr{U}$ or $\mathscr{V}$, by setting $A = \mathbb{C}$ we have constructed a $\mathbb{C}$-linear endomorphism of $\mathrm{H}^0(\mathscr{X}_0(N)_{\mathbb{C}}, \omega_{\mathbb{C}}^{\otimes k})$. It is a straightforward matrix calculation with the standard $\Gamma_0(N)$-structure on the universal Weierstrass family over $\mathbb{C} - \mathbb{R}$ to verify that the algebraically-defined operator $w_{Q,k/\mathbb{C}}$ coincides with the analytic Atkin–Lehner involution on $\mathrm{M}_k(\Gamma_0(N))$, as follows. For $\tau \in \mathbb{C} - \mathbb{R}$ and $(E; C_Q, C'_Q) = (\mathbb{C}/L_\tau, \langle 1/Q \rangle, \langle 1/Q' \rangle)$ with $L_\tau = \mathbb{Z} \oplus \mathbb{Z}\tau$, if we pick $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ such that $Q'|c$ and $Q|d$ then multiplication by $1/(c\tau + d)$ induces an isomorphism of triples

$$(E/C_Q; E[Q]/C_Q, (C_Q + C'_Q)/C_Q) = (\mathbb{C}/L_{Q\tau}, \langle \tau \rangle, \langle 1/Q' \rangle) \simeq (\mathbb{C}/L_{\gamma(Q\tau)}, \langle 1/Q \rangle, \langle 1/Q' \rangle).$$

Hence, $w_{Q,k/\mathbb{C}}$ acting on $\mathrm{H}^0(\mathscr{X}_0(N)_{\mathbb{C}}, \omega_{\mathbb{C}}^{\otimes k}) \simeq \mathrm{M}_k(\Gamma_0(N))$ is the operator

$$f \mapsto f|_k \begin{pmatrix} aQ & b \\ cQ & d \end{pmatrix},$$

and since $N = QQ'$ divides cQ this is indeed the analytic Atkin–Lehner involution $w_{Q,k}$. Thus, to conclude the proof of Theorem A.1 it remains to prove:

Lemma A.4. *If $p \nmid Q$ and $R \subseteq \mathbb{C}$ is any $\mathbb{Z}_{(p)}$-subalgebra then the subset $\mathrm{H}^0(\mathscr{U}_R, \omega_R^{\otimes k}) \subseteq \mathrm{M}_k(\Gamma_0(N))$ is precisely the subset of modular forms whose q-expansion at ∞ lies in $R[\![q]\!]$.*

Proof: One containment is obvious by the compatibility of the algebraic and analytic theories of q-expansion at ∞. For the reverse inclusion, suppose a modular form $f \in \mathrm{M}_k(\Gamma_0(N))$ satisfies $f_\infty(q) \in R[\![q]\!] \subseteq \mathbb{C}[\![q]\!]$, so at least by the q-expansion principle over $R_{\mathbb{Q}} = R[1/p]$ we may identify f with a section of $\omega_{R_{\mathbb{Q}}}^{\otimes k}$ over $\mathscr{X}_0(N)_{R_{\mathbb{Q}}} = \mathscr{U}_{R[1/p]}$. We need to show that this section extends (necessarily uniquely) to a section of $\omega_R^{\otimes k}$ over $\mathscr{U}_R$. By chasing p-powers in the denominator, it is equivalent to show that if a section σ of $\omega_R^{\otimes k}$ over $\mathscr{U}_R$ has all q-expansion coefficients in pR then σ/p is also a section of $\omega_R^{\otimes k}$ over $\mathscr{U}_R$. A standard argument due to Katz reduces this to the case $R = \mathbb{Z}_{(p)}$, as follows. Since the q-expansion lies in the subset $R \otimes_{\mathbb{Z}} \mathbb{Z}[\![q]\!] \subseteq R[\![q]\!]$ and this inclusion induces an injection modulo p, it is equivalent to prove exactness of the complex

$$\mathrm{H}^0(\mathscr{U}, \omega_{\mathbb{Z}_{(p)}}^{\otimes k} \otimes_{\mathbb{Z}_{(p)}} R) \xrightarrow{p} \mathrm{H}^0(\mathscr{U}, \omega_{\mathbb{Z}_{(p)}}^{\otimes k} \otimes_{\mathbb{Z}_{(p)}} R) \to (R/pR) \otimes_{\mathbb{Z}} \mathbb{Z}[\![q]\!].$$

By $\mathbb{Z}_{(p)}$-flatness of R and the compatibility of quasi-coherent cohomology with flat base change, this complex is the scalar extension by $\mathbb{Z}_{(p)} \to R$ of the analogous such complex for the coefficient ring $\mathbb{Z}_{(p)}$, so indeed it suffices to treat the case $R = \mathbb{Z}_{(p)}$.

Consider the map $\operatorname{Spec} \mathbb{Z}_{(p)}[\![q]\!] \to \mathscr{X}_0(N)_{\mathbb{Z}_{(p)}}$ associated to $(\underline{\mathrm{Tate}}, \mu_N)$. This lands inside of the open substack $\mathscr{U}$ and sends the closed point to ∞ in characteristic p. I claim that the resulting morphism $\operatorname{Spec} \mathbb{Z}_{(p)}[\![q]\!] \to \mathscr{U}$ is flat. To prove this, it suffices to check flatness of the composition of φ with the faithfully flat map $\operatorname{Spec} W(\overline{\mathbb{F}}_p)[\![q]\!] \to \operatorname{Spec} \mathbb{Z}_{(p)}[\![q]\!]$. Since $\mathscr{U}$ is Deligne–Mumford there is a well-defined complete local ring at each of its geometric points (namely, the universal deformation ring of the structure corresponding to the geometric point), and $(\underline{\mathrm{Tate}}, \mu_N)$ over $W(\overline{\mathbb{F}}_p)[\![q]\!]$ is the unique algebraization of the universal deformation of $(C_1, \mu_N)_{/\overline{\mathbb{F}}_p}$ (proof: it is harmless to drop the multiplicative μ_N in this deformation-theoretic claim since $C_1^{\mathrm{sm}} = \mathbf{G}_m$, and on underlying generalized elliptic curves the claim is part of [4, Lemma 3.3.5]). Thus, $\operatorname{Spec} W(\overline{\mathbb{F}}_p)[\![q]\!] \to \mathscr{U}$ is flat, so the morphism $\varphi : \operatorname{Spec} \mathbb{Z}_{(p)}[\![q]\!] \to \mathscr{U}$ is indeed flat.

To exploit this flatness, we need one further property: the image of φ hits each irreducible component of $\mathscr{U}_{\mathbb{F}_p}$. In fact, $\mathscr{U}_{\mathbb{F}_p}$ is irreducible. Let us briefly recall the proof. Since the cuspidal substack in $\mathscr{X}_0(N)$ is a relative effective Cartier divisor over $\mathbb{Z}$, the cuspidal substack in $\mathscr{U}_{\mathbb{F}_p}$ is a Cartier divisor. Hence, it suffices to prove irreducibility of the complement of the cusps in $\mathscr{U}_{\mathbb{F}_p}$. This complement is the open substack of $\mathscr{Y}_0(N)_{\mathbb{F}_p}$ whose geometric points have level structure with multiplicative p-part, and to prove that this is irreducible it suffices to check the irreducibility of the corresponding open set in the coarse moduli space. The case $p \nmid N$ follows from the fact [4, Thms. 3.2.7, 4.2.1(1)] that the proper map $\mathscr{X}_0(N)_{\mathbb{Z}[1/N]} \to \operatorname{Spec} \mathbb{Z}[1/N]$ is smooth with fibers that are geometrically connected (and so geometrically irreducible), and if $p|N$ then the irreducible components of the coarse moduli space of $\mathscr{Y}_0(N)_{\mathbb{F}_p}$ are worked out in [17, Ch. 13] where it is proved that one of these components contains the locus with multiplicative p-part in the level structure as a dense open subset. This furnishes the desired irreducibility.

It now remains to prove a general result on extending sections of line bundles over normal Artin stacks by working generically on the closed fiber. To be precise, let $\mathscr{S}$ be a normal locally noetherian Artin stack that is flat over a discrete valuation ring R with fraction field K, and let $\varphi : S \to \mathscr{S}$ be a flat map from an algebraic space S whose image hits each irreducible component of the closed fiber of $\mathscr{S} \to \operatorname{Spec} R$. If $\mathscr{F}$ is an $\mathscr{O}_{\mathscr{S}}$-flat quasi-coherent sheaf and $\sigma_\eta \in \mathscr{F}_K(\mathscr{S}_K)$ is a section such that the pullback section $\varphi_K^*(\sigma_\eta) \in (\varphi_K^* \mathscr{F}_K)(\mathscr{S}_K)$ lies in the subset $(\varphi^* \mathscr{F})(\mathscr{S})$ then I claim that σ_η lies in the subset $\mathscr{F}(\mathscr{S}) \subseteq \mathscr{F}_K(\mathscr{S}_K)$. Using a smooth covering of $\mathscr{S}$ by an algebraic space, descent theory reduces us to the case when $\mathscr{S}$ is an algebraic space, and we then similarly reduce to the case when $\mathscr{S}$ and S are schemes. Working Zariski-locally then permits us to assume $\mathscr{S} = \operatorname{Spec} A$ and $S = \operatorname{Spec} A'$ are affine.

Letting M be the flat A-module associated to $\mathscr{F}$, we seek to prove that if $m_\eta \in M_K$ has image in $M_K \otimes_{A_K} A'_K = (M \otimes_A A')_K$ lying in $M \otimes_A A'$ then $m_\eta \in M \subseteq M_K$. By Lazard's theorem we can express M as a direct limit of finite free A-modules, so we reduce to the case $M = A$. Hence, if π is a uniformizer of R then denominator-chasing on m_η reduces us to checking that $A/\pi A \to A'/\pi A'$ is injective. Since $\operatorname{Spec} A' \to \operatorname{Spec} A$ is flat and hits every irreducible component of the special fiber of $\operatorname{Spec} A$ over $\operatorname{Spec} R$, for each generic point $\mathfrak{p}$ of this special fiber there is a point $\mathfrak{p}'$ of $\operatorname{Spec} A'$ over $\mathfrak{p}$. The local map $A_{\mathfrak{p}} \to A'_{\mathfrak{p}'}$ is flat, so it is faithfully flat. Hence, if $a \in A$ becomes divisible by π in A' then a is divisible by π in $A_{\mathfrak{p}}$. By R-flatness of A we conclude that the rational function a/π on $\operatorname{Spec} A$ is defined in codimension ≤ 1, so by normality of A we get $a/\pi \in A$ as desired. ■

Remark A.5. The reason we had to work with $\mathscr{U}$ rather than $\mathscr{V}$ in the above analysis is that we only imposed an integrality condition at one cusp, namely ∞ (and $\mathscr{U}_{\mathbb{F}_p}$ is the irreducible and connected component of $\mathscr{V}_{\mathbb{F}_p}$ passing through ∞). The need to work with $\mathscr{U}$ rather than $\mathscr{V}$ is the reason we had to require $p \nmid Q$ in Theorem A.1.

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