LIFTING GLOBAL REPRESENTATIONS WITH LOCAL PROPERTIES

BRIAN CONRAD

1. Introduction

Let k be a global field, with Galois group G_k and Weil group W_k relative to a choice of separable closure k_s/k. Let Γ be either G_k or W_k, and H a linear algebraic group over $F = \mathbf{C}$ or $\overline{\mathbf{Q}}_p$ with $p \neq \mathrm{char}(k)$. Let $\rho : \Gamma \to H(F)$ be a (continuous) representation, always understood to be ramified at only finitely many places (as is automatic for commutative H, but not for $H = \mathrm{GL}_2$ with $k = \mathbf{Q}$ and $F = \overline{\mathbf{Q}}_p$, even assuming semistability at p; see [KR, Thm. 25(b)].) Let $f : H' \to H$ be a quotient map between linear algebraic F-groups, with $Z := \ker f$ central of multiplicative type; e.g., an isogeny between connected H and H'. Consider the problem of lifting a given ρ to a representation $\rho' : \Gamma \to H'(F)$.

In the absence of local lifting obstructions, the global obstruction lies in a Tate–Shafarevich group that can be analyzed via Tate duality. But we want more, namely to preserve local properties of ρ at finitely many places and construct "optimal" (and explicit) counterexamples. For $F = \overline{\mathbf{Q}}_p$ and $\mathrm{char}(k) = 0$, by [W2] the local lifting obstruction at $v|p$ when requiring semistability at v is that $\rho|_{I_v}$ admits a Hodge–Tate lift; for finite Z this amounts to lifting 1-parameter subgroups (see Theorem 6.2 and Corollary 6.7).

The study of finite Z rests on killing obstructions using central pushouts along an inclusion of Z into a torus, so the special case $H = H' = \mathbf{G}_m$ (forcing $f(t) = t^n$ for some nonzero $n \in \mathbf{Z}$) controls the general case. Class field theory suggests that this *local-global problem* for characters (i.e., if $\chi : W_k \to F^\times$ is an nth power on W_{k_v} for all v then is χ an nth power?) is "dual" to the *classical Grunwald–Wang problem*: are the nth powers in $k^\times$ characterized by local conditions away from a fixed finite set S of places of k? The Grunwald–Wang theorem (see Appendix A) characterizes the triples (k, S, n) for which there are counterexamples to the classical problem, and describes the counterexamples explicitly.

There is a "universal formula" for counterexamples to the classical Grunwald–Wang problem, but apparently no "universal formula" for counterexamples to the local-global problem for characters, nor any direct link between counterexamples to the two problems; see Remark 2.2. However, the two problems have finite obstruction spaces that are dual to each other (thereby identifying the pairs (k, n) to consider for counterexamples to the latter problem):

Remark 1.1. For any finite set Σ of places of k and finite G_k-module M, let $Ш^i_\Sigma(k, M)$ denote the kernel $\ker(\mathrm{H}^i(k, M) \to \prod_{v \notin \Sigma} \mathrm{H}^i(k_v, M))$. (We also use this for finite commutative k-group schemes M via fppf cohomology.) By Tate's result that $\mathrm{H}^2(k, \mathbf{Q}/\mathbf{Z}) = 0$ (see [S3,

Date: December 11, 2011.

I am grateful to T. Barnet–Lamb, K. Buzzard, F. Calegari, S. Dasgupta, M. Emerton, K. Kedlaya, S.W. Shin, A. Snowden, and especially J. Tate for illuminating discussions. This work was partially supported by NSF grant DMS-0917686.

§6.5] for a proof when k is a number field; it adapts to function fields), the connecting map $\delta : \mathrm{H}^1(k, \mathbf{Q}/\mathbf{Z}) \to \mathrm{H}^2(k, \mathbf{Z}/n\mathbf{Z})$ is surjective. Thus, the cokernel of multiplication by n on the space $\mathrm{H}^1(k, \mathbf{Q}/\mathbf{Z})$ of finite-order characters is $\mathrm{H}^2(k, \mathbf{Z}/n\mathbf{Z})$. Hence, $\text{Ш}^2_\emptyset(k, \mathbf{Z}/n\mathbf{Z})$ is the space of obstructions to the local-global problem for nth roots of finite-order characters.

When $\mathrm{char}(k) \nmid n$, $\text{Ш}^2_\emptyset(k, \mathbf{Z}/n\mathbf{Z})$ is dual to $\text{Ш}^1_\emptyset(k, \mu_n)$ (see [NSW, Thm. 8.6.8], with S there taken to be the set of all places of k). The structure of $\text{Ш}^1_\emptyset(k, \mu_n)$ for any $n > 0$ is the content of the Grunwald–Wang theorem. In particular, $\text{Ш}^1_\emptyset(k, \mu_n)$ has order 1 or 2, so when $\mathrm{char}(k) \nmid n$ a finite-order counterexample to the local-global principle for nth roots of characters is unique up to nth powers if it exists and such a counterexample exists if and only if the classical Grunwald–Wang problem for the triple $(k, \emptyset, n)$ has a negative answer.

It is well-known that the element 16 in $\mathbf{Q}(\sqrt{7})$ is a counterexample to the classical Grunwald–Wang problem for 8th powers. Thus, by Remark 1.1, there must be a finite-order character on $G_{\mathbf{Q}(\sqrt{7})}$ that is locally an 8th power but not globally so. There is no apparent way to exploit 16 to find such a character, but here is an example.

Example 1.2. Let $k = \mathbf{Q}(\sqrt{\alpha})$ with $\alpha^2 = 7$. The prime $p = 113$ splits as $(15+4\alpha)(15-4\alpha)$. The character $\chi : G_k \twoheadrightarrow \langle -1 \rangle$ associated to the totally real quadratic extension $k(\sqrt{15+4\alpha})$ of k is unramified at all places away from $\mathfrak{p} = (15 + 4\alpha)$. The restriction of χ to every decomposition group G_{k_v} is an 8th power of a character $G_{k_v} \to \mu_{16}$ (this is obvious except at $\mathfrak{p}$), but χ is not the 8th power of a character on G_k. (See Example 2.1.)

Here is an example in which local properties cannot be preserved under the lifting process.

Example 1.3. Let K be a quadratic field ramified at 2, with v its unique 2-adic place. Assume K_v is $\mathbf{Q}_2(\sqrt{3})$ or $\mathbf{Q}_2(\sqrt{\pm 6})$ (i.e., -1 and ± 2 are not squares in K_v), so $(K, \{v\}, 8)$ is a counterexample to the classical Grunwald–Wang problem. There exists a character $\psi : G_K \to \mu_2$ split at v (i.e., $\psi(G_{K_v}) = 1$) such that ψ is the 8th power of a character $G_K \to \mu_{16}$ but every 8th root of ψ (as a character) is ramified at v. (See Example 3.6.)

Two experts independently asked me about the special case $H = H' = \mathbf{G}_m$, so it seems worthwhile to address the problem in general and to understand counterexamples.

Theorem 1.4. *Let $F = \mathbf{C}$ or $\overline{\mathbf{Q}}_p$ for a prime $p \neq \mathrm{char}(k)$, and let $1 \to Z \to H' \xrightarrow{f} H \to 1$ be a central extension of linear algebraic F-groups, with Z of multiplicative type. Let $n \geq 1$ be the exponent of Z/Z^0. For $\Gamma = G_k$ or W_k, let $\rho : \Gamma \to H(F)$ be a representation, and for every place v of k that is archimedean or ramified in ρ, assume that $\rho|_{\Gamma_v}$ lifts to a representation $\Gamma_v \to H'(F)$.*

(1) *If $(k, \emptyset, n)$ is not in the special case (see Definition* A.1*) then there exists a representation $\rho' : \Gamma \to H'(F)$ lifting ρ.*
(2) *Let S and T be finite disjoint sets of places of k with no archimedean places in T, and assume that ρ is unramified at S and tame at T. Assume $(k, S \cup T, n)$ is not in the special case. Then ρ' in* (1) *can be chosen to be unramified at S and tame at T.*
(3) *Assume* $\mathrm{char}(k) = 0$*, $F = \overline{\mathbf{Q}}_p$, and Z is finite. Let Σ be a finite set of places $v|p$ of k disjoint from $S \cup T$, and for each $v \in \Sigma$ let $\mathbf{P}_v$ denote one of the conditions: Hodge–Tate, deRham, semistable, or crystalline* ([F2], [F3]). *Assume that $(k, S \cup T \cup \Sigma, n)$*

is not in the special case and that $\rho|_{I_v}$ *satisfies* $\mathbf{P}_v$ *and admits a Hodge–Tate lift to* H' *for all* $v \in \Sigma$. *Then* ρ' *in* (1) *can be chosen to be unramified at* S, *tame at* T, *and to satisfy* $\mathbf{P}_v$ *at each* $v \in \Sigma$.

The main content in part (1) is the case $Z = \mathbf{G}_m$ with $F = \overline{\mathbf{Q}}_p$, for which the Grunwald–Wang theorem intervenes in the proof, but in an entirely different way than for finite Z.

Example 1.5. Let k be a number field unramified over 2. The special case in part (1) cannot occur, and likewise for part (2) if also some 2-adic place of k is not in $S \cup T$. For such (k, S, T) there is also no global obstruction in part (3) when $p \neq 2$. More generally, in part (3) if $(k, \emptyset, n)$ is not in the special case then the necessary conditions for there to be a nontrivial global obstruction can be made more restrictive; see Proposition 6.9.

Remark 1.6. It is not clear in what generality there should be a version of Theorem 1.4(3) for $\dim Z > 0$, avoiding counterexamples as in Example 6.8, even if H' is connected reductive, $\rho(\Gamma)$ is Zariski-dense, and $\mathbf{P}_v$ is "crystalline" for all $v|p$.

For $H = H' = \mathbf{G}_m$ we can go beyond Theorem 1.4(1) to characterize exactly when an nth root exists. (A refinement of Theorem 1.4(2) is given in Proposition 3.5(2) for $H = H' = \mathbf{G}_m$, with $f(t) = t^n$, where we provide necessary and sufficient conditions for the existence of an nth root satisfying the desired local conditions when $(k, S \cup T, n)$ is in the special case and some nth root exists.) The characterization requires some notation, as follows.

For a number field k, Appendix A defines an integer $s_k \geq 2$, a (possibly empty) set S_k of 2-adic places of k, and an element $\eta_{s_k} = \zeta_{2^{s_k}} + \zeta_{2^{s_k}}^{-1} \in k$. (For example, $s_{\mathbf{Q}(\sqrt{7})} = 2$, $S_{\mathbf{Q}(\sqrt{7})} = \emptyset$, and $\eta_2 = 0$.) Note that $\mathrm{ord}_v(2) \geq 2^{s_k - 2}$ for all places $v|2$ of k. Let $a_{k,n} = (2 + \eta_{s_k})^{n/2}$; this admits an nth root $a_{k,n,v}^{1/n} \in k_v^\times$ for all $v \notin S_k$ (described in Remark A.2). The choices of $\eta_{s_k} \in k$ and $a_{k,n,v}^{1/n}$ will not matter in what follows.

For a global field k, the classical Grunwald–Wang problem for $(k, \emptyset, n)$ has a negative answer if and only if $(k, \emptyset, n)$ is in the *special case* (see Definition A.1). In this case we have:

Theorem 1.7. *Choose* $n \geq 1$ *and assume* $(k, \emptyset, n)$ *is in the special case. Let* $F = \mathbf{C}$ *or* $\overline{\mathbf{Q}}_p$. *Consider a character* $\chi : \mathbf{A}_k^\times / k^\times \to F^\times$ *that is everywhere locally an* n*th power.*

(1) *The product* $P_{\chi,n} := \prod_v \chi_v(a_{k,n,v}^{1/n})$ *is equal to* 1 *if and only if* χ *admits an* n*th root, and otherwise* $P_{\chi,n} = -1$.
(2) *There exists a quadratic choice for* χ *that satisfies* $P_{\chi,n} = -1$, *and it can be chosen to be split at any desired finite set of places* v *of* k.

Remark 1.8. In Corollary 3.4 we characterize the quadratic extensions k'/k such that $G_k \twoheadrightarrow \mathrm{Gal}(k'/k) = \langle -1 \rangle$ is a counterexample to the local-global problem for nth roots of characters.

Here is an outline of the contents. In §2 we discuss explicit quadratic counterexamples to the local-global problem for characters, and we prove Theorem 1.7 in §3. In §4 we treat p-adic Hodge theory properties for characters. The proof of Theorem 1.4 is given in §5–§6; this uses Theorem 1.7 along with a result of Wintenberger (see Theorem 6.2) and some geometric local class field theory. In Appendix A we discuss the Grunwald–Wang theorem. In Appendix B we prove two results on abelian semi-simple crystalline representations, extending theorems of Tate; one of them is used in §4.

2. Explicit quadratic counterexamples

Here is an explicit quadratic character that is locally an 8th power but not globally so.

Example 2.1. Let $k = \mathbf{Q}(\alpha)$ with $\alpha^2 = 7$, and let $n = 8$ (so $a_{k,n} = 16$). The quadratic character of G_k associated to $k(\sqrt{15+4\alpha})/k$ will be our example. The field k has class number 1 and a totally positive fundamental unit $u_0 = 8 + 3\alpha$, so

$$\mathbf{A}_k^\times/k^\times = (\mathbf{R}_{>0}^\times \times \mathbf{R}^\times \times \prod_{v\nmid\infty} \mathscr{O}_v^\times)/u_0^{\mathbf{Z}}.$$

Let p be a rational prime such that $p \equiv 1 \bmod 8$ and $p \equiv 1, 2, 4 \bmod 7$ (e.g., $p = 113$, 137, 177, 193, ...), so $(\frac{2}{p}) = 1$ and $(\frac{7}{p}) = 1$. Clearly p is split in k. Let v be one of the places of k over p, $\mathfrak{p}_v$ the corresponding prime ideal of $\mathscr{O}_k$, and $r := \mathrm{ord}_2(p-1) \geq 3$. Assume that $u_0 \bmod \mathfrak{p}_v$ has multiplicative order divisible by 8. This says exactly that u_0 is not a 2^{r-2}th-power in the residue field $\kappa(v) = \mathbf{F}_p$ at v.

Since $(3+\alpha)^2 = 2u_0$, and 2 mod p is a square, the residue class $u_0 \bmod \mathfrak{p}_v$ is a square. Its order is assumed to be divisible by 8, so $p \equiv 1 \bmod 16$ (ruling out the possibility $p = 137$). It also follows that there exists a (unique) *quadratic* character χ of G_k ramified exactly at v (and nowhere else, not even the infinite places). Clearly χ is locally an 8th power away from v, and at v it is an 8th power if and only if χ kills the cyclic subgroup $\mu_8(\kappa(v)) \subseteq \kappa(v)^\times$ of order 8. This is in the subgroup generated by the square $u_0 \bmod \mathfrak{p}_v$, which in turn is killed by the quadratic χ_v, so χ kills $\mu_8(\kappa(v))$ and hence is locally an 8th power at all places.

The identity $(3+\alpha)^2 = 2u_0$ implies $(3+\alpha)^8 = 16u_0^4$, so u_0^4 is locally an 8th power in k but not globally an 8th power. Writing $u_0^4 = x_w^8$ in each completion k_w, $\chi_w(x_w)$ is independent of the choice of x_w since the local 8th power χ_w must be trivial on 8th roots of unity in $k_w^\times$. Hence, the product $\prod_w \chi_w(x_w)$ is well-defined (with all but finitely many factors equal to 1). If $\chi = \psi^8$ for an idele class character ψ of k then $\prod_w \chi_w(x_w) = \psi((x_w^8)) = \psi(u_0^4) = 1$. But by construction we have $\prod_w \chi_w(x_w) = \chi_v(x_v)$, so this forces x_v to be a square in $\kappa(v)$, and hence u_0^4 to be a 16th power in $\kappa(v) = \mathbf{F}_p$. Thus, χ is not globally an 8th power if u_0 is not a 4th power in $\kappa(v)$, since $p \equiv 1 \bmod 16$. (Note that if $p \equiv 1 \bmod 16$ and u_0 is not a 4th power in $\kappa(v)$ then necessarily $u_0 \bmod \mathfrak{p}_v$ has multiplicative order divisible by 8. Among all $p < 400,000$ with $p \equiv 17 \bmod 32$, approximately half of them have u_0 not a 4th power at one of the two places of k over p.)

The first such p is 113. Indeed, take v corresponding to the ideal $(15 + 4\alpha)$ with norm 113. Since $113 - 1 = 4 \cdot 28$ and $u_0^{28} \equiv -1 \bmod \mathfrak{p}_v$, u_0 is not a 4th power in $\kappa(v)$ and its multiplicative order is divisible by 8. The character χ corresponds to a quadratic extension $k(\sqrt{\theta})/k$ with a totally positive θ that is a unit multiple of the totally positive $15+4\alpha$. Thus, we must be able to take θ to be exactly one of $15 + 4\alpha$ or $(15 + 4\alpha)/u_0$, and the one that works is the unique one for which $k(\sqrt{\theta})/k$ is unramified at the unique 2-adic place w_0 of k. It therefore suffices to show that $15 + 4\alpha$ is a square multiple of 5 in $k_{w_0} \simeq \mathbf{Q}_2(\sqrt{-1})$, since the extension $\mathbf{Q}_2(\sqrt{5}) = \mathbf{Q}_2(\sqrt{-3})$ of $\mathbf{Q}_2$ is unramified.

It is easy to check that in $R := \mathbf{Z}_2[\alpha] = \mathbf{Z}_2[\sqrt{-1}]$ we have $(15+4\alpha)/(1-4\alpha) \equiv -1 \bmod 16$, so it is the same to say that $5(1-4\alpha)$ is a square in R. But $\pi := 1 - \alpha$ is a uniformizer of R since its norm into $\mathbf{Z}_2$ is 6, and $5(1 - 4\alpha) = -25 + 20\pi \equiv -25 \bmod 4\pi$. Thus, we just need to observe that all elements of $1 + 4(\pi)$ are squares in R (since $(1 + 2\pi x)^2 \equiv 1 + 4\pi x \bmod 8$).

Remark 2.2. Assume $(k, \emptyset, n)$ is in the special case (in the sense of Definition A.1). There is a "universal formula" for a counterexample to the classical Grunwald–Wang problem for (k, n), namely $a_{k,n} = (2+\eta_{s_k})^{n/2}$. Thus, it is natural to wonder if a "universal formula" exists for a quadratic counterexample to the local-global problem for nth roots of characters.

More specifically, for a choice of $\eta_{s_k} \in k$, consider the quadratic character χ of G_k associated to one of the quadratic extensions $K = k(\sqrt{\pm(2+\eta_{s_k})})$ of k (ramified only at 2-adic places). Is χ everywhere locally an nth power but not globally so, at least for the minimal choice $n = 2^{s_k+1}$? Note that, in contrast with K/k, the quadratic counterexample in Example 2.1 for $k = \mathbf{Q}(\sqrt{7})$ is ramified away from 2 and is unramified at all 2-adic places.

We now show that this attempted "universal quadratic counterexample" works for $k = \mathbf{Q}(\sqrt{7})$ with $n = 8$ but fails in general with $n = 2^{s_k+1}$. This explains why it is nontrivial to construct quadratic counterexamples to the local-global problem for nth roots of characters when $(k, \emptyset, n)$ is in the special case (as will be done in Proposition 3.3 in general).

Let $k = L(\sqrt{7})$ for a number field L totally split over 2 with $\mathbf{Q}(i) \not\subset L$. The number of 2-adic places v of k is $[L : \mathbf{Q}]$, and all satisfy $k_v = \mathbf{Q}_2(\sqrt{-1}) \neq \mathbf{Q}_2(\sqrt{\pm 2})$. Thus, $\mathbf{Q}(\sqrt{\pm 2}) \not\subset k$, so $s_k = 2$, $\eta_{s_k} = 0$, and $(k, \emptyset, 8)$ is in the special case. For each $v|2$, K/k induces the extension $\mathbf{Q}_2(\zeta_8)/\mathbf{Q}_2(i)$ whose associated norm group contains $\mu_8(\mathbf{Q}_2(i)) = \langle i \rangle$, so K/k is ramified at v and χ is an 8th power at v. Hence, χ is everywhere locally an 8th power.

By the criterion in Corollary 3.4 (whose first three parts encode the condition of being everywhere locally an 8th power), χ is not an 8th power if and only if the number of $v|2$ such that $\mathrm{N}_{K_v/k_v}(K_v^\times)$ contains $1 + i = a_{k,8}^{1/8} \in k_v$ is odd. But $k_v = \mathbf{Q}_2(i)$ and $K_v = \mathbf{Q}_2(\zeta_8)$ for all $v|2$, and $1 + \zeta_8$ has norm $1 - \zeta_8^2 = 1 \pm i$, so χ is *not* an 8th power if and only if the number $[L : \mathbf{Q}]$ of 2-adic places of k is odd (e.g., $L = \mathbf{Q}$). Thus, if L is real quadratic and 2 splits in L then χ is an 8th power on $G_{L(\sqrt{7})}$.

3. Duality and ramification arguments

Let k be a global field, and let F denote $\mathbf{C}$ or $\overline{\mathbf{Q}}_p$ with $p \neq \mathrm{char}(k)$. Consider the local-global problem for nth roots of $F^\times$-valued characters of W_k. For finite-order characters it is clearly immaterial whether we use $\mathbf{C}$ or $\overline{\mathbf{Q}}_p$, and in Remark 1.1 we saw that when $\mathrm{char}(k) \nmid n$ a finite-order counterexample exists if and only if $(k, \emptyset, n)$ is in the special case. Our first goal in this section is to remove the finite-order restriction and the hypothesis on $\mathrm{char}(k)$, as in Theorem 1.4(1) for $H = H' = \mathbf{G}_m$. By the following standard lemma, for this purpose as well as to prove Theorem 1.7 it suffices to treat the case of $\mathbf{C}^\times$-valued characters of W_k (so we now may and do focus on that case).

Lemma 3.1. *Every character χ from G_k or W_k to $\overline{\mathbf{Q}}_p^\times$ is a finite-order twist of an nth power.*

The same obviously holds for $\mathbf{C}^\times$-valued characters when $\mathrm{char}(k) > 0$.

Proof. It suffices to the analogous assertion for χ on any commutative profinite group C (since G_k is the quotient of W_k by its identity component when k is a number field). The image of χ lies in $K^\times$ for a finite extension $K/\mathbf{Q}_p$ by a Baire category argument. Some open subgroup U of C is carried onto a finite free $\mathbf{Z}_p$-submodule of $(1 + \mathfrak{m}_K)^n$, so $\chi|_U$ admits a continuous nth root valued in $K^\times$. It is elementary to extend this nth root to a character $C \to \mathscr{O}_{K'}^\times$ for some finite extension K'/K. ■

Every $\mathbf{C}^\times$-valued character of W_k naturally decomposes as a unitary character times a character valued in $\mathbf{R}^\times_{>0}$, so we lose nothing by restricting attention to unitary characters. Writing $\mathrm{X}(G)$ to denote the Pontryagin dual of a locally compact Hausdorff abelian group G (this is the group of unitary characters), and letting $C_k := \mathbf{A}_k^\times/k^\times \simeq W_k^{\mathrm{ab}}$, we want to determine whether or not triviality in $\mathrm{X}(C_k)/n\mathrm{X}(C_k)$ can be detected by local conditions.

Since the nth power map $C_k \to C_k$ is proper, it factors through a topological quotient map onto the closed subgroup of nth powers in C_k. Hence, by Pontryagin duality, $\mathrm{X}(C_k)/n\mathrm{X}(C_k) = \mathrm{X}(C_k[n])$. That is, a character of C_k is an nth power if and only if it is trivial on $C_k[n]$. A local version of this argument, using properness of the nth-power map on $k_v^\times$, shows that an element of $\mathrm{X}(k_v^\times)$ is an nth power if and only if it is trivial on $k_v^\times[n] = \mu_n(k_v)$.

The local-global problem for nth roots of $\mathbf{C}^\times$-valued characters of W_k concerns the triviality or not of the kernel of the composite map

$$\mathrm{X}(C_k)/n\mathrm{X}(C_k) \to \mathrm{X}(\mathbf{A}_k^\times)/n\mathrm{X}(\mathbf{A}_k^\times) \hookrightarrow \prod_v \mathrm{X}(k_v^\times)/n\mathrm{X}(k_v^\times). \tag{3.1}$$

We will call this kernel the *obstruction group* to the local-global problem for such characters. The composite map (3.1) is the same as the map

$$\mathrm{X}(C_k[n]) \to \prod_v \mathrm{X}(\mu_n(k_v)) = \mathrm{X}(\prod_v \mu_n(k_v)) = \mathrm{X}(\mathbf{A}_k^\times[n])$$

that is Pontryagin dual to the natural continuous map

$$\phi_{k,n} : \mathbf{A}_k^\times[n] = \prod_v \mu_n(k_v) \to C_k[n]$$

between compact groups. Hence, the obstruction group is Pontryagin dual to $\operatorname{coker} \phi_{k,n}$.

But [AT, Ch. X, Thm. 2] describes $\operatorname{coker} \phi_{k,n}$ explicitly. Indeed, by applying the snake lemma to the nth-power endomorphism of the canonical exact sequence

$$1 \to k^\times \to \mathbf{A}_k^\times \to C_k \to 1,$$

the cokernel of $\mathbf{A}_k^\times[n] \to C_k[n]$ is identified with the kernel of $k^\times/(k^\times)^n \to \mathbf{A}_k^\times/(\mathbf{A}_k^\times)^n$. This kernel is identified with the group $\text{Ш}^1_\emptyset(k, \mu_n)$ that we know is trivial except when $(k, \emptyset, n)$ is in the special case, and in the special case it is explicitly described of order 2. Concretely, $\mathbf{A}_k^\times[n] \to C_k[n]$ is surjective except when $(k, \emptyset, n)$ is in the special case, in which case a collection of local nth roots of $a_{k,n}$ represents the nontrivial class in the order-2 cokernel.

We have just proved by non-cohomological means that the obstruction group to the local-global problem for nth roots of $\mathbf{C}^\times$-valued characters of W_k is dual to the group $\text{Ш}^1_\emptyset(k, \mu_n)$ that is trivial except when $(k, \emptyset, n)$ is in the special case (thereby proving Theorem 1.4(1) for $H = H' = \mathbf{G}_m$), and that in the special case the obstruction group has order 2 and its elements are characterized by the product criterion in Theorem 1.7(1).

Remark 3.2. The product criterion in Theorem 1.7(1) is exactly what arose in Example 2.1 (with $n = 8$), except that we replaced $a_{k,n}$ with $(3+\sqrt{7})^8/a_{k,n} = u_0^4$, a harmless modification. In Proposition 3.5 we will address the issue of controlling local properties of nth roots of characters when such an nth root exists.

Let's make more explicit how the preceding proof of Theorem 1.7 yields a counterexample to the local-global problem for nth roots of $\mathbf{C}^\times$-valued characters of W_k whenever $(k, \emptyset, n)$ is in the special case. Consider the unique nontrivial quadratic character ψ_0 of $C_k[n]$ that kills the (open) index-2 image of $\mathbf{A}_k^\times[n]$. Since $C_k[n]$ is closed in C_k, by Pontryagin duality ψ_0 extends to a unitary character ψ of C_k, and this ψ is well-defined up to multiplication by nth powers of unitary characters of C_k.

By construction such a character ψ has pullback to $\mathbf{A}_k^\times$ that is trivial on $\mathbf{A}_k^\times[n]$, so on $\mathbf{A}_k^\times$ this character is an nth power. This says exactly that ψ viewed as a character of W_k is everywhere locally an nth power, but it cannot be so globally since as a character on C_k it is not trivial on $C_k[n]$. Such ψ are precisely all of the counterexamples to the Grunwald-Wang problem for nth roots of unitary characters of W_k when $(k, \emptyset, n)$ is in the special case.

A finer result along these lines (suggested by Tate) gives an optimal result on the existence and properties of exceptional finite-order counterexamples, in the spirit of the classical Grunwald–Wang theorem, thereby completing the proof of Theorem 1.7:

Proposition 3.3. *Assume $(k, \emptyset, n)$ is in the special case (in the sense of Definition* A.1*), so k is a number field, $S_k = \emptyset$, and $2^{s_k+1}|n$ with $s_k \geq 2$. There exists a quadratic character ψ on G_k that is locally an nth power but not globally an nth power, and it is unique modulo nth powers of characters $G_k \to \mu_{2n}$.*

Moreover, given any finite set S of places of k, there exists such a ψ that is split at all $v \in S$ (i.e., $\psi|_{G_{k_v}} = 1$ for all $v \in S$).

For χ as in Theorem 1.7 and any ψ as above, exactly one of χ or its quadratic twist $\psi\chi$ is an nth power on W_k; e.g., apply the product criterion in Theorem 1.4(1).

Proof. The preceding Pontryagin duality calculations show that the existence problem is exactly to extend the nontrivial quadratic character ψ_0 on $C_k[n]$ to a quadratic character on C_k. The following construction of such an extension is due to Tate. Since the norm-1 idele class group is compact, $C_k[n]$ is compact and the mth-power map on C_k is proper for all $m \geq 1$. In particular, C_k^2 is closed in C_k. It suffices to prove that

$$C_k^2 \cap C_k[n] \subseteq \ker \psi_0 = k^\times \mathbf{A}_k^\times[n]/k^\times$$

inside C_k. Indeed, granting this we have that ψ_0 factors through $C_k[n]/(C_k^2 \cap C_k[n])$, yet the natural continuous bijective homomorphism

$$C_k[n]/(C_k^2 \cap C_k[n]) \to C_k^2 C_k[n]/C_k^2$$

is a topological isomorphism due to the compactness of $C_k[n]$ and closedness of C_k^2 in C_k. Hence, the nontrivial ψ_0 can be uniquely viewed as a character of the closed subgroup $C_k^2 C_k[n]/C_k^2$ of C_k/C_k^2, so it extends to a character of C_k/C_k^2, which in turn is exactly a quadratic character of C_k of the desired type.

Since $C_k^2 \cap C_k[n] = C_k[2n]^2$ (as for any abelian group in place of C_k), we have to prove that for any $c \in C_k[2n]$, the idele class c^2 is represented by an element of $\mathbf{A}_k^\times[n]$. Pick any idele $a \in \mathbf{A}_k^\times$ representing c, so $a^{2n} = \alpha$ for some $\alpha \in k^\times$. In other words, α is everywhere locally a $2n$th power in k. Since $(k, \emptyset, n)$ is in the special case by hypothesis, so is $(k, \emptyset, dn)$ for any integer $d \geq 1$. Taking $d = 2$, the Grunwald–Wang theorem for $(k, \emptyset, 2n)$ implies that either α is a $2n$th-power in k or α is a ${k^{\times}}^{2n}$-multiple of $a_{k,2n} = (2+\eta_{s_k})^n \in {k^{\times}}^n$, so either way $\alpha = \beta^n$ for some $\beta \in k^{\times}$. This gives

$$\beta^n = \alpha = a^{2n} = (a^2)^n,$$

so a^2/β is a representative of c^2 that lies in $\mathbf{A}_k^{\times}[n]$.

Now consider the task of constructing ψ that is split at a specified finite set S of places of k. Since $(k, \emptyset, n)$ is in the special case, S_k is empty. Fix a quadratic character $\theta : G_k \to \mu_2$ that is locally an nth power but not globally an nth power. We seek a character $\xi : G_k \to \mu_{2n}$ such that the quadratic character $\theta\xi^{-n}$ is trivial on the decomposition group at each $v \in S$. By hypothesis each $\theta|_{G_{k_v}}$ admits an nth root ξ_v valued in μ_{2n}. By Proposition A.3, there is a global character $\xi : G_k \to \mu_{2n}$ extending the ξ_v since S_k is empty. ■

What are the exceptional quadratic characters in Proposition 3.3? Given one such ψ, the set of all such characters is $\{\psi\chi^n\}$ for characters $\chi : G_k \to \mu_{2n}$. These correspond to certain quadratic extensions of k (one example of which was computed explicitly for $k = \mathbf{Q}(\sqrt{7})$ and $n = 8$ in Example 2.1). Here is a characterization of such extensions in terms of local behavior at real and ramified places, together with a global parity condition on the set of ramified non-archimedean places.

Corollary 3.4. *Assume $(k, \emptyset, n)$ is in the special case (so k is a number field), and let $t = \mathrm{ord}_2(n) > s_k$ (so $t \geq 3$). Fix a choice of $\eta_{s_k} \in k$ and let $a_{k,n} = (2+\eta_{s_k})^{n/2}$. For any place $v \nmid \infty$ of k, let q_v denote the size of the residue field $\kappa(v)$ at v and let $a_{k,n,v}^{1/n}$ denote an nth root of $a_{k,n}$ in k_v.*

A quadratic character $\psi : G_k \twoheadrightarrow \mathrm{Gal}(k'/k) = \mu_2$ is locally an nth power but not globally an nth power if and only if the following conditions all hold:

1. *k'/k is split at all real places of k;*
2. *for all places $v \nmid 2\infty$ of k such that k'/k is ramified at v, $q_v \equiv 1 \bmod 2^{t+1}$;*
3. *for all places $v|2$ of k such that k'/k is ramified at v and k_v does not contain a 2^{t+1}th root of unity, the 2-power roots of unity in k_v are norms from k'_v;*
4. *the parity of the number of ramified places $v \nmid 2\infty$ of k for which $a_{k,n}^{(q_v-1)/2^{t+1}} \bmod v$ has even order in $\kappa(v)^{\times}$ is opposite to the parity of the number of ramified places $v|2$ of k for which the element $a_{k,n,v}^{1/n} \in k_v$ is a norm from k'_v.*

Before proving the corollary, we make some observations. Conditions (1), (2), and (3) amount to local constraints on the structure of k'/k at all ramified places, whereas (4) is a "global" constraint. Remark A.2 describes an explicit nth root of $a_{k,n}$ in each k_v (since $(k, \emptyset, n)$ is in the special case), and the proof below shows that in (4) the choice among nth roots of $a_{k,n}$ in k_v for $v|2$ does not matter (given that (3) holds). An example of this corollary is provided by Example 2.1 with $n = 8$ and $t = 3$, for which there is no 2-adic ramification and exactly one non-archimedean ramified place v of $k = \mathbf{Q}(\sqrt{7})$, with $\#\kappa(v) - 1 = p - 1$ divisible by $16 = 2^{t+1}$.

Proof. The quadratic character ψ is an nth power if and only if it is a 2^tth power, so we may assume $n = 2^t$. Since n is even, (1) says exactly that ψ is an nth power at the archimedean places. Now consider the local nth power condition at a non-archimedean place v, so we may assume v is ramified in k'/k. Suppose $v \nmid 2$. Then $\psi|_{I_v}$ factors through the unique surjective character $\xi_v : \kappa(v)^\times \twoheadrightarrow \mu_2$, so ψ is a 2^tth power locally at v precisely when there exists a surjective character $\kappa(v)^\times \twoheadrightarrow \mu_{2^{t+1}}$. This is condition (2), which implies that $\zeta_{2^{s_k+2}} \in k_v^\times$ (since $t \geq s_k + 1$) and hence $a_{k,n,v}^{1/n}$ can be chosen to be either square root of $2 + \eta_{s_k}$. Note also that the property of $\sqrt{2+\eta_{s_k}}$ being a square in $\kappa(v)$ is independent of the choice of square root since -1 is a square in $\kappa(v)$, and this property is equivalent to the element $a_{k,n}^{(q_v-1)/2^{t+1}} \in \kappa(v)^\times$ having even order since $a_{k,n}^{(q_v-1)/2^{t+1}}$ is an odd power of $(\sqrt{2+\eta_{s_k}})^{(q_v-1)/2}$.

Now suppose $v|2$. If k_v contains $\mu_{2^{t+1}}$ then μ_{2^t} consists of squares in k_v and hence is killed by $\psi|_{I_v}$, so condition (3) is exactly the local nth power condition on ψ at the ramified 2-adic places. Thus, the combined conditions (1), (2), and (3) say exactly that ψ is everywhere locally an nth power. In view of our preceding considerations with (2), if the conditions (1), (2), and (3) all hold then (4) is exactly the product criterion in Theorem 1.7(1) (since $a_{k,n}$ is totally positive at real places and is a local unit at all non-archimedean places away 2). ■

We can control ramification of nth roots when they exist:

Proposition 3.5. *Let k be a global field, $n \geq 1$ an integer, and $F = \mathbf{C}$ or $\overline{\mathbf{Q}}_p$ with $p \neq \mathrm{char}(k)$. Let $\chi : W_k \to F^\times$ be a character that admits an nth root, and let S, Σ, and T be disjoint finite sets of places of k such that at each $v \in S$ the character χ is split (i.e., $\chi(G_{k_v}) = 1$) and T consists of non-archimedean places at which χ is at worst tamely ramified.*

1. *There is a (possibly trivial) character $\psi : G_k \to \mu_2$ and an nth root χ' of $\psi\chi$ such that ψ is split at each $v \in \Sigma$, ψ and χ' are split at each $v \in S$, ψ and χ' are tame at each $v \in T$, and ψ and χ' are unramified at each $v \in T$ where χ is unramified.*
2. *The character $\psi = 1$ does not work in (1) precisely when the following all hold: $(k, S \cup T, n)$ is in the special case (so k is a number field and $S_k \subset S \cup T$), $S_k \cap T \neq \emptyset$, $2^{s_k-1} | \mathrm{ord}_v(2)$ for all $v \in S_k \cap T$, and*

$$\prod_{v \notin S_k} \chi_v(a_{k,n,v}^{1/n}) \cdot \prod_{v \in S_k \cap T} \chi_v(x_v) \neq 1, \tag{3.2}$$

where $x_v^2 = (2 + \eta_{s_k}) \bmod (1 + \mathfrak{m}_v)$ for $v \in S_k \cap T$ and $a_{k,n,v}^{1/n} \in k_v^\times$ is an nth root of $a_{k,n}$ for $v \notin S_k$. In such exceptional cases, (3.2) equals -1.

The product in (3.2) makes sense because (i) $\chi_v(\mu_n(k_v)) = 1$ for all v and (ii) if $v \in S_k \cap T$ then the existence and uniqueness of x_v modulo 1-units follows from the evenness of $\mathrm{ord}_v(2 + \eta_{s_k}) = \mathrm{ord}_v(2)/2^{s_k-2}$ (since $v|2$). Note also that the obstruction to using $\psi = 1$ requires the presence of at least one "bad" 2-adic place in T (i.e. $S_k \bigcap T \neq \emptyset$).

Proof. Fix an initial choice of nth root $\chi' : W_k \to F^\times$ of χ. For each $v \in S$ the restriction $\chi'|_{W_{k_v}}$ is valued in μ_n. Likewise, if $v \in T$ then χ' is μ_n-valued on the wild inertia subgroup P_v in W_{k_v}, and χ' is μ_n-valued on the inertia subgroup I_v when χ is unramified at v.

Since $k_v^\times = \mathbf{Z} \times \kappa(v)^\times \times (1 + \mathfrak{m}_v)$, we may define a character $\xi_v : W_{k_v} \to \mu_n$ as follows. If $v \in S$ then $\xi_v = \chi'|_{W_{k_v}}$. If $v \in \Sigma$ then $\xi_v = 1$. If $v \in T$ and χ is unramified at v then choose ξ_v corresponding to a character $k_v^\times \to \mu_n$ whose restriction to $\mathscr{O}_{k_v}^\times$ corresponds to $\chi'|_{I_v}$. Finally, if $v \in T$ and $\chi(I_v) \neq 1$ then choose ξ_v corresponding to a character $k_v^\times \to \mu_n$ whose restriction to the group $1 + \mathfrak{m}_v$ of 1-units corresponds to $\chi'|_{P_v}$.

First assume $(k, S \cup \Sigma \cup T, n)$ is not in the special case. By Proposition A.3 there is a character $\xi : W_k \to \mu_n$ whose W_{k_v}-restriction is ξ_v for each $v \in S \cup \Sigma \cup T$, so $\chi'\xi^{-1}$ is an nth root of χ. By construction, $\chi'\xi^{-1}$ is split at every $v \in S$. Likewise, for each $v \in T$ the character $\chi'\xi^{-1}$ is tame at v, and even unramified when χ is. We have succeeded with $\psi = 1$.

Now suppose instead that $(k, S \cup \Sigma \cup T, n)$ is in the special case, so k is a number field and $S \cup \Sigma \cup T$ contains S_k. By Proposition A.3 we can construct ξ as above if and only if $\prod_{v \in S_k} \xi_v(a_{k,n}) = 1$. Suppose otherwise, so S_k is *non-empty* and $\prod_{v \in S_k} \xi_v(a_{k,n}) = -1$. In this case there exists ξ valued in μ_{2n} (and not μ_n), so $\psi := \xi^n$ is a quadratic character such that $\chi'\xi^{-1}$ is an nth root of $\chi\psi$. Moreover, by construction ψ is split at every $v \in \Sigma$, and the characters ψ and $\chi'\xi^{-1}$ are split at every $v \in S$, tame at every $v \in T$, and unramified at those $v \in T$ where χ is unramified. This completes the proof of (1).

To succeed with $\psi = 1$, we may take $\Sigma = \emptyset$ without loss of generality and may suppose $(k, S \cup T, n)$ is in the special case. It is necessary and sufficient to find characters $\theta_v : G_{k_v} \to \mu_n$ for $v \in S \cup T$ such that: $\prod_{v \in S_k} \theta_v(a_{k,n}) = -1$, $\theta_v = 1$ for all $v \in S$, θ_v is tame for all $v \in T$, and θ_v is unramified at all $v \in T$ where χ is unramified. Indeed, this is exactly the same as saying that replacing ξ_v with $\theta_v\xi_v$ fixes the problem.

By definition $a_{k,n} = (2 + \eta_{s_k})^{n/2}$, so $\theta_v(a_{k,n}) = \theta_v^{n/2}(2 + \eta_{s_k}) = \pm 1$. Since any $v \in S_k$ is 2-adic, for such v a tamely ramified quadratic character on G_{k_v} must be unramified. Thus, failure to succeed with $\psi = 1$ means exactly that for some $v \in S_k \cap T$ the unramified quadratic character of G_{k_v} is nontrivial on $2 + \eta_{s_k}$. This says that there exists $v \in S_k \cap T$ such that $\mathrm{ord}_v(2 + \eta_{s_k})$ is odd. The extension $\mathbf{Q}(\eta_{s_k})/\mathbf{Q}$ of degree $2^{s_k - 2}$ is totally ramified at 2 with uniformizer $2 + \eta_{s_k}$. Thus, for each $v \in S_k$ the subfield $\mathbf{Q}_2(\eta_{s_k}) \subseteq k_v$ is totally ramified over $\mathbf{Q}_2$ with degree 2^{s_k-2} and uniformizer $2 + \eta_{s_k}$, so if $e(v) := \mathrm{ord}_v(2)$ then $\mathrm{ord}_v(2 + \eta_{s_k}) = e(v)/2^{s_k-2}$. This is even precisely when $2^{s_k-1} | e(v)$. In such cases, we express the product $\prod_{v \in S_k} \xi_v(a_{k,n})$ more directly in terms of χ, as follows.

Consider $v \in S_k \cap T$, so $\xi_v = \chi'_v\eta_v$ where η_v is tame and ξ_v is valued in μ_n. Since $2 + \eta_{s_k} = x_v^2 u_v$ for some $u_v \in 1 + \mathfrak{m}_v$ (due to v being 2-adic and $\mathrm{ord}_v(2 + \eta_{s_k})$ being even), $\xi_v(a_{k,n}) = \chi'_v(a_{k,n})\eta_v(a_{k,n})$ with $\eta_v(a_{k,n}) = \eta_v(2 + \eta_{s_k})^{n/2} = \eta_v(x_v)^n = \chi_v(x_v)^{-1}$. Thus,

$$\prod_{v \in S_k} \xi_v(a_{k,n}) = \prod_{v \in S_k} \chi'_v(a_{k,n}) \cdot \prod_{v \in S_k \cap T} \chi_v(x_v)^{-1} = \prod_{v \notin S_k} \chi'_v(a_{k,n})^{-1} \cdot \prod_{v \in S_k \cap T} \chi_v(x_v)^{-1}$$

since χ' is an idele class character. But for $v \notin S_k$ there is an nth root $a_{k,n,v}^{1/n} \in k_v^\times$, so $\chi'_v(a_{k,n}) = \chi_v(a_{k,n,v}^{1/n})$ for such v. ∎

Example 3.6. Let k be a quadratic field ramified at 2, and v its unique 2-adic place. Assume that -1 and ± 2 are not squares in k_v, so $S_k = \{v\}$. (Explicitly, k_v is either $\mathbf{Q}_2(\sqrt{3})$, $\mathbf{Q}_2(\sqrt{6})$, or $\mathbf{Q}_2(\sqrt{-6})$.) We claim that there exists a character $\chi : G_k \to \mu_2$ that is an 8th power and *split* at v yet for which every 8th root of χ is ramified at v.

We have $s_k = 2$, so $\eta_{s_k} = 0$. Let $n = 8$, so $(k, S_k, n) = (k, \{v\}, n)$ is in the special case. Consider characters $\chi' : G_k \to \mu_{16}$ carrying I_v into μ_8, so the 8th power $\chi := \chi'^8$ is quadratic and unramified at v. (The quadratic twist of any such χ by itself is trivial, so Proposition 3.5(1) obviously holds for such χ.) We will prove the existence of such χ' for which every quadratic lift $\phi : k_v^\times \to \mu_2$ of $\chi'^4 : I_v \to \mu_2$ is non-trivial at the element $2 \in k_v^\times$ with even order, so then Proposition 3.5(2) with $S = \Sigma = \emptyset$ and $T = \{v\}$ implies that such $\chi := {\chi'}^8$ admits no 8th root on G_k that is unramified at v. We will arrange χ' to carry $k_v^\times$ into μ_8, so χ is even split at v.

By Proposition A.3, any character $\chi'_v : k_v^\times \to \mu_8$ extends to a character $G_k \to \mu_{16}$. Since $k_v^\times = \pi_v^{\mathbf{Z}} \times \mathscr{O}_{k_v}^\times$ for a uniformizer π_v at v (e.g., π_v may be taken to be $\sqrt{\pm 6}$ or $1 + \sqrt{3}$ when $k_v = \mathbf{Q}_2(\sqrt{\pm 6})$ or $k_v = \mathbf{Q}_2(\sqrt{3})$ respectively), it suffices to construct a character $\chi'_v : k_v^\times \to \mu_8$ such that every $\phi : k_v^\times \to \mu_2$ extending ${\chi'_v}^4 : \mathscr{O}_{k_v}^\times \to \mu_2$ is nontrivial at 2. Writing $2 = \pi_v^2 u_v$ for $u_v \in \mathscr{O}_{k_v}^\times$, we must have $\phi(2) = \phi(\pi_v)^2 \phi(u_v) = {\chi'_v}^4(u_v)$, so we just have to construct a character $\mathscr{O}_{k_v}^\times \to \mu_8$ that is nontrivial at u_v^4. This amounts to the condition that u_v^4 has nontrivial image in $\mathscr{O}_{k_v}^\times / (\mathscr{O}_{k_v}^\times)^8$. Since -1 is not a square in k_v by hypothesis, it is equivalent to prove that $\pm u_v$ are non-squares in k_v, or in other words that ± 2 are non-squares in k_v. This was one of our hypotheses on k_v.

4. The p-adic case for number fields

For a finite-dimensional continuous $\mathbf{Q}_p$-linear representation V of the Galois group of a p-adic local field, a *basic p-adic Hodge theory property* for V means any of the conditions: Hodge–Tate, deRham, semistable, or crystalline. This makes sense for representations on finite-dimensional $\overline{\mathbf{Q}}_p$-vector spaces by descent to a field of definition K of finite degree over $\mathbf{Q}_p$ and consideration of the resulting underlying finite-dimensional $\mathbf{Q}_p$-linear representation space. (The choices of K and descent to a K-linear representation do not matter).

Let k be a number field and $F = \overline{\mathbf{Q}}_p$. Choose a character $\chi : G_k \to F^\times$ and a set Σ_0 of p-adic places of k. Assume that for each $v \in \Sigma_0$, the restriction χ_v of χ at v satisfies a basic p-adic Hodge theory property $\mathbf{P}_v$, and assume that χ is an nth power. Can we find an nth root of χ as in Proposition 3.5 that also satisfies $\mathbf{P}_v$ at each $v \in \Sigma_0$? The main properties of interest are the Hodge–Tate and crystalline properties, due to the well-known:

Lemma 4.1. *Let L be a finite extension of $\mathbf{Q}_p$. An abelian semi-simple linear representation $\rho : G_L \to \mathrm{GL}(V)$ on a finite-dimensional $\mathbf{Q}_p$-vector space is Hodge–Tate if and only if it is deRham, and is semistable if and only if it is crystalline.*

See [W2, Prop. 1.5.2] for the equivalence of the semistable and crystalline conditions in the abelian case without a semisimplicity hypothesis.

Proof. The problem is to prove that if χ is Hodge–Tate (resp. semistable) then it is deRham (resp. crystalline). Since ρ is semisimple as a $\mathbf{Q}_p$-linear representation space, we may assume it is irreducible over $\mathbf{Q}_p$. Thus, the image of $\mathbf{Q}_p[G_L^{\mathrm{ab}}]$ in $\mathrm{GL}(V)$ is a commutative field K of finite degree over $\mathbf{Q}_p$, so V is equipped with a structure of K-vector space (over its $\mathbf{Q}_p$-linear structure) such that ρ acts through a continuous character $\chi : G_L \to K^\times$. By irreducibility, $\dim_K V = 1$. Thus, we can suppose $V = K$ and G_L acts K-linearly. By a theorem of Tate (see [S6, III, A.7]), χ is Hodge–Tate if and only if it is locally algebraic (in the sense of Definition B.1).

Let $r_L : L^\times \to G_L^{\mathrm{ab}}$ denote the local Artin map (under either choice of normalization), and for a finite extension E of $\mathbf{Q}_p$ let $\underline{E}^\times$ denote the Weil restriction torus $\mathrm{R}_{E/\mathbf{Q}_p}(\mathbf{G}_m)$. A refinement of the theory of local algebraicity (Proposition B.4) gives that the composite map

$$\mathscr{O}_L^\times \stackrel{r_L}{\to} G_L^{\mathrm{ab}} \stackrel{\chi}{\to} \mathscr{O}_K^\times$$

is induced by the map on $\mathbf{Q}_p$-points arising from a homomorphism of $\mathbf{Q}_p$-tori $\underline{L}^\times \to \underline{K}^\times$ if and only if χ is crystalline.

This shows that if χ is Hodge–Tate then it is potentially crystalline, and hence deRham. Likewise, if χ is semistable (hence Hodge–Tate) then the preceding shows that it is potentially crystalline and hence has vanishing monodromy operator, so it is crystalline. ■

For the existence of an nth root that is Hodge–Tate, clearly it is necessary that the given character be Hodge–Tate with all weights divisible by n (when using a descent to a character valued in a finite extension $K/\mathbf{Q}_p$; the choice of K does not matter). This is also sufficient:

Proposition 4.2. *Consider the setup in Proposition* 3.5 *with k a number field and $F = \overline{\mathbf{Q}}_p$. Let $\Sigma_{\mathrm{HT}} \subseteq \Sigma$ be a set of p-adic places in Σ at which χ is Hodge–Tate with weights divisible by n, and let $\Sigma_{\mathrm{cr}} \subseteq \Sigma_{\mathrm{HT}}$ be a subset of places at which χ is crystalline. Drop the requirement in part* (1) *that ψ is split at $v \in \Sigma_{\mathrm{cr}}$.*

Any χ' as in part (1) *is Hodge–Tate at every $v \in \Sigma_{\mathrm{HT}}$, and it can be arranged in part* (1) *that χ' is crystalline at every $v \in \Sigma_{\mathrm{cr}}$. Likewise, part* (2) *carries over with the additional requirement for χ' to be crystalline at the places $v \in \Sigma_{\mathrm{cr}}$, except that in the characterization of the exceptional cases the conditions are that $(k, S \cup T \cup \Sigma_{\mathrm{cr}}, n)$ is in the special case, $2^{s_k-1} | \mathrm{ord}_v(2)$ for $v \in S_k \cap (T \cup \Sigma_{\mathrm{cr}})$, and*

$$\prod_{v \notin S_k} \chi_v(a_{k,n,v}^{1/n}) \cdot \prod_{v \in S_k \cap T} \chi_v(x_v) \cdot \prod_{v \in S_k \cap \Sigma_{\mathrm{cr}}} \eta_v(a_{k,n}) \neq 1 \tag{4.1}$$

for a crystalline nth root η_v of χ_v for $v \in S_k \cap \Sigma_{\mathrm{cr}}$. In such cases, (4.1) *equals* -1.

We will see below that the crystalline nth root η_v exists for $v \in S_k \cap \Sigma_{\mathrm{cr}}$, and that the choice of such nth root does not matter. Note that $S_k \cap \Sigma_{\mathrm{cr}} = \emptyset$ when $p \neq 2$.

Proof. The Hodge–Tate aspect is a special case of the following general local assertion. Let L and K be finite extensions of $\mathbf{Q}_p$, $d = [K : \mathbf{Q}_p]$, and $\xi : G_L \to K^\times$ a character such that the d-dimensional $\mathbf{Q}_p$-linear representation space underlying ξ^n is Hodge–Tate with all Hodge–Tate weights $\{w_1, \dots, w_d\}$ divisible by n. Then the underlying $\mathbf{Q}_p$-linear representation space of ξ is Hodge–Tate with weights $\{w_j/n\}$. Indeed, the Sen operator for ξ is semisimple (using the canonical decomposition of $K \otimes_{\mathbf{Q}_p} \mathbf{C}_L$ into $\mathbf{C}_L$-lines), and the Tate–Sen weights are $\{w_j/n\}$, which lie in $\mathbf{Z}$.

We next check that after increasing K to contain a primitive nth root of unity, if ξ^n is also crystalline then it admits a crystalline nth root valued in $K^\times$. In other words, we seek $\theta : G_L \to \mu_n$ such that $\theta\xi$ is crystalline. Note that if such a θ exists, it is unique up to an unramified twist since any two choices have ratio that is crystalline of finite order and hence unramified. Pick a finite extension L'/L such that $\xi|_{G_{L'}}$ is crystalline. By Proposition B.4, there is a (unique) map of $\mathbf{Q}_p$-tori $\xi' : \underline{L'}^\times \to \underline{K}^\times$ such that the resulting map $L'^\times \to K^\times$ on $\mathbf{Q}_p$-points has restriction to $\mathscr{O}_{L'}^\times$ that agrees with $\xi|_{G_{L'}}$ under local class field theory. Our task is to construct a (necessarily unique) factorization of ξ' as a map of $\mathbf{Q}_p$-tori

$$\underline{L'}^{\times} \overset{\mathrm{N}_{L'/L}}{\twoheadrightarrow} \underline{L}^{\times} \overset{h}{\to} \underline{K}^{\times}$$

and a character $\theta : G_L^{\mathrm{ab}} \to \mu_n$ such that the composite map

$$\mathscr{O}_L^{\times} \overset{r_L}{\to} G_L^{\mathrm{ab}} \overset{\theta\xi}{\to} K^{\times}$$

coincides with h on $\mathbf{Q}_p$-points (restricted to $\mathscr{O}_L^{\times} \subset L^{\times}$).

The crystalline hypothesis on ξ^n implies (by Proposition B.4) that ξ'^n factors through the map $\mathrm{N}_{L'/L}$ of $\mathbf{Q}_p$-tori via a map which recovers $\xi^n|_{I_L}$. Hence, ξ' carries the $\mathbf{Q}_p$-group $\ker \mathrm{N}_{L'/L}$ into the finite n-torsion $\mathbf{Q}_p$-subgroup of $\underline{K}^{\times}$. But $\ker \mathrm{N}_{L'/L}$ is a torus, hence connected, so ξ' kills it. In other words, we do get a factorization $\xi' = h \circ \mathrm{N}_{L'/L}$ as maps of $\mathbf{Q}_p$-tori. But it is obvious that $(\xi \circ r_L)^{-1} \cdot h : \mathscr{O}_L^{\times} \to K^{\times}$ is killed by the nth power (since $\xi^n \circ \mathrm{N}_{L'/L} = \xi'^n$ as algebraic maps, and $\xi'^n = h^n \circ \mathrm{N}_{L'/L}$ as algebraic maps), so this is valued in μ_n. Extending it to a μ_n-valued character of $L^{\times}$ provides an associated homomorphism $\theta : G_L \to \mu_n$ that does the job.

Returning to the global setting, we modify the proof of Proposition 3.5 as follows. For each $v \in \Sigma_{\mathrm{cr}}$, we only require that $\xi_v : G_{k_v} \to \mu_n$ makes the nth root $\chi'_v \xi_v^{-1}$ of χ_v a crystalline character (rather than that $\xi_v = 1$). This makes such ξ_v unique up to an unramified μ_n-valued twist. The proof of (1) now carries over without requiring ψ to split at Σ_{cr}.

Let $e(v) = \mathrm{ord}_v(2)$ for $v|2$. If $v \in S_k \cap \Sigma_{\mathrm{cr}}$, the value $\xi_v^{n/2}(2 + \eta_{s_k})$ must be 1 when $\mathrm{ord}_v(2 + \eta_{s_k})$ is even (equivalently, when $2^{s_k - 1}|e(v)$) and otherwise it can be arranged to be -1 by applying a degree-n unramified twist to ξ_v. Thus, if $2^{s_k-1} \nmid e(v_0)$ for some $v_0 \in S_k \cap \Sigma_{\mathrm{cr}}$ then we can change the sign of $\xi_{v_0}(a_{k,n})$ if necessary to kill the global obstruction to taking $\psi = 1$. Hence, in the revised version of part (2) incorporating conditions at Σ_{cr}, for the characterization of the exceptional cases we also require that $2^{s_k-1}|e(v)$ for all $v \in S_k \cap \Sigma_{\mathrm{cr}}$ and only demand that $(k, S \cup T \cup \Sigma_{\mathrm{cr}}, n)$ (rather than $(k, S \cup T, n)$) is in the special case.

The modified requirement on θ_v for $v \in \Sigma_{\mathrm{cr}}$ is that $\theta_v : G_{k_v} \to \mu_n$ is unramified, yet the additional condition that $2^{s_k-1}|e(v)$ for such v forces $\theta_v(a_{k,n}) = \theta_v^{n/2}(2 + \eta_{s_k}) = 1$. When expressing $\prod_{v \in S_k} \xi_v(a_{k,n})$ in terms of χ, we replace $S_k \cap T$ with $S_k(T \cup \Sigma_{\mathrm{cr}})$, and for $v \in S_k \cap \Sigma_{\mathrm{cr}}$ we have $\xi_v = \chi'_v \eta_v$ for a crystalline nth root η_v of χ_v (which presently depends on the choice of global nth root χ' of χ). This leads to the extra factor $\prod_{v \in S_k \cap \Sigma_{\mathrm{cr}}} \eta_v(a_{k,n})$ for the crystalline nth root η_v of χ_v. But for such v, η_v is unique up to an unramified μ_n-twist, and $\eta_v(a_{k,n}) = \eta_v^{n/2}(2+\eta_{s_k})$ with $\mathrm{ord}_v(2+\eta_{s_k}) \in 2\mathbf{Z}$, so $\eta_v(a_{k,n})$ is *independent* of the choice of crystalline nth root η_v of χ_v. Hence, this choice can be made locally at v, without reference to the abstract global χ', so (4.1) is obtained. ■

5. Representations into linear algebraic groups

Let $f : H' \to H$ be a surjective homomorphism between linear algebraic F-groups and assume that $\ker f$ is central in H' (e.g., an isogeny between connected reductive groups). For a representation $\rho : G_k \to H(F)$, does ρ factor through f via a representation $\rho' : G_k \to H'(F)$ in the absence of local obstructions? If ρ' exists, can we control its local properties at finitely many places in terms of the local properties of ρ? These questions also make sense using W_k rather than G_k. For $H' = H = \mathbf{G}_m$ this is the local-global problem for $F^{\times}$-valued characters, together with its local refinements. The case of central isogenies with $F = \overline{\mathbf{Q}}_p$ and k a number field was considered in [W1] in a slightly different setting (with compatible families).

In this section we take up these matters except for the p-adic Hodge theory aspects when k is a number field and $F = \overline{\mathbf{Q}}_p$, which we address in §6. For ease of reference later, we begin by recording some standard lemmas.

Lemma 5.1. *Let $K_0 \subset \overline{\mathbf{Q}}_p$ a subfield of finite degree over $\mathbf{Q}_p$, and H_0 a K_0-group of finite type. Any continuous homomorphism $\rho_0 : W_k \to H_0(\overline{\mathbf{Q}}_p)$ factors through a continuous homomorphism $W_k \to H_0(K)$ for a subfield $K \subset \overline{\mathbf{Q}}_p$ of finite degree over K_0, and if k is a number field then this factors through the maximal profinite quotient G_k of W_k.*

Proof. The target is totally disconnected, so in the number field case ρ factors through G_k since $W_k \to G_k$ is a topological quotient map whose kernel is the identity component [T, 1.4.4]. If k is a function field then $W_k = W_k^1 \rtimes \mathbf{Z}$ for a profinite group W_k^1. Thus, a Baire category argument applied to compact subgroups of $H_0(\overline{\mathbf{Q}}_p) = \varinjlim H_0(K)$ does the job. ■

Lemma 5.2. *Let $F = \mathbf{C}$ or $\overline{\mathbf{Q}}_p$ with $p \neq \mathrm{char}(k)$, let $f : H' \to H$ be a central quotient of an F-group of finite type, and let $\rho' : W_k \to H'(F)$ be a continuous homomorphism. If $F = \mathbf{C}$ then ρ' is unramified at all but finitely many places of k, and if $F = \overline{\mathbf{Q}}_p$ and $\rho := f \circ \rho'$ is unramified at all but finitely many places of k then the same is true for ρ'.*

When $F = \overline{\mathbf{Q}}_p$, the condition involving the central quotient ρ of ρ' cannot be removed; an interesting counterexample with $k = \mathbf{Q}$, $H' = \mathrm{GL}_2$, and $H = 1$ is given in [KR, Thm. 25(b)].

Proof. First assume $F = \mathbf{C}$ and $\mathrm{char}(k) > 0$. We have $W_k = W_k^1 \rtimes \mathbf{Z}$ with W_k^1 a profinite group containing all inertia subgroups. The profinite image $\rho'(W_k^1)$ in the Lie group $H'(\mathbf{C})$ must be finite, but a cofinal system of finite-index closed subgroups of W_k^1 is given by the subgroups $W_{k'} \cap W_k^1 = W_{k'}^1$ for finite Galois extensions k'/k, so there exists such a k'/k so that ρ' is unramified at all places of k that are unramified in k'.

Next consider $F = \mathbf{C}$ and $\mathrm{char}(k) = 0$. We may replace H' with the Zariski closure of $\rho'(W_k)$ so that ρ' has Zariski-dense image. The preimage $\rho'^{-1}(H'^0)$ is an open normal subgroup of W_k with finite index, so by replacing k with the corresponding finite Galois extension we may arrange that H' is connected. We have $W_k = W_k^1 \rtimes \mathbf{R}$ for a compact group W_k^1 that is an extension of G_k by a commutative connected compact group Δ_k centralized by the action of $\mathbf{R}$ on W_k^1 [T, 1.4.4]. The image $K = \rho'(\Delta_k)$ is a compact connected commutative subgroup of the Lie group $H'(\mathbf{C})$, so it is a compact torus and is normalized by $\rho'(W_k)$. The complexification $K_{\mathbf{C}}$ has image in $H'(\mathbf{C})$ equal to $T'(\mathbf{C})$ for a torus $T' \subseteq H'$, and $\rho'(W_k)$ normalizes T'. But $\rho'(W_k)$ is Zariski-dense in the connected H', so T' is centralized by H'. The composite map $W_k \to (H'/T')(\mathbf{C})$ factors through the quotient $W_k/\Delta_k = G_k \rtimes \mathbf{R}$ that is a direct product. The image of G_k in $(H'/T')(\mathbf{C})$ must be finite, so the image of W_k^1 in $H'(\mathbf{C})$ has finite image in $(H'/T')(\mathbf{C})$. Once again replacing k with a finite Galois extension brings us to the case that W_k^1 lands in $T'(\mathbf{C})$. But every $\mathbf{C}^\times$-valued character of W_k is finitely ramified, so this case is settled.

Now suppose $F = \overline{\mathbf{Q}}_p$ (and let $\mathrm{char}(k) \neq p$ be arbitrary). We may and do choose a subfield K_0 of F of finite degree over $\mathbf{Q}_p$ such that there is a central extension

$$1 \to Z_0 \to H_0' \to H_0 \to 1$$

of finite type K_0-groups descending the given central extension over F. Lemma 5.1 provides a finite extension K/K_0 inside F such that ρ' factors through $H_0'(K)$. Since $H_0'(K)$ is locally profinite, the compact subgroup $\rho'(W_k^1)$ is profinite. Hence, by replacing k with a sufficiently large finite extension we may arrange that $\rho'(W_k^1)$ is contained in any desired open subgroup of $H_0'(K)$ around the identity. In particular, it is torsion-free and pro-p.

Let v be a non-archimedean place of k that is unramified in ρ and does not divide p; this accounts for all but finitely many places of k. Since the inertia group I_v has trivial image in $H_0(K)$, $\rho'(I_v)$ lies in $Z_0(K)$. The centrality of Z_0 in H_0' implies that $\rho'(W_{k_v})$ is abelian. But $I_v \subset W_k^1$, so $\rho'(I_v)$ is torsion-free and pro-p. By local class field theory, the inertia subgroup of $W_{k_v}^{\mathrm{ab}}$ is the product of a finite group and a pro-p_v group, where p_v is the residue characteristic at v. We have arranged that $p_v \neq p$, so $\rho'(I_v) = 1$. ■

Proposition 5.3. *Consider a central extension $1 \to Z \to H' \to H \to 1$ of finite type affine F-groups, with Z an F-torus. Let $\Gamma = G_k$ or W_k, and $\rho : \Gamma \to H(F)$ a representation. There exists a representation $\rho' : \Gamma \to H'(F)$ lifting ρ.*

The affineness hypothesis can be removed, at the cost of some technical complications when $F = \overline{\mathbf{Q}}_p$. We only require the affine case in what follows.

Proof. Since we require representations to be unramified at all but finitely many places, by Lemma 5.2 it suffices to lift ρ merely as a continuous homomorphism. By induction on $\dim Z$, we may and do assume $Z = \mathbf{G}_m$. Also, note that H' may be disconnected.

Step 1. We first review how to build an isogeny-complement $\widetilde{H}$ to Z in H' (i.e., a closed subgroup $\widetilde{H} \subset H'$ such that $\widetilde{H} \to H$ is surjective with finite kernel). If N' is a connected normal subgroup of H' that has finite intersection with Z and is carried isogenously onto its image in H, it suffices to find an isogeny-complement in H'/N' for the isogenous image of Z (since the preimage under $H' \to H'/N'$ of such an isogeny-complement is an isogeny-complement to Z in H'). By taking N' to be the unipotent radical of ${H'}^0$, we may assume ${H'}^0$ is reductive. Then by taking $N' = \mathscr{D}({H'}^0)$ we may assume $T' := {H'}^0$ is a torus.

The finite group $\pi_0(H') = H'(F)/T'(F)$ acts on T' and thereby acts on the cocharacter group $X' = \mathrm{X}_*(T')$. The line $\mathrm{X}_*(Z)_{\mathbf{Q}} \subset X'_{\mathbf{Q}}$ has trivial $\pi_0(H')$-action, and complete reducibility provides a $\pi_0(H')$-equivariant retraction $X'_{\mathbf{Q}} \to \mathrm{X}_*(Z)_{\mathbf{Q}}$. Multiplying by a sufficiently divisible nonzero integer provides a homomorphism $H' \to Z$ whose kernel is the desired isogeny-complement to Z.

Step 2. Returning to the original H', let $\widetilde{H} \subset H'$ be an isogeny-complement to Z, so $\widetilde{H} \to H$ is a surjection with finite kernel $\widetilde{H} \bigcap Z = Z[n_0] = \mu_{n_0} \subset \mathbf{G}_m = Z$ for some $n_0 > 0$. For positive integral multiples n of n_0, let $H_n' = \widetilde{H} \cdot Z[n]$, so the natural map $f_n : H_n' \to H$ is a surjection with kernel $Z[n]$. We will show that for sufficiently divisible n, ρ admits a continuous lift through f_n except possibly when $F = \mathbf{C}$ and $\Gamma = W_k$ with $\mathrm{char}(k) = 0$.

Consider the case $\Gamma = G_k$. For any $n \in n_0 \cdot \mathbf{Z}^+$, the induced map $H_n'(F) \to H(F)$ is a local homeomorphism. Thus, by the total disconnectedness of Γ, the obstruction to lifting through the central isogeny f_n is a continuous cohomology class $c_n \in \mathrm{H}^2(\Gamma, \mu_n(F))$ (where the discrete coefficients have trivial Γ-action). Moreover, if n' is a multiple of n then the natural map $\mathrm{H}^2(\Gamma, \mu_n(F)) \to \mathrm{H}^2(\Gamma, \mu_{n'}(F))$ carries c_n to $c_{n'}$. But $\varinjlim \mathrm{H}^2(G_k, \mu_n(F)) = \mathrm{H}^2(k, \mathbf{Q}/\mathbf{Z})$, and this vanishes by a result of Tate (as we noted in Remark 1.1).

Step 3. Suppose instead that $\Gamma = W_k$. First we treat the case $\mathrm{char}(k) > 0$ (so W_k is totally disconnected). Let $I_k := \ker(G_k \twoheadrightarrow \widehat{\mathbf{Z}})$. A choice of arithmetic Frobenius element $\phi \in W_k$ provides compatible identifications $G_k = I_k \rtimes \widehat{\mathbf{Z}}$ and $W_k = I_k \rtimes \mathbf{Z}$. Let $\widetilde{c}_n : W_k \times W_k \to \mu_{n_0}(F)$ be a continuous 2-cocycle (relative to the natural topology of W_k) representing the obstruction to the existence of a (continuous) homomorphism $\rho' : W_k \to H'_{n_0}(F)$ lifting ρ, so for $n \in n_0 \cdot \mathbf{Z}^+$ the cocycle $\widetilde{c}_n$ defined via $\mu_{n_0}(F) \hookrightarrow \mu_n(F)$ represents the obstruction to lifting to $H'_n(F)$. We claim that $\widetilde{c}_{n_0}$ is continuous for the profinite topology on W_k (even though ρ may not be). To check this, we may replace k with a finite Galois extension.

If $F = \mathbf{C}$ then $\rho(I_k)$ is finite, so we may pass to the trivial case when $\rho(I_k) = 1$. If $F = \overline{\mathbf{Q}}_p$ then we may increase k so that $\rho(I_k)$ lies in an open subgroup Ω of $H(F)$ over which $H'_{n_0}(F) \to H(F)$ admits a topological group splitting. That is, the preimage Ω'_{n_0} of Ω in $H'_{n_0}(F)$ is identified with $\Omega \times \mu_{n_0}(F)$ as topological groups, so we can uniquely lift $\rho|_{I_k}$ to $\rho'_{n_0} : I_k \to \Omega \subset \Omega \times \mu_{n_0}(F) = \Omega'_{n_0}$ and the obstruction to lifting ρ is controlled by the $\mu_{n_0}(F)$-factor of the conjugation action on ρ'_{n_0} by a chosen lift $h'_{n_0} \in H'_{n_0}(F)$ of $\rho(\phi) \in H(F)$. By replacing the constant field with a degree-d extension we may replace h'_{n_0} with its dth power, so by choosing $d = n_0$ we can kill the $\mu_{n_0}(F)$-component of its conjugation action on ρ'_{n_0}. This completes the proof that each $\widetilde{c}_n$ is continuous for the profinite topology on W_k in general (using our original k). The resulting system of classes $c_n \in \mathrm{H}^2(k, \mu_n(F))$ for n varying through multiples of n_0 is compatible with change in n, so the vanishing of $\mathrm{H}^2(k, \mathbf{Q}/\mathbf{Z})$ again provides the desired lift.

Step 4. Suppose $\Gamma = W_k$ and $\mathrm{char}(k) = 0$. The case $F = \overline{\mathbf{Q}}_p$ immediately reduces to the settled G_k-case by the final part of Lemma 5.1. Finally, suppose $F = \mathbf{C}$, so neither $H(F)$ nor W_k is totally disconnected. We cannot build continuous obstruction cocycles as easily as in the other cases, and we cannot expect to lift ρ through any f_n (since $\rho(W_k)$ may contain circles, and S^1 has nontrivial central extensions by μ_n; fortunately, S^1 has no nontrivial central extensions by $\mathbf{C}^\times$).

Let W_k^1 and Δ_k be as in the proof of Lemma 5.2, so Δ_k is normal in $W_k \simeq W_k^1 \rtimes \mathbf{R}$ with $W_k/\Delta_k = G_k \times \mathbf{R}$. More specifically, W_k has identity component of the form $W_k^0 \simeq \Delta_k \rtimes \mathbf{R}$ and the method of proof of Lemma 5.2 (using that a continuous homomorphism from a profinite group into a Lie group has finite image) shows that $\rho(W_k)$ contains the compact connected normal subgroup $\rho(W_k^0)$ with finite index. Grant for a moment that the subgroup $\rho(W_k^0)$ lifts *isomorphically* through the quotient map $H'(\mathbf{C}) \to H(\mathbf{C})$, so a choice of such lifting defines a lift $\widetilde{\rho}$ of the restriction of ρ to the finite-index closed (hence open) normal subgroup $\rho^{-1}(\rho(W_k^0)) \subset W_k$. Such a normal subgroup has the form $W_{k'}$ for a finite Galois extension k'/k, and the obstruction to extending $\widetilde{\rho}$ to a lift of ρ on the entirety of W_k is given by a class in $\mathrm{H}^2(W_k/W_{k'}, \mathbf{C}^\times) = \mathrm{H}^2(k'/k, \mathbf{C}^\times)$, where we may equip $\mathbf{C}^\times$ with the discrete topology. Increasing k' to some k'' has the effect of computing the obstruction to finding a lift of ρ that extends $\widetilde{\rho}|_{W_{k''}}$. Passing to the limit over such k'' yields an obstruction in $\varinjlim \mathrm{H}^2(k''/k, \mathbf{C}^\times) = \mathrm{H}^2(k, \mathbf{C}^\times) = 1$, so the desired lift ρ' of ρ exists.

It remains to show that $\rho(W_k^0)$ lifts isomorphically into $H'(\mathbf{C})$, or more intrinsically that any topological central extension of $\rho(W_k^0)$ by $\mathbf{C}^\times$ splits. The image $\rho(\Delta_k) \subseteq H(\mathbf{C})$ is a compact connected commutative subgroup, so it is a compact torus. Hence, $\rho(W_k^0)$ is either a compact torus or an extension of $\mathbf{R}$ by such a torus. But every topological central extension of a compact torus or $\mathbf{R}$ by $\mathbf{C}^\times$ is split, so we are done. ∎

Corollary 5.4. *In the setting of Proposition* 5.3*, let S and T be finite disjoint sets of places of k with no archimedean places in T, and assume ρ is unramified at S and tame at T. The lift ρ' of ρ can be chosen to be unramified at S and tame at T.*

Proof. Choose some ρ' lifting ρ. We may and do assume $Z = \mathbf{G}_m$ and that S contains no complex places, and we seek $\chi : \Gamma^{\mathrm{ab}} \to F^\times = Z(F)$ such that $\chi^{-1}\rho'$ is unramified at S and tame at T. For non-archimedean or real v, let $J_v \subset k_v^\times$ denote the maximal compact subgroup. If $v \in S$ then $\rho'(W_{k_v})$ is abelian, due to the unramifiedness of ρ at v and the centrality of Z, so ρ' induces a homomorphism $k_v^\times \to F^\times$ whose restriction $J_v \to F^\times$ is trivial if and only if ρ' is unramified at v.

Fix $v \in T$. Although $\rho(I_v)$ is abelian, due to tameness, $\rho(\Gamma_v)$ may not be abelian for such v. Nonetheless, we claim that $\rho|_{\Gamma_v}$ admits a tame lifting $\rho'_v : \Gamma_v \to H'(F)$. To prove this, let p_v be the residue characteristic and q_v the size of the residue field at v. Let τ_v denote a topological generator of the tame inertia group at v, and σ_v an arithmetic Frobenius element at v, so σ_v conjugates τ_v via the q_vth power map. For any $\phi \in H'(F)$ lifting $\rho(\sigma_v)$ and any $h' \in H'(F)$ lifting $\rho(\tau_v)$, we have $\phi h' \phi^{-1} \in h'^{q_v} c$ for some $c \in Z(F) = F^\times$. Writing $c = b^{q_v - 1}$ for some $b \in F^\times$, if we replace h' with bh', we get $\phi h' \phi^{-1} = h'^{q_v}$; note that this is unaffected by changing the choice of ϕ.

First consider that case that $\rho(\tau_v)$ has finite order (as is automatic except for possibly when $F = \overline{\mathbf{Q}}_p$ and $p_v \neq p$). By tameness its order $m = m_v$ is not divisible by p_v. Thus, $a := h'^m \in F^\times$ in $H'(F)$ and $a^{q_v} = (h'^{q_v})^m = \phi h'^m \phi^{-1} = a$, so $a^{q_v - 1} = 1$. Hence, in such cases h' has order not divisible by p_v, so $\rho'_v(\tau_v) := h'$ and $\rho'_v(\sigma_v) := \phi$ defines a tame homomorphism $\rho'_v : W_{k_v} \to H'(F)$ lifting $\rho|_{W_{k_v}}$. If $\rho(\sigma_v)$ also has finite order then we can modify the choice of ϕ so that ϕ has finite order.

Thus, when $F = \mathbf{C}$ we can always find a tame lifting $\rho'_v : \Gamma_v \to H'(F)$ of $\rho|_{\Gamma_v}$, and likewise for $F = \overline{\mathbf{Q}}_p$ and the W_k-cases except possibly when $\rho(I_v)$ is infinite (forcing $p_v \neq p$). In such exceptional W_k-cases (with $F = \overline{\mathbf{Q}}_p$) we can find a tame local lift as follows. The image $\rho(I_v)$ is the product of $\mathbf{Z}_p$ and a finite cyclic group of order not divisible by p or by p_v, so by a variant of the preceding considerations (now using that p-adic Lie groups are pro-p near the identity) we can lift $\rho(I_v)$ to a pro-cyclic group C' in $H'(F)$ that is pro-p near the identity and has "order" not divisible by p_v. For $h' \in C'$ lifting $\rho(\tau_v)$ and any $\phi \in H'(F)$ lifting $\rho(\sigma_v)$, $\phi h' \phi^{-1} = h'^{q_v} c$ in $H'(F)$ for some $c \in C' \bigcap F^\times$. We may choose $m > 0$ so that c^m is a 1-unit that is trivial or multiplicatively generates a $\mathbf{Z}_p$. Any $(q_v - 1)$th root b of c in $F^\times$ satisfies the same property (with m replaced by $m(q_v - 1)$), so as above we can replace h' with bh' to get to the case $c = 1$. We have constructed a tame local lift when $F = \overline{\mathbf{Q}}_p$ in all W_k-cases.

In the G_k-cases with $F = \overline{\mathbf{Q}}_p$, we may use the preceding constructions on tame inertia and just have to arrange that a pro-cyclic subgroup of $H(F)$ (such as the one generated by the image of a Frobenius element of G_{k_v}) always lifts to a pro-cyclic subgroup of the central cover $H'(F)$. Such a pro-cyclic lifting is built by an easier version of the construction of a tame inertia lift when the inertial image under ρ is infinite (the present situation is easier since we don't need to keep track of divisibility by p_v).

To summarize (allowing $\Gamma = G_k$ or W_k, and any F), for each $v \in S \bigcup T$ there exists a continuous lifting $\rho'_v : \Gamma_v \to H'(F)$ of $\rho|_{\Gamma_v}$ that is tame at $v \in T$ and unramified at $v \in S$. Since $\rho'|_{\Gamma_v}$ is another lift of $\rho|_{\Gamma_v}$ valued in $H'(F)$, we have $\rho'|_{\Gamma_v} = \chi_v \rho'_v$ for a character $\chi_v : \Gamma_v^{\rm ab} \to F^\times$. It suffices to construct a character $\chi : \Gamma^{\rm ab} \to F^\times$ agreeing with χ_v on the maximal compact subgroup J_v of $\Gamma_v^{\rm ab}$ for all $v \in S \bigcup T$ (as then $\chi^{-1}\rho'$ is a lift of ρ with the desired local properties at $S \cup T$).

If $F = \mathbf{C}$ or if $F = \overline{\mathbf{Q}}_p$ and $\mathrm{char}(k) > 0$ then all $\chi_v(J_v)$ are finite, so we can construct a finite-order χ via Proposition A.3. Now suppose $F = \overline{\mathbf{Q}}_p$ and k is a number field. Since the $\overline{\mathbf{Q}}_p^\times$-valued character group of W_k coincides with that of G_k, we can assume $\Gamma = W_k$. If $v \nmid p$ then $\chi_v(J_v)$ is again finite (since J_v has finite pro-p part), so by finite-order twisting we may assume $\chi_v = 1$ for all $v \in S \cup T$ that are not p-adic. Let S'_p denote the set of p-adic places in $S \cup T$. Choose a subfield $K_0 \subset \overline{\mathbf{Q}}_p$ of finite degree over $\mathbf{Q}_p$ and a central extension

$$1 \to \mathbf{G}_m \to H'_0 \to H_0 \to 1$$

of affine K_0-groups of finite type such that scalar extension to F recovers the given central extension over F and $\rho'(W_k) \subset H'_0(K_0)$ and $\chi_v(W_k^{\rm ab}) \subset K_0^\times$ for all $v \in S'_p$ (see Lemma 5.1).

The unipotent radical U' of H'_0 maps isomorphically onto the unipotent radical U of H_0, and the natural map $H'_0 \to (H'_0/U') \times_{H_0/U} H_0$ is an isomorphism, so we may pass to the case when H'_0 and H_0 are reductive (but possibly disconnected). The identity component H'_0 is an almost direct product of its maximal central torus Z'_0 and its derived group G'_0. By replacing K_0 with a finite extension, we may assume that Z'_0 is K_0-split. The finite component group π_0 of H'_0 naturally acts on Z'_0, with trivial action on the central $\mathbf{G}_m$, so the semisimplicity of the π_0-action on $\mathrm{X}_*(Z'_0)_{\mathbf{Q}}$ provides a K_0-torus S'_0 in Z_0 complementary to the central $\mathbf{G}_m$ and normal in H'_0. Thus, $T'_0 := H'_0/(S'_0 \cdot G'_0)$ is a quotient of H'_0 with identity component $\mathbf{G}_m$ onto which the central $\mathbf{G}_m$ in H'_0 maps via an isogeny.

By increasing K_0, we may assume that the component group Δ_0 of T'_0 is constant. The extension structure on $T'_0(F)$ is classified by an element of $\mathrm{H}^2(\Delta_0, F^\times)$. This cohomology group is killed by the order N of Δ_0, so using a central pushout of T'_0 along the Nth-power endomorphism of $\mathbf{G}_m$ provides an isogeny $\theta : T'_0 \to \mathbf{G}_m$. Replacing θ with θ^m for a sufficiently divisible $m > 0$ ensures that the compact image of the character

$$\overline{\rho}' : W_k \stackrel{\rho'}{\to} H'_0(K_0) \to T'_0(K_0) \stackrel{\theta}{\to} K_0^\times \tag{5.1}$$

has torsion-free pro-p image. Thus, it may be identified with a character $\xi : C_k = \mathbf{A}_k^\times/k^\times \to K_0^\times$ that factors through the maximal pro-p torsion-free quotient and hence is unramified outside p. Composing θ with the natural map to T'_0 from the central $\mathbf{G}_m$ in H'_0 defines an isogeny $\mathbf{G}_m \to \mathbf{G}_m$ given by $x \mapsto x^d$ for some nonzero $d \in \mathbf{Z}$.

Since the image of ξ is torsion-free and pro-p, it consists of 1-units in K_0 and is multiplicatively finite free over $\mathbf{Z}_p$. By choosing a $\mathbf{Z}_p$-basis and extracting dth roots as 1-units, we get a continuous dth root ξ' of ξ valued in a torsion-free group of 1-units in $K^\times$ for a finite extension K of K_0. Clearly ξ' is also unramified away from p, and by viewing it with values in the central $\mathbf{G}_m \subseteq H'_0$ we can replace ρ' with ${\xi'}^{-1}\rho'$ without affecting the inertial restriction at places away from p. We rename K as K_0 and replace χ_v with $\chi_v \xi'|_{\Gamma_v}$ to arrive at the case that $\xi = 1$. The map $q_0 : H'_0 \to H_0 \times \mathbf{G}_m$ whose components are $f_0 : H'_0 \to H_0$ and $H'_0 \to T'_0 \stackrel{\theta}{\to} \mathbf{G}_m$ is an isogeny between identity components, and $q_0 \circ \rho' = (1, \rho)$. Thus, for p-adic places $v \in S \cup T$ we see that $\rho'(I_v)$ is finite. But $\chi_v^{-1}\rho'|_{\Gamma_v}$ is unramified for $v \in S$ and is tame for $v \in T$. Hence, for a p-adic place $v \in S$ the image $\chi_v(I_v)$ is finite and for a p-adic place $v \in T$ the image of χ_v on wild inertia is finite. In the latter case, $\chi_v(I_v)$ must be finite since the inertial part $\mathscr{O}_{k_v}^{\times}$ of Γ_v^{ab} has finite tame part.

For all p-adic places $v \in S \cup T$, we have arranged that $\chi_v(I_v)$ is finite. By Proposition A.3 we can construct a finite-order character $\chi : G_k \to F^{\times}$ whose inertial restriction at each p-adic place of $S \cup T$ agrees with χ_v and that is split at all other places of $S \cup T$. The twist $\chi^{-1}\rho'$ is a lift of ρ to H' that satisfies the desired local properties. ■

Next, we generalize Proposition 5.3 by considering Z that is of multiplicative type but possibly disconnected; this includes all *central* isogenies $H' \to H$ between linear algebraic F-groups. Via the factorization $H' \to H'/Z^0 \to H$ and the settled case of connected Z, the lifting problem (without consideration of local conditions) can be understood via the Grunwald–Wang theorem by using an elementary cohomological argument (as in Remark 1.1) provided that $\#(Z/Z^0)$ is not divisible by $\mathrm{char}(k)$. However, to control local properties of the lift (and to avoid restrictions on $\mathrm{char}(k)$) we need to use the connected case in another way to relate the problem to our earlier results on the local-global problem for characters (and its local refinements). Here is the general result, which establishes Theorem 1.4(1) beyond the case $H = H' = \mathbf{G}_m$, as well as Theorem 1.4(2).

Theorem 5.5. *Let $1 \to Z \to H' \to H \to 1$ be a central extension of linear algebraic F-groups with Z of multiplicative type, and let $\Gamma = G_k$ or W_k. Let $\rho : \Gamma \to H(F)$ be a representation, and for each place v of k that is archimedean or ramified for ρ, assume that $\rho|_{\Gamma_v}$ lifts to a representation $\Gamma_v \to H'(F)$. Let $n \geq 1$ denote the exponent of Z/Z^0.*

(1) *There exists a representation $\rho' : \Gamma \to H'(F)$ lifting ρ except possibly when k is a number field and $(k, \emptyset, n)$ is in the special case.*
(2) *Assume that a lift ρ' of ρ exists. Let S and T be finite disjoint sets of places of k with no archimedean places in T, and assume that ρ is unramified at S and tame at T. Then ρ' can be chosen to be unramified at S and tame at T except for possibly when k is a number field, $(k, S \cup T, n)$ is in the special case, $S_k \neq \emptyset$, and $2^{s_k - 1} | \mathrm{ord}_v(2)$ for all $v \in S_k$.*

For a number field k, the set S_k of "bad" 2-adic places of k and the integer $s_k \geq 2$ are defined in Appendix A. Also, when $F = \overline{\mathbf{Q}}_p$ and v is a p-adic place of k such that $\rho|_{\Gamma_v}$ is semistable, a sufficient criterion for $\rho|_{\Gamma_v}$ to admit a lift (even a semistable lift) is that $\rho|_{I_v}$ admits a Hodge–Tate lift $I_v \to H'(F)$; see Corollary 6.7.

Proof. By applying Proposition 5.3 to the central quotient map $H' \to H'/Z^0$, we may and do assume that Z is finite.

Here is a short proof of (1) when $\Gamma = G_k$ with $\mathrm{char}(k) \nmid \#Z(F)$. In such cases, the obstruction to the existence of ρ' lies in $\mathrm{H}^2(k, Z(F))$, where the discrete finite coefficient group

$Z(F)$ has trivial G_k-action. Since we assume the absence of local obstructions to the existence of ρ', the global obstruction lies in the subgroup $Ш^2_{\emptyset}(k, Z(F))$ whose vanishing is well-understood via Tate global duality and the Grunwald–Wang theorem since $\mathrm{char}(k) \nmid \#Z(F)$.

To handle part (2), and to give a uniform proof of part (1) in all cases (e.g., $F = \mathbf{C}$ with $\Gamma = W_k$ for a number field k, or the case $\mathrm{char}(k) | \#Z(F)$ that often occurs when $\mathrm{char}(k) = 2$ and H' is a connected semisimple group), we will analyze the global obstruction in another way. A central pushout along an inclusion of Z into a torus will reduce the assertions to the understood local-global problem for $F^{\times}$-valued characters (and its local refinements).

The finite commutative Z admits an inclusion into an F-torus $\mathscr{T}$. Choose such an inclusion and form a central pushout diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z & \longrightarrow & H' & \xrightarrow{f} & H & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \mathscr{T} & \longrightarrow & \mathscr{H}' & \longrightarrow & H & \longrightarrow & 1 \end{array} \tag{5.2}$$

for a finite type F-group $\mathscr{H}'$ generated by H' and $\mathscr{T}$ as subgroups, with $\mathscr{T}$ central in $\mathscr{H}'$. By Proposition 5.3, there exists a representation $\widetilde{\rho} : \Gamma \to \mathscr{H}'(F) = \mathscr{T}(F)H'(F)$ lifting ρ, and (by Corollary 5.4) it can be chosen to be unramified at S and tame at T.

The possibilities for $\rho' : \Gamma \to H'(F)$ are precisely the representations $\chi\widetilde{\rho}$ taking values in the subgroup $H'(F) \subseteq \mathscr{H}'(F)$, or equivalently having trivial image in $(\mathscr{H}'/H')(F)$. But $\mathscr{H}'/H' = \mathscr{T}/Z$, so the condition on χ is exactly that it lifts the composite representation

$$\psi : \Gamma \xrightarrow{\widetilde{\rho}} \mathscr{H}'(F) \to (\mathscr{T}/Z)(F). \tag{5.3}$$

We conclude that ρ' exists if and only if the character ψ admits a continuous lift through the isogeny of tori $\mathscr{T} \to \mathscr{T}/Z$, and if $\widetilde{\rho}$ is chosen to be unramified at S and tame at T then ψ satisfies these local properties and for part (2) we seek to preserve these properties when lifting ψ to $\mathscr{T}$.

Relative to suitable bases of the character groups of $\mathscr{T}$ and $\mathscr{T}/Z$, the isogeny between these tori takes the form

$$(c_1, \dots, c_r) \mapsto (c_1^{n_1}, \dots, c_r^{n_r}) \tag{5.4}$$

where $n_1 | \dots | n_r$ are the invariant factors of $Z(F)$ (so $n = n_r$). Hence, if we write $\psi : \Gamma \to (\mathscr{T}/Z)(F) = (F^{\times})^r$ in the form $(\psi_1, \dots, \psi_r)$ then the task is to construct an n_ith root of ψ_i for every $1 \le i \le r$, moreover preserving the desired local conditions at S and T when they are satisfied by ψ. The absence of local obstructions to the existence of ρ' says exactly that ψ_i is locally an n_ith power for all i. Thus, we deduce part (1) in general by applying Theorem 1.4(1) in the settled case $H = H' = \mathbf{G}_m$, and we deduce part (2) by applying Proposition 3.5 (with S and Σ there taken to be empty, and T there taken to be $S \cup T$ in our present circumstances). ■

Remark 5.6. Let $f : H' \to H$ be an isogeny between connected *semisimple* F-groups, and suppose k is a number field. The only Killing-Cartan type having fundamental group with order divisible by 8 is type A_r with $r \equiv -1 \bmod 8$. Thus, since $8 | 2^{s_k+1}$, if H has no simple factors of type A_r with $r \equiv -1 \bmod 2^{s_k+1}$ then the obstructions in both parts of Theorem 5.5 for this f are necessarily trivial.

Corollary 5.7. *Let H be a connected reductive F-group, H' its semisimple derived group, and n the exponent of the center $Z_{H'}$ of H'. Let ρ be a representation of G_k or W_k valued in $H(F)$, and at the archimedean and ramified places for ρ assume there is no obstruction to the existence of a central twist valued in $H'(F)$.*

(1) *There exists a global central twist ρ' of ρ valued in H' except possibly when k is a number field and $(k, \emptyset, n)$ is in the special case.*
(2) *Assume ρ' exists, and let S and T be sets of places of k as in Theorem* 5.5(2)*. Then ρ' can be chosen to be unramified at S and tame at T except possibly when $(k, S \cup T, n)$ is in the special case, $S_k \neq \emptyset$, and $2^{s_k - 1} | \mathrm{ord}_v(2)$ for all $v \in S_k$.*

If $(k, \emptyset, n)$ is in the special case then H' has a simple factor of type A_r with $r \equiv -1 \bmod 2^{s_k+1}$ (as in Remark 5.6).

Proof. Let $\overline{\rho} = \rho \bmod Z_H(F)$ be the quotient of ρ valued in the adjoint semisimple group H/Z_H. Since $H'/Z_{H'} = H/Z_H$, the central twist in part (1) is precisely a lift of $\overline{\rho}$ through the central isogeny $H' \to H'/Z_{H'}$. Thus, the assertion in part (1) is a special case of Theorem 5.5(1). Similarly, part (2) is a special case of Theorem 5.5(2). ■

Example 5.8. Consider Corollary 5.7 with $H = \mathrm{GSp}_{2g}$. The problem is to find an $F^\times$-valued character χ such that $\chi\rho$ is valued in Sp_{2g}. This amounts to constructing a square root of the homothety character of ρ.

Example 5.9. Suppose $H = \mathrm{GO}_r$, and let $\theta : G_k \to F^\times$ be the homothety character of ρ, so $q_r \circ \rho(g) = \theta(q) q_r$ for the standard rank-r split quadratic form q_r on F^r. A necessary and sufficient condition for ρ to admit a central twist valued in SO_r is the existence of $\chi : G_k \to F^\times$ such that $\chi^2 = \theta$ and $\chi^r = \det \rho$ (so $\chi^{-1}\rho$ is valued in O_r with trivial determinant). For odd r it suffices to find a square root of θ since $\mathrm{O}_r = \mu_2 \times \mathrm{SO}_r$ for such r. For even r it is necessary and sufficient to find a square root of θ and to verify the equality $\theta^{r/2} = \det \rho$ (which obviously can be checked locally).

6. Lifting p-adic Hodge theory properties

Let L be a finite extension of $\mathbf{Q}_p$, and let Γ denote G_L, W_L, or I_L. Consider a linear algebraic group H over $\overline{\mathbf{Q}}_p$, a continuous representation $\sigma : \Gamma \to H(\overline{\mathbf{Q}}_p)$, and a basic p-adic Hodge theory property $\mathbf{P}$ (crystalline, semi-stable, deRham, or Hodge–Tate). For any linear representation $\xi : H \to \mathrm{GL}(V)$ over $\overline{\mathbf{Q}}_p$, we get a linear action of Γ on V via $\xi \circ \sigma$. Exactly as in §4, we can ask if this action satisfies $\mathbf{P}$ (using a descent to a subfield of $\overline{\mathbf{Q}}_p$ of finite degree over $\mathbf{Q}_p$, the choice of which does not matter).

Definition 6.1. The representation σ *satisfies* $\mathbf{P}$ if $\xi \circ \sigma$ satisfies $\mathbf{P}$ for all ξ.

By standard Tannakian arguments, it suffices to consider a single faithful ξ. The condition that σ satisfies $\mathbf{P}$ is preserved under composition with homomorphisms of linear algebraic groups, so a semistable representation $I_L \to H(\overline{\mathbf{Q}}_p)$ always lands in $H^0(\overline{\mathbf{Q}}_p)$ since projection to $(H/H^0)(\overline{\mathbf{Q}}_p)$ is a finite-image representation that is semistable and hence unramified.

We seek to generalize Theorem 5.5(2) and Corollary 5.7 by incorporating basic p-adic Hodge theory properties at p-adic places of k away from $S \cup T$ when $F = \overline{\mathbf{Q}}_p$. To do this, we first recall a useful construction attached to any linear algebraic group H_0 over $\mathbf{Q}_p$ and representation $\sigma_0 : I_L \to H_0(\mathbf{Q}_p)$. By a variant of the Tannakian formalism, there is a unique element $\Theta_{\sigma_0} \in (\mathfrak{h}_0)_{\mathbf{C}_L}$ such that for every linear representation $\xi_0 : H_0 \to \mathrm{GL}(V_0)$ over $\mathbf{Q}_p$, the Sen operator of the semilinear I_L-representation $(\xi_0)_{\mathbf{C}_L} \circ \sigma_0$ on $(V_0)_{\mathbf{C}_L}$ is $\mathrm{Lie}(\xi_0)(\Theta_{\sigma_0})$.

The representation σ_0 is Hodge–Tate if and only if Θ_{σ_0} is semisimple (relative to the identification of $\mathfrak{h}_0$ as the Lie algebra of H_0) and has eigenvalues in $\mathbf{Z}$ under all linear representations ξ_0 of H_0 over $\mathbf{Q}_p$, in which case $\Theta_{\sigma_0} = \mathrm{Lie}(\lambda_0)(t\partial_t)$ for a unique 1-parameter subgroup $\lambda : \mathbf{G}_m \to (H_0)_{\mathbf{C}_L}$ over $\mathbf{C}_L$. Equivalently, $(\xi_0)_{\mathbf{C}_L} \circ \lambda$ defines the Hodge–Tate decomposition of $(V_0)_{\mathbf{C}_L}$ for all ξ_0 over $\mathbf{Q}_p$. These 1-parameter subgroups provide a convenient lifting criterion for representations of I_L:

Theorem 6.2 (Wintenberger)**.** *Let $f_0 : H_0' \to H_0$ be a homomorphism between linear algebraic groups over $\mathbf{Q}_p$ such that $\mathrm{Lie}(f_0)$ is an isomorphism. Let $\sigma_0 : I_L \to H_0(\mathbf{Q}_p)$ be a Hodge–Tate representation, and $\lambda : \mathbf{G}_m \to (H_0)_{\mathbf{C}_L}$ the 1-parameter $\mathbf{C}_L$-subgroup classifying the Hodge–Tate structure on σ_0.*

(1) *If a continuous lift $\sigma_0' : I_L \to H_0'(\mathbf{Q}_p)$ of σ_0 exists then σ_0' is Hodge–Tate if and only if λ lifts to a 1-parameter subgroup λ' of $(H_0')_{\mathbf{C}_L}$. In such cases, σ_0' is deRham if and only if σ_0 is deRham.*

(2) *Assume σ_0 is semistable. There is a semistable representation $\sigma_0' : I_L \to H_0'(\mathbf{Q}_p)$ lifting σ_0 if and only if λ lifts to a 1-parameter subgroup λ' of $(H_0')_{\mathbf{C}_L}$. In such cases σ_0' is unique if it exists, and if it exists then it is crystalline if and only if σ_0 is crystalline.*

Proof. For the Hodge–Tate case of (1), let $\Theta_0' \in (\mathfrak{h}_0')_{\mathbf{C}_L}$ be the associated Sen element. The isomorphism $\mathrm{Lie}(f_0)_{\mathbf{C}_L}$ carries Θ_0' to the Sen element $\Theta_0 \in (\mathfrak{h}_0)_{\mathbf{C}_L}$ corresponding to the Hodge–Tate representation σ_0, so Θ_0' inherits semisimplicity from Θ_0 due to the functoriality of the Jordan decomposition with respect to f_0. The hypothesis on lifting of 1-parameter $\mathbf{C}_L$-subgroups through $(f_0)_{\mathbf{C}_L}$ implies that Θ_0' has eigenvalues in $\mathbf{Z}$ under any linear representation of H_0', so σ_0' is Hodge–Tate.

For the deRham case of (1) (assuming a Hodge–Tate σ_0' exists), note that it is harmless to replace L with a finite extension, so we can assume that σ_0' and σ_0 land in the respective identity components of H_0' and H_0. Hence, we may assume that H_0 and H_0' are connected, so $\ker f_0$ is central in H_0. Any two continuous lifts of σ_0 are therefore related by a finite-order central twist, so if one such lift is deRham then all lifts are deRham. Hence, the deRham case is reduced to (2) since deRham representations are potentially semistable.

It remains to prove (2). We may and do replace H_0' and H_0 with their identity components. Now f_0 is a central isogeny, so any two continuous lifts of σ_0 are related by a finite-order central twist. Thus, for any two semistable lifts, the finite-order central twist relating them must also be semistable and therefore trivial. This proves the uniqueness of a semistable σ_0'. The existence of a semistable lift is (an immediate consequence of) the main result of [W2, §2]. To check that if σ_0 is crystalline then σ_0' is crystalline, consider a linear representation $\xi_0 : H_0' \to \mathrm{GL}(V)$ over $\mathbf{Q}_p$. We want to prove that $\xi_0 \circ \sigma_0'$ is crystalline, and it is the same to check this for the underlying $\mathbf{Q}_p$-linear representation of the K-linear representation V_K of I_L for a finite extension $K/\mathbf{Q}_p$. The finite central $\mathbf{Q}_p$-subgroup $\mu = \ker f_0$ in H_0' splits over a finite extension $K/\mathbf{Q}_p$, and we work with such a K.

Decompose V_K according to the K-linear characters of the finite constant K-group μ_K. It suffices to show that for each such character χ, the χ-isotypic component $W = (V_K)_\chi$ is crystalline as a $\mathbf{Q}_p$-linear I_L-representation space. Since W is semistable, the problem is to prove that the operator N on the covariantly associated filtered (ϕ, N)-module $D = D_{\rm st}(W)$ vanishes. Let $\mathbf{F}$ denote the residue field of $(L^{\rm un})^\wedge$ and $E = W(\mathbf{F})[1/p]$, so D is naturally a module over $K \otimes_{\mathbf{Q}_p} E$. Letting W^* denote the (semistable) K-linear dual of W, and D^* denote the E-linear dual of D (equipped with its compatible K-linear structure), the nilpotent $K \otimes_{\mathbf{Q}_p} E$-linear monodromy operator on $D_{\rm st}(W \otimes_K W^*) = D \otimes_{K \otimes_{\mathbf{Q}_p} E} D^*$ is $N \otimes 1 - 1 \otimes N^*$, where N^* denotes the E-linear dual of N (with action on D^* that is $K \otimes_{\mathbf{Q}_p} E$-linear). The H'_K-action on $W \otimes_K W^*$ factors through an H_K-action, so via the inclusion $H \hookrightarrow {\rm R}_{K/\mathbf{Q}_p}(H_K)$ we see that the $\mathbf{Q}_p$-linear I_L-action on $W \otimes_K W^*$ factors as the composition of σ_0 and the algebraic H-action, so it is crystalline. Hence, $N \otimes 1 - 1 \otimes N^* = 0$. It is easy to check (by considering the factor fields of $K \otimes_{\mathbf{Q}_p} E$ separately) that this forces N and N^* to be scalar endomorphisms over $K \otimes_{\mathbf{Q}_p} E$ and hence 0 (due to nilpotence). ■

We need a version of Theorem 6.2 over $\overline{\mathbf{Q}}_p$, with I_L replaced by W_L or G_L and with $\ker f$ central but possibly of positive dimension. This will be given in Corollary 6.7, resting on a preliminary version for central torus kernels in Proposition 6.5. An analogue for isogenies in the crystalline case is [W1, §1.2, Prop. 2], but that case rests on uniqueness properties which are not available when $\dim \ker f > 0$. We will use geometric local class field theory to handle the non-uniqueness aspects of the problem:

Lemma 6.3. *Let L be a finite extension of $\mathbf{Q}_p$ and let $c_\phi : I_L^{\rm ab} \to I_L^{\rm ab}$ be the automorphism $g \mapsto \phi g \phi^{-1}$ induced by conjugation by any arithmetic Frobenius element $\phi \in W_L$. Let φ be the endomorphism of $\mathrm{Hom}(I_L^{\rm ab}, \overline{\mathbf{Q}}_p^\times)$ dual to c_ϕ.*

For every $\theta \in \mathrm{Hom}(I_L^{\rm ab}, \overline{\mathbf{Q}}_p^\times)$ there exists $\chi \in \mathrm{Hom}(I_L^{\rm ab}, \overline{\mathbf{Q}}_p^\times)$ such that $\varphi(\chi)/\chi = \theta$. If θ has finite order then χ can also be chosen to have finite order.

Proof. Let $F = \overline{\mathbf{Q}}_p$. Any $\theta \in \mathrm{Hom}(I_L^{\rm ab}, F^\times)$ is valued in $K^\times$ for a finite extension $K/\mathbf{Q}_p$, and the abelian profinite $I_L^{\rm ab}$ uniquely decomposes as the direct product of its wild (pro-p) part and its tame quotient $\prod_{\ell \neq p} \mathbf{Z}_\ell$ on which c_ϕ induces multiplication by the size q of the residue field κ of L. Thus, θ uniquely decomposes as a product of a finite tame part and a pro-p-part. Since $F^\times$ is $(q-1)$-divisible, the tame part is easily handled by extracting a $(q-1)$th root in $F^\times$. To handle the wild part of the character group, we will use geometric local class field theory for $L' := (L^{\rm un})^\wedge = W(\mathbf{F}) \otimes_{W(\kappa)} L$ (where $\mathbf{F}$ is an algebraic closure of the residue field κ of L). We will lift a wild θ through the operation $\chi \mapsto \varphi(\chi)/\chi$ (preserving the finite-order property when θ has finite order) at the expense of replacing the target $K^\times$ with the multiplicative group of a finite extension containing p^nth roots of all 1-units in K, where p^n is the order of the group of p-power roots of unity in L.

Following [S1, 5.3], define the functor π_1 on the category of commutative pro-algebraic $\mathbf{F}$-groups to be the first left-derived functor of the right-exact functor π_0 of connected components. By [S1, 10.2], π_1 is exact on connected pro-algebraic groups over $\mathbf{F}$. Furthermore, its formation commutes with inverse limits [S1, 6.2, Prop. 3, Prop. 4]. By the main theorem of [S2], we have canonically $I_L^{\rm ab} = G_{L'}^{\rm ab} \simeq \pi_1(U_{L'})$, where $U_{L'}$ is the pro-algebraic $\mathbf{F}$-group of integral units of L'. In particular, the wild part of $I_L^{\rm ab}$ is identified with the pro-unipotent radical $\pi_1(U^1_{L'})$ of $\pi_1(U_{L'})$, where $U^j_{L'}$ is the pro-unipotent $\mathbf{F}$-group associated to $1+\mathfrak{m}^j_{L'}$ (so $U^1_{L'}/U^j_{L'}$ is an ordinary smooth connected unipotent $\mathbf{F}$-group). The filtration $\{U^j_{L'}\}$ of $U^1_{L'}$ defines a composition series on $\pi_1(U^1_{L'})$ whose successive quotients are $\pi_1(U^j_{L'}/U^{j+1}_{L'})$ with $U^j_{L'}/U^{j+1}_{L'} \simeq \mathbf{G}_a$ using the jth power of a uniformizer of L, or even any κ-basis of $\mathfrak{m}^j_L/\mathfrak{m}^{j+1}_L$. (Changing the basis has the effect of $\kappa^\times$-scaling on $\mathbf{G}_a$.)

Our task is to prove the surjectivity of the endomorphism of $\mathrm{Hom}(\pi_1(U^1_{L'}), F^\times)$ induced by $c_\phi - 1$ on $\pi_1(U^1_{L'})$ via its isomorphism to $G^{\mathrm{ab}}_{L'}$ from geometric local class field theory. This latter isomorphism is functorial in L' (see [S2, §2.3, Prop. 7]), so we may compute c_ϕ as the effect on $U^1_{L'}$ by the chosen q-Frobenius element $\phi \in G_L$. In particular, the effect on $U^j_{L'}/U^{j+1}_{L'} \simeq \mathbf{G}_a$ is the usual q-power endomorphism ϕ_q on the $\mathbf{F}$-group $\mathbf{G}_a$ because the isomorphism $U^j_{L'}/U^{j+1}_{L'} \simeq \mathbf{G}_a$ is defined via a κ-basis of $\mathfrak{m}^j_L/\mathfrak{m}^{j+1}_L$ (the choice of which does not matter). Thus, $\phi - 1 : U^1_{L'} \to U^1_{L'}$ is a surjective morphism (equivalently, injective on Hopf algebras) since $t^q - t$ is a surjective endomorphism of $\mathbf{G}_a$, so we have a short exact sequence of pro-algebraic groups

$$0 \to U^1_L \to U^1_{L'} \stackrel{\phi-1}{\to} U^1_{L'} \to 0$$

where U^1_L denotes the 0-dimesional pro-algebraic $\mathbf{F}$-group associated to the profinite group of 1-units in L.

Passing to the "homotopy sequence" and using that π_1 vanishes on 0-dimensional objects [S1, 5.3, Prop. 5] and π_0 is the identity functor on 0-dimensional objects, we get a short exact sequence of profinite groups

$$0 \to \pi_1(U^1_{L'}) \stackrel{\phi-1}{\to} \pi_1(U^1_{L'}) \to U^1_L \to 0.$$

Pushing out along a character $\theta : \pi_1(U^1_{L'}) \to U^1_K \subset F^\times$ for a finite extension K/K_0 gives a short exact sequence of pro-p groups

$$0 \to U^1_K \to E \to U^1_L \to 0. \tag{6.1}$$

If θ has finite order then we get an analogous exact sequence

$$0 \to \mu_{p^\infty}(K) \to E \to U^1_L \to 0 \tag{6.2}$$

with $\mu_{p^\infty}(K)$ a finite cyclic p-group. Note that in both cases the pro-p-group E is necessarily a *finitely generated* $\mathbf{Z}_p$-module. To lift θ through the injective endomorphism $\pi_1(\phi-1)$ of $\pi_1(U_{L'})$, it suffices to show that (6.1) splits after pushout to the 1-units of some finite extension K' of K, and to preserve the finite-order property when θ has finite order it suffices to split (6.2) after pushout to $\mu_{p^\infty}(K')$ for some K'/K.

Now it remains to show that any class $\xi \in \mathrm{Ext}^1_{\mathbf{Z}_p}(U^1_L, U^1_K)$ is killed under pushout to $\mathrm{Ext}^1_{\mathbf{Z}_p}(U^1_L, U^1_{K'})$ for some finite extension K'/K, and similarly upon replacing the 1-units in K and K' with the finite groups of p-power roots of unity in K and K' respectively. Decomposing U^1_L into the product of its finite cyclic torsion subgroup μ and a finite free $\mathbf{Z}_p$-module, we may instead consider the analogous assertions for $\mathrm{Ext}^1_{\mathbf{Z}_p}(\mu, U^1_K)$ and $\mathrm{Ext}^1_{\mathbf{Z}_p}(\mu, \mu_{p^\infty}(K))$. Upon choosing an isomorphism $\mu \simeq \mathbf{Z}_p/(p^n)$ we get $\mathrm{Ext}^1_{\mathbf{Z}_p}(\mu, M) \simeq M/(p^n)M$ naturally in $\mathbf{Z}_p$-modules M. The map $U^1_K/(U^1_K)^{p^n} \to U^1_{K'}/(U^1_{K'})^{p^n}$ vanishes by taking K' large enough to contain p^nth roots of all 1-units of K, and similarly with finite groups of p-power roots of unity in place of 1-units. ∎

Proposition 6.4. *Let L be a finite extension of $\mathbf{Q}_p$, and let Γ denote I_L, G_L, or W_L. Consider a central extension $1 \to Z \to H' \xrightarrow{f} H \to 1$ of affine $\overline{\mathbf{Q}}_p$-groups of finite type, with Z a torus. Any representation $\sigma : \Gamma \to H(\overline{\mathbf{Q}}_p)$ admits a lift $\sigma' : \Gamma \to H'(\overline{\mathbf{Q}}_p)$.*

This is a local version of Proposition 5.3, for which Lemma 6.3 will enable us to bypass the absence of a W_L-analogue of Tate's vanishing theorem for $\mathrm{H}^2(k, \mathbf{Q}/\mathbf{Z})$ with k a global field. As with Proposition 5.3, the affineness hypothesis can be removed (but we will not discuss it).

Proof. Since Γ is totally disconnected, by arguing as in the proof of Proposition 5.3 we reduce to proving $\varinjlim \mathrm{H}^2(\Gamma, (1/n)\mathbf{Z}/\mathbf{Z}) = 0$. If $\Gamma = G_L$, this limit is Tate-dual to the total Tate module of $\mu_\infty(L)$, which vanishes since $\mu_\infty(L)$ is finite. If $\Gamma = I_L$ then it vanishes since I_L has cohomological dimension 1 [S5, II, §3.1].

It remains to consider the case $\Gamma = W_L$. Choose a finite extension K_0 of $\mathbf{Q}_p$ and a map $f_0 : H_0' \to H_0$ between affine finite type K_0-groups that descends the given central quotient map $f : H' \to H$ and satisfies $\ker f_0 \simeq \mathbf{G}_m$ over K_0. Increasing K_0 by a finite amount, we may assume that $\sigma(\Gamma) \subseteq H_0(K_0)$. The map $H_0'(K_0) \to H_0(K_0)$ is surjective (Hilbert 90), and it admits continuous (even K_0-analytic) local sections through any $h_0 \in H_0(K_0)$ since $H_0' \to H_0$ is smooth. By taking K_0 large enough, we may pick a continuous lift $\widetilde{\sigma}_0'$ of $\sigma|_{I_L}$ valued in $H_0'(K_0)$. Let $\phi \in W_L$ be a Frobenius element and choose $h_0' \in H_0'(K_0)$ lifting $\sigma(\phi)$. To construct σ_0' we seek a character $\chi : I_L \to F^\times$ such that $\phi \mapsto h_0'$ and $\chi\widetilde{\sigma}_0' : I_L \to H'(F)$ defines a representation $W_L \to H'(F)$. That is, we want $h_0'(\chi\widetilde{\sigma}_0')(g){h_0'}^{-1} \stackrel{?}{=} (\chi\widetilde{\sigma}_0')(\phi g \phi^{-1})$ for $g \in I_L$. The h_0'-conjugate of $\widetilde{\sigma}_0'$ is a lift of $g \mapsto \sigma(\phi g \phi^{-1})$, as is $g \mapsto \widetilde{\sigma}_0'(\phi g \phi^{-1})$, so $h_0'\widetilde{\sigma}_0'(g){h_0'}^{-1} = \eta(g)\widetilde{\sigma}_0'(\phi g \phi^{-1})$ for some character $\eta : I_L \to F^\times$. Hence, we seek χ such that $\eta(g) = \chi(\phi g \phi^{-1})/\chi(g)$ for all $g \in I_L$. This is part of Lemma 6.3, which also incorporates a finer assertion for η of finite order. ∎

The local obstruction to lifting p-adic Hodge theory properties through a central extension by a torus is encoded on inertia in terms of the Hodge–Tate condition:

Proposition 6.5. *In the setup of Proposition 6.4, consider a representation $\sigma : \Gamma \to H(\overline{\mathbf{Q}}_p)$ satisfying a basic p-adic Hodge theory property $\mathbf{P}$. There exists a lift $\sigma' : \Gamma \to H'(\overline{\mathbf{Q}}_p)$ that satisfies $\mathbf{P}$ if and only if $\sigma|_{I_L}$ admits a Hodge–Tate lift $I_L \to H'(\overline{\mathbf{Q}}_p)$.*

Proof. The "only if" implication is obvious. For the converse, pick a Hodge–Tate lift σ' of $\sigma|_{I_L}$. Let $K_0/\mathbf{Q}_p$ be finite inside $F := \overline{\mathbf{Q}}_p$ over which there is a central extension

$$1 \to Z_0 \to H_0' \to H_0 \to 1$$

descending the given central extension over F (with $Z_0 = \mathbf{G}_m^r$), and such that $\sigma(\Gamma) \subseteq H_0(K_0)$ and $\sigma'(I_L) \subseteq H_0'(K_0)$. For the Weil restrictions $\mathscr{H}' = \mathrm{R}_{K_0/\mathbf{Q}_p}(H_0')$ and $\mathscr{H} = \mathrm{R}_{K_0/\mathbf{Q}_p}(H_0)$ we have a central extension of linear algebraic $\mathbf{Q}_p$-groups

$$1 \to \mathscr{Z} \to \mathscr{H}' \xrightarrow{\pi} \mathscr{H} \to 1$$

with the torus $\mathscr{Z} = \mathrm{R}_{K_0/\mathbf{Q}_p}(Z_0)$. In particular, $\sigma : I_L \to \mathscr{H}(\mathbf{Q}_p)$ lifts to $\sigma' : I_L \to \mathscr{H}'(\mathbf{Q}_p)$.

Since Z_0 is a central torus in H_0', there exists an isogeny splitting $\psi_0 : H_0' \to Z_0 = \mathbf{G}_m^r$ over K_0. Passing to Weil restrictions provides $\psi : \mathscr{H} \to \mathscr{Z}$ whose restriction to the central $\mathscr{Z} \hookrightarrow \mathscr{H}$ is an isogeny. The composite homomorphism $\chi = \psi \circ \sigma' : I_L \to \mathscr{Z}(\mathbf{Q}_p)$ is Hodge–Tate and semisimple, so we may apply the following lemma.

Lemma 6.6. *Let $\rho : I_L \to \mathrm{GL}(V)$ be an abelian semisimple Hodge–Tate representation on a finite-dimensional $\mathbf{Q}_p$-vector space. This is potentially crystalline, and there exists $m > 0$ such that $g \mapsto \rho(g^m)$ is crystalline.*

Proof. Let $L' = (L^{\mathrm{un}})^\wedge$, so $I_L = G_{L'}$. The image of $\mathbf{Q}_p[G_{L'}^{\mathrm{ab}}]$ in $\mathrm{End}(V)$ under ρ is a semisimple commutative $\mathbf{Q}_p$-subalgebra, which is to say $\prod E_i$ for finite extension fields E_i of $\mathbf{Q}_p$, so we may suppose ρ is a character $G_{L'}^{\mathrm{ab}} \to E^\times$ for a finite extension $E/\mathbf{Q}_p$. This character is locally algebraic, due to the Hodge–Tate property, so Proposition B.4(i) implies the potentially crystalline property (though this can also be seen more directly with Barsotti–Tate groups, using [S6, Prop. 4, III, A.4] and [S6, Cor. Thm. 2, III, A.5]).

To find $m > 0$ such that the mth power of ρ is crystalline, it is harmless to replace ρ with a twist by a finite-order character (valued in a finite extension of E). We will first construct such a twist that descends to a representation of some G_{L_n} (valued in a finite extension of E), where L_n is the degree-n unramified extension of L. Consider the endomorphism φ of $\mathrm{Hom}(I_L^{\mathrm{ab}}, \overline{\mathbf{Q}}_p^\times)$ as in Lemma 6.3; this is induced by the conjugation c_ϕ on I_L by an arithmetic Frobenius element $\phi \in W_L$. For $n > 0$ sufficiently divisible so that the element $\phi^n \in G_L$ is the identity on the Galois closure of E in $\overline{L}$, $\varphi^n(\rho)/\rho$ is Hodge–Tate with all Hodge–Tate weights equal to 0. Hence, this ratio has finite order on I_L, so Lemma 6.3 provides a character χ of finite order on I_L such that $\varphi^n(\rho)/\rho = \varphi^n(\chi)/\chi$. The resulting twist $\chi^{-1}\rho$ is φ^n-invariant, so it descends to a character of G_{L_n} (valued in a finite extension of E).

By replacing E with a finite extension and ρ with a finite-order twist as above, we may assume ρ descends to a character $\rho_0 : G_{L_n}^{\mathrm{ab}} \to E^\times$. This is locally algebraic; let $\chi : \mathrm{R}_{L_n/\mathbf{Q}_p}(\mathbf{G}_m) \to \mathrm{R}_{E/\mathbf{Q}_p}(\mathbf{G}_m)$ be the homomorphism of $\mathbf{Q}_p$-tori that agrees with ρ_0 on an open subgroup U of $\mathscr{O}_{L_n}^\times$ via local class field theory. If $m = [\mathscr{O}_{L_n}^\times : U]$ then $\rho_0^m = \chi^m$ on $\mathscr{O}_{L_n}^\times$. Hence, $\rho_0^m|_{G_{L_n}}$ is crystalline by Proposition B.4(i), so ρ^m is crystalline on I_L. ∎

By replacing ψ with ψ^N for a sufficiently divisible $N > 0$, we may assume that $\psi \circ \sigma'$ is crystalline and therefore satisfies property $\mathbf{P}$. The map

$$(\psi, \pi) : \mathscr{H}' \to \mathscr{Z} \times \mathscr{H}$$

is an isogeny between linear algebraic $\mathbf{Q}_p$-groups, and its composite $(\psi\circ\sigma', \sigma)$ with σ' satisfies $\mathbf{P}$. But σ' is Hodge–Tate, so by Theorem 6.2 there is a lift $\widetilde{\sigma} : I_L \to \mathscr{H}'(\mathbf{Q}_p) = H_0'(K_0)$ of

$$(\psi \circ \sigma', \sigma) : I_L \to (\mathscr{Z} \times \mathscr{H})(\mathbf{Q}_p) = (K_0^\times)^r \times H_0(K_0)$$

satisfying $\mathbf{P}$. Extending scalars to F gives a type-$\mathbf{P}$ lift $\widetilde{\sigma} : I_L \to H'(F)$ of $\sigma|_{I_L} : I_L \to H(F)$.

We have solved the lifting problem on I_L, and we need to find a solution on Γ. Let $Z' = \mathbf{G}_m^{r-1}$ be the product of the first $r-1$ factors of Z, so H'/Z' is a central extension of H by $Z/Z' = \mathbf{G}_m$. Assuming the case $r = 1$ is settled, there is a type-$\mathbf{P}$ lift $\overline{\sigma}' : \Gamma \to (H'/Z')(F)$ of σ. By hypothesis, $\sigma|_{I_L}$ has a Hodge–Tate lift $\rho : I_L \to H'(F)$, so the representations $\overline{\rho} = \rho \bmod Z'(F)$ and $\overline{\sigma}'|_{I_L}$ are Hodge–Tate lifts of $\sigma|_{I_L}$. Thus, there is a character $\chi : I_L \to$

$(Z/Z')(F) = F^\times$ such that $\chi\overline{\rho} = \overline{\sigma}'|_{I_L}$. Necessarily χ is Hodge–Tate since $\overline{\rho}$ and $\overline{\sigma}'$ are Hodge–Tate. By identifying Z/Z' with the evident $\mathbf{G}_m$-factor of Z complementary to Z', we may view χ as taking values in $Z(F)$, so $\chi\rho$ is a Hodge–Tate lift of $\overline{\sigma}'|_{I_L}$. Induction on r then implies that $\overline{\sigma}'$ admits a type-$\mathbf{P}$ lift $\Gamma \to H'(F)$, hence likewise for σ. Thus, we may may and do now assume $Z = \mathbf{G}_m$.

Pick an arithmetic Frobenius element $\phi \in W_L$. By Proposition 6.4, there is a lift $\sigma' : \Gamma \to H'(F)$ of σ, so the existence of a type-$\mathbf{P}$ lift on I_L provides a character $\chi : I_L \to F^\times$ such that $\chi\sigma'|_{I_L}$ satisfies $\mathbf{P}$. Let $h' = \sigma'(\phi)$. Enlarge K_0 so that χ takes values in $K_0^\times$. Letting c_ϕ denote the endomorphism $g \mapsto \phi g \phi^{-1}$ of I_L, h'-conjugation carries $(\chi \circ c_\phi)\sigma'|_{I_L}$ to $(\chi\sigma'|_{I_L}) \circ c_\phi$. This latter representation is type-$\mathbf{P}$ (by functoriality of p-adic Hodge theory relative to extension of the ground field), so $(\chi \circ c_\phi)\sigma'|_{I_L}$ satisfies $\mathbf{P}$. But $\chi\sigma'|_{I_L}$ satisfies $\mathbf{P}$, so $(\chi \circ c_\phi)/\chi$ twists the type-$\mathbf{P}$ representation $\chi\sigma'|_{I_L}$ into the type-$\mathbf{P}$ representation obtained by h'-conjugation. Hence, $(\chi \circ c_\phi)/\chi$ satisfies $\mathbf{P}$.

Let φ be the endomorphism of $\mathrm{Hom}(I_L^{\mathrm{ab}}, F^\times)$ defined by $\theta \mapsto \theta \circ c_\phi$. By Lemma 6.6, the type-$\mathbf{P}$ character $\varphi(\chi)/\chi$ is potentially crystalline. Thus, if $\mathbf{P}$ is "crystalline" or "semistable" then $\varphi(\chi)/\chi$ is crystalline since a semistable and potentially crystalline representation has vanishing monodromy operator (cf. end of the proof of Lemma 4.1). We claim that $\varphi(\chi)/\chi$ has *finite* order. (Note that χ may not be Hodge–Tate.) This character is valued in a central torus, yet we saw above that it scales a representation (namely, $\chi\sigma'|_{I_L}$) to a conjugate and thus is valued in the derived group. In any linear algebraic group, the maximal central torus has finite intersection with the derived group (as we may check in the maximal reductive quotient). This gives the asserted finiteness. Thus, $\varphi(\chi)/\chi$ is trivial in the crystalline and semistable cases, and has finite image in the deRham and Hodge–Tate cases.

In the crystalline and semistable cases we conclude that $\varphi(\chi) = \chi$, so $\chi : I_L^{\mathrm{ab}} \to F^\times$ is invariant under ϕ-conjugation on I_L. Thus, in such cases χ extends to a character $\widetilde{\chi} : G_L^{\mathrm{ab}} \to F^\times$ (carrying ϕ to 1, for example). But then $\widetilde{\chi}\sigma' : G_L^{\mathrm{ab}} \to H'(F)$ is a lift of σ that satisfies $\mathbf{P}$ since its I_L-restriction $\chi\sigma'|_{I_L}$ does. This settles the crystalline and semistable cases.

Now consider the deRham and Hodge–Tate cases, so χ is potentially crystalline and $\varphi(\chi)/\chi : I_L \to F^\times$ has finite order. By Lemma 6.3, there exists a character $\eta : I_L \to F^\times$ of finite order such that $\varphi(\chi)/\chi = \varphi(\eta)/\eta$. Hence, $\chi/\eta : I_L \to F^\times$ is invariant under c_ϕ and so extends to a character $G_L \to F^\times$ carrying ϕ to 1. It is harmless to replace χ with χ/η since a finite-order twist has no effect on the deRham or Hodge–Tate properties on I_L. Thus, we conclude as in the crystalline and semistable cases. ■

Consider the variant of Proposition 6.5 in which $Z := \ker(H' \twoheadrightarrow H)$ is merely of multiplicative type rather than a torus (e.g., a central isogeny). An additional hypothesis is required in the deRham and Hodge–Tate cases, as the following example shows. Consider a tame character $\psi : G_L \to \overline{\mathbf{Q}}_p^\times$. This is finitely ramified (by local class field theory), so it is deRham, and it admits an nth root on I_L for any $n > 0$ since I_L^{tame} is torsion-free. However, if ψ is ramified then it does not admit an nth root on W_L or G_L when n is divisible by $q - 1$. This provides counterexamples for lifting the Hodge–Tate property through the central isogenies $t^n : \mathbf{G}_m \to \mathbf{G}_m$ and $\mathrm{SL}_n \to \mathrm{PGL}_n$ for such n. But in these examples there is no lift even without the p-adic Hodge theory constraint. That turns out to be the only further obstruction when Z is permitted to be disconnected:

Corollary 6.7. *The lifting criterion in Proposition* 6.5 *remains valid if Z is merely assumed to be of multiplicative type, provided that when $\mathbf{P}$ is "deRham" or "Hodge–Tate" we assume ρ admits a lifting as a Γ-representation.*

Before proving this corollary, we make some observations. Theorem 6.2(2) provides a useful reformulation of the lifting criterion for finite Z in the semistable and crystalline cases (and geometric local class field theory shows that there is no analogue in the deRham or Hodge–Tate cases). Also, by Proposition 6.4, the additional hypothesis imposed in Corollary 6.7 in the deRham and Hodge–Tate cases is automatically satisfied when Z is a torus (and is necessary when Z is disconnected and $\Gamma = G_L$, as is easily seen via local class field theory).

Proof. Let us first grant the case of finite Z and settle the general case. Since $H'/Z^0 \to H$ has finite central kernel, by hypothesis there is a type-$\mathbf{P}$ lift $\rho : \Gamma \to (H'/Z^0)(F)$ of σ. By hypothesis there is a Hodge–Tate lift $\sigma' : I_L \to H'(F)$ of σ, so σ' mod Z^0 and $\rho|_{I_L}$ are related through twisting by a character $\chi : I_L \to (Z/Z^0)(F)$. But $Z \to Z/Z^0$ admits a section since Z is of multiplicative type, so we can lift χ to a character $I_L \to Z(F)$ of finite order. Thus, we may replace σ' with a finite-order twist so that it lifts $\rho|_{I_L}$, and this procedure preserves the Hodge–Tate property. Proposition 6.5 lifts ρ to a type-$\mathbf{P}$ representation $\Gamma \to H'(F)$.

Now we may and do assume that Z is finite. Let $Z \hookrightarrow \mathscr{T}$ be an inclusion into an F-torus, and form the central pushout as in (5.2). By hypothesis, $\rho|_{I_L}$ admits a Hodge–Tate lift to $H'(F)$, and hence to $\mathscr{H}'(F)$. Thus, Proposition 6.5 provides a type-$\mathbf{P}$ representation $\widetilde{\rho} : \Gamma \to \mathscr{H}'(F)$ lifting ρ. It is necessary and sufficient to prove that the composite character $\psi : \Gamma \to (\mathscr{T}/Z)(F)$ as in (5.3) has a type-$\mathbf{P}$ lift through the isogeny of tori $\mathscr{T} \to \mathscr{T}/Z$. The assumption that $\rho|_{I_L}$ admits a Hodge–Tate lift to $H'(F)$ implies that $\psi|_{I_L}$ admits a Hodge–Tate lift to $\mathscr{T}(F)$. By choosing compatible bases of the character groups of $\mathscr{T}$ and $\mathscr{T}/Z$ as in (5.4), we are reduced to showing that if a type-$\mathbf{P}$ character $\psi : \Gamma \to F^\times$ admits a Hodge–Tate nth root on I_L then it admits an nth root of type $\mathbf{P}$, provided that when $\mathbf{P}$ is "deRham" or "Hodge–Tate" there is an nth root on Γ. That is, we have reduced to the special case of lifting through the isogeny $t \mapsto t^n$ on $\mathbf{G}_m$.

The case $\Gamma = W_L$ easily reduces to $\Gamma = G_L$ (due to the structure of $W_L^{\rm ab}$ and $G_L^{\rm ab}$), and any Hodge–Tate character $I_L \to F^\times$ is potentially crystalline (Lemma 6.6). Since ψ is crystalline if and only if it is semistable, we just need to treat the crystalline cases for $\Gamma = G_L$ or I_L and the Hodge–Tate cases on G_L (when there exists an nth root on G_L in this latter case).

Consider a Hodge–Tate character $\psi : G_L \to F^\times$ such that $\psi|_{I_L} = \xi^n$ for some Hodge–Tate character $\xi : I_L \to F^\times$. The Hodge–Tate weights of ψ are divisible by n, so by (the proof of) Proposition 4.2, if ψ admits an nth root on G_L then this nth root is Hodge–Tate and can be chosen to be crystalline when ξ is crystalline.

For a crystalline character $\psi : I_L \to F^\times$ that admits a Hodge–Tate nth root, the argument near the end of the proof of Proposition 6.5 provides a descent $\psi_0 : G_L \to F^\times$ of ψ. Thus, the settled crystalline case on G_L provides a crystalline nth root of ψ. ■

Now we incorporate p-adic Hodge theory conditions into Theorem 5.5(2) and Corollary 5.7 when $F = \overline{\mathbf{Q}}_p$ and k is a number field. The following example (illustrating a phenomenon brought to my attention by F. Calegari) shows that an analogue of Theorem 5.5(2) using

p-adic Hodge theory conditions with $Z = \mathbf{G}_m$ requires additional hypotheses on ρ. Such an analogue seems quite deep, so in what follows we will only consider finite Z.

Example 6.8. Let k be a real quadratic field in which a fundamental unit ε satisfies $\mathrm{N}_{k/\mathbf{Q}}(\varepsilon) = -1$ (i.e., ε has opposite signs at the two real places), and assume it is unramified at 2 and 3. For instance, $k = \mathbf{Q}(\sqrt{5})$ and $\varepsilon = (1+\sqrt{5})/2$ (with 3 inert), or $k = \mathbf{Q}(\sqrt{13})$ and $\varepsilon = (3+\sqrt{13})/2$ (with 3 split). Let $p > 2$ be a prime split in k, with $\{v, v'\}$ the places of k over p. Let Ad^0 denote the 3-dimensional space of traceless matrices in Mat_2, viewed as a faithful representation for PGL_2 in the usual manner.

Let $L_0/\mathbf{Q}$ be a totally real cyclic extension of degree $p-1$ that is disjoint from $k/\mathbf{Q}$ and totally ramified at p with p-adic completion $\mathbf{Q}_p(\zeta_p)$. Let $L = k \otimes_{\mathbf{Q}} L_0$. By using induction of idele class characters, we will construct a representation $\rho : G_L \to \mathrm{PGL}_2(\overline{\mathbf{Q}}_p)$ that is unramified at the unique place $\widetilde{v}$ over v, crystalline at the unique place $\widetilde{v}'$ over v' with Hodge–Tate weights $-1, 0, 1$ on Ad^0, and has local restriction at $\widetilde{v}'$ admitting a crystalline lift to $\mathrm{GL}_2(\overline{\mathbf{Q}}_p)$, yet there is no representation $\rho' : G_L \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ lifting ρ that is Hodge–Tate at both $\widetilde{v}$ and $\widetilde{v}'$. (Recall that we require representations to be unramified at all but finitely many places.) Note that by Proposition 5.3 and Corollary 5.4 there exist representations ρ' lifting ρ unramified at any desired finite set of places where ρ is unramified, and we will arrange that all such ρ' are odd and absolutely irreducible. However, due to the construction using induction from a quadratic extension of k, ρ will be reducible on Ad^0 (as is necessary for ρ to fail to admit a lift ρ' that is Hodge–Tate at $\widetilde{v}$ and $\widetilde{v}'$).

Let k'/k be a quadratic extension obtained as the compositum of k with an imaginary quadratic field that is unramified at 2 and 3 and split at p. Thus, k' has as its roots of unity only ± 1, and $k'/\mathbf{Q}$ is totally split at p. Since $k(\sqrt{\varepsilon})$ and $k(\sqrt{-\varepsilon})$ each have a real place, whereas k' is totally complex, it follows that $\mathscr{O}_{k'}^\times = \pm\varepsilon^{\mathbf{Z}}$.

Let $\{w_1, w_2\}$ be the places of k' over v, and $\{w_1', w_2'\}$ be the places of k' over v'. We have canonically $\mathbf{Q}_p = k_v = k'_{w_j}$ and $\mathbf{Q}_p = k_{v'} = k'_{w_j'}$, with $\varepsilon_v \in k_v = \mathbf{Q}_p$ and $\varepsilon_{v'} \in k_{v'} = \mathbf{Q}_p$ satisfying $\varepsilon_{v'} = -1/\varepsilon_v$. The pro-$p$ group $(1+\mathfrak{m}_v) \times (1+\mathfrak{m}_{v'})$ meets $\mathscr{O}_k^\times$ in an infinite cyclic group $u^{\mathbf{Z}}$ for some u. The maximal pro-p torsion-free quotient of the idele class group $C_{k'}$ of k' contains as a finite-index open subgroup the maximal torsion-free quotient of $(\prod_{w|p}(1+\mathfrak{m}_w))/u^{\mathbf{Z}_p}$, where $1+\mathfrak{m}_w = 1+p\mathbf{Z}_p \simeq \mathbf{Z}_p$.

Using the canonical square root on $1+p\mathbf{Z}_p$, consider the character

$$\prod_{w|p}(1+\mathfrak{m}_w) \to 1+p\mathbf{Z}_p$$

defined by $(c_{w_1}, c_{w_2}, c_{w_1'}, c_{w_2'}) \mapsto \sqrt{c_{w_1'}/c_{w_2'}}$. This kills the diagonally embedded u, so it extends to a character $\psi : C_{k'} \to K^\times$ valued in some finite extension K of $\mathbf{Q}_p$ and unramified away from p. At the places of k' over v, the character ψ^2 is trivial on wild inertia; at the places w_1' and w_2' of k' over v', ψ^2 agrees on wild inertia with $\mathbf{Q}_p(-1)$ and $\mathbf{Q}_p(1)$ respectively. In particular, ψ is not Hodge–Tate at either w_j'.

The representation $\rho' = \mathrm{Ind}_{G_{k'}}^{G_k}(\psi) : G_k \to \mathrm{GL}_2(K) \subset \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ is clearly unramified at v and away from $p\infty$, and is odd at the real places. Also, ρ' is (absolutely) irreducible since ψ does not descend to a $\overline{\mathbf{Q}}_p^\times$-valued character of G_k (as all such characters factor through

$G_{\mathbf{Q}}$ near 1, due to the global unit ε, yet ψ is unramified over v and infinitely ramified over v'). Thus, $\rho'|_{G_L}$ is also absolutely irreducible (since L/k is cyclic) and odd at the real places of L. Moreover, the restriction of ρ' to $G_{k_{v'}}$ is a direct sum $\eta_1 \oplus \eta_2$ of $\overline{\mathbf{Q}}_p^\times$-valued characters that are trivial on tame inertia and have restriction to wild inertia $1 + \mathfrak{m}_{v'} = 1 + p\mathbf{Z}_p$ given by $c \mapsto \sqrt{c}$ and $c \mapsto 1/\sqrt{c}$. In particular, ρ' is not Hodge–Tate at v'.

For the natural quotient map $f : \mathrm{GL}_2 \to \mathrm{PGL}_2$ modulo the central torus $Z = \mathbf{G}_m$, the composite representation $\overline{\rho}' = f \circ \rho' : G_k \to \mathrm{PGL}_2(\overline{\mathbf{Q}}_p)$ computes the 3-dimensional adjoint representation on Ad^0. In this way, $\overline{\rho}'|_{G_{k_{v'}}}$ is identified with a direct sum of the 1-dimensional trivial representation and characters that are trivial on tame inertia and respectively agree on wild inertia with with $\mathbf{Q}_p(-1)$ and $\mathbf{Q}_p(1)$. We conclude that $\rho := \overline{\rho}'|_{G_L}$ is crystalline at the unique place $\widetilde{v}'$ of L over v' (with inertial restriction $\mathbf{Q}_p(-1) \oplus \mathbf{Q}_p(1) \oplus \mathbf{Q}_p$ on Ad^0) and it is unramified at $\widetilde{v}$. The representation $I_{v'} \to \mathrm{GL}_2(\mathbf{Q}_p)$ defined by $\mathbf{Q}_p(1) \oplus \mathbf{Q}_p$ has adjoint representation $\mathbf{Q}_p(1) \oplus \mathbf{Q}_p(-1) \oplus \mathbf{Q}_p$ on Ad^0, so $\rho|_{I_{\widetilde{v}'}}$ admits a Hodge–Tate (even crystalline) lift to $\mathrm{GL}_2(\overline{\mathbf{Q}}_p)$. Hence, $\rho|_{G_{L_{\widetilde{v}'}}}$ admits a crystalline lift to $\mathrm{GL}_2(\overline{\mathbf{Q}}_p)$, by Proposition 6.5.

Finally, we claim that $\rho : G_L \to \mathrm{PGL}_2(\overline{\mathbf{Q}}_p)$ has no lift $\widetilde{\rho} : G_L \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ that is Hodge–Tate at both p-adic places $\widetilde{v}$ and $\widetilde{v}'$. Indeed, the continuous lifts have the form $\chi\rho'|_{G_L}$ for $\chi : G_L \to \overline{\mathbf{Q}}_p^\times$, but ρ' is unramified at $\widetilde{v}$ and is not Hodge–Tate at $\widetilde{v}'$. Thus, if $\chi\rho'|_{G_L}$ is Hodge–Tate at both $\widetilde{v}$ and $\widetilde{v}'$ then χ is Hodge–Tate at $\widetilde{v}$ but not at $\widetilde{v}'$. The number field L is totally real and abelian over $\mathbf{Q}$, so its $\mathbf{Z}_p$-rank is 1 and hence its $\overline{\mathbf{Q}}_p^\times$-valued characters of infinite order are finite-order twists of p-adic powers of the p-adic cyclotomic character. Hence, the Hodge–Tate property for χ at $\widetilde{v}$ implies that some Tate twist of χ has finite order. This is inconsistent with χ not being Hodge–Tate at $\widetilde{v}'$.

In view of the preceding example, we now consider a central extension $1 \to Z \to H' \to H \to 1$ of linear algebraic groups over $F = \overline{\mathbf{Q}}_p$ with *finite* Z. Let $\rho : G_k \to H(F)$ be a representation. Choose finite disjoint sets S and T of places of k with no archimedean places in T, and assume that ρ is unramified at S and tame at T.

Proposition 6.9. *Using notation as above, let Σ be a set of p-adic places of k disjoint from $S \cup T$. Assume that for each $v \in \Sigma$ the restriction $\rho|_{I_v}$ satisfies a basic p-adic Hodge theory property $\mathbf{P}_v$ and admits a Hodge–Tate lift $\rho'_v : I_v \to H'(F)$. Let Σ_{sst} be the set of $v \in \Sigma$ for which $\mathbf{P}_v$ is either "semistable" or "crystalline".*

Let n denote the exponent of Z, and assume that $(k, \emptyset, n)$ is not in the special case. There exists a representation $\rho' : G_k \to H'(F)$ lifting ρ that is unramified at S, tame at T, and satisfies $\mathbf{P}_v$ at v for all $v \in \Sigma$ except for possibly when $(k, S \cup T \cup \Sigma_{\mathrm{sst}}, n)$ is in the special case, $S_k \neq \emptyset$ and $2^{s_k - 1} | \mathrm{ord}_v(2)$ for all $v \in S_k$.

Proof. We may and do assume that we are not in the exceptional situations described at the end of the proposition. Since $(k, \emptyset, n)$ is not in the special case, ρ lifts to some ρ' as a G_k-representation (by Theorem 5.5(1)). Hence, by Corollary 6.7 each $\rho|_{G_{k,v}}$ admits a lift $\rho'_v : G_{k,v} \to H'(F)$ that satisfies $\mathbf{P}_v$ for $v \in \Sigma$. Thus, by finiteness of Z, we can use Lemma 4.1 and Proposition 4.2 (with S there taken to be empty, and T there taken to be $S \cup T$ in the present circumstances) to carry over the proof of Theorem 5.5(2) incorporating the condition

$\mathbf{P}_v$ at each $v \in \Sigma$. (The divisibility condition on the Hodge–Tate weights in Proposition 4.2 is satisfied, due to the Hodge–Tate lifting hypothesis at each $v \in \Sigma$.) ■

Using Proposition 6.9 instead of Theorem 5.5(2), we can refine Corollary 5.7(2):

Corollary 6.10. *Let k be a number field, H a connected reductive $\overline{\mathbf{Q}}_p$-group, H' its semisimple derived group, and n the exponent of $Z_{H'}$. Assume that $(k, \emptyset, n)$ is not in the special case.*

Let $\rho : G_k \to H(F)$ and S, T, Σ be as in Proposition 6.9, and assume that there is no local obstruction to the existence of a central twist ρ' of ρ valued in $H'(\overline{\mathbf{Q}}_p)$. Let $\Sigma_{\rm sst}$ be the set of $v \in \Sigma$ for which the local property $\mathbf{P}_v$ is either "semistable" or "crystalline".

There exists an H'-valued central twist ρ' of ρ that is unramified at S, tame at T, and satisfies $\mathbf{P}_v$ at each $v \in \Sigma$ except for possibly when $(k, S \cup T \cup \Sigma_{\rm sst}, n)$ is in the special case, $S_k \neq \emptyset$, and $2^{s_k - 1} | \mathrm{ord}_v(2)$ for all $v \in S_k$.

Appendix A. The Grunwald–Wang theorem

For any $m \geq 2$, let $\eta_m \in \mathbf{Q}(\mu_{2^m})^+$ denote an element of the form $\zeta_{2^m} + \zeta_{2^m}^{-1}$ for some primitive 2^mth root of unity in $\mathbf{Q}(\mu_{2^m})$. (Note that $\eta_2 = 0$, and $\mathbf{Q}(\mu_{2^{m+1}})/\mathbf{Q}(\eta_m)$ is biquadratic with its three quadratic subfields respectively generated by $\sqrt{-1}$ and $\sqrt{\pm(2+\eta_m)}$ [AT, Ch. X, §1].) For a number field k, let s_k denote the maximal $s \geq 2$ such that k contains $\mathbf{Q}(\mu_{2^s})^+$. The elements $\eta_{s_k} \in k$ are not uniquely determined up to $\mathrm{Aut}(k/\mathbf{Q})$-conjugacy in general if $s_k > 2$. Any use of $\eta_{s_k} \in k$ will be independent of the choice made.

Let S be a finite (possibly empty) set of places of a global field k. The Grunwald-Wang theorem [AT, Ch. X, Thm. 1] computes the obstruction group for the local-to-global principle for nth roots in k when we ignore local information at S; i.e., the group

$$\ker(k^\times/(k^\times)^n \to \prod_{v \notin S} (k_v^\times)/(k_v^\times)^n) \simeq \text{Ш}_S^1(k, \mu_n).$$

It says that $\text{Ш}_S^1(k, \mu_n) = 1$ except when (k, S, n) satisfies the following condition.

Definition A.1. The triple (k, S, n) is in the *special case* when

- k is a number field,
- $k(\mu_{2^{s_k+1}})/k$ is biquadratic (equivalently, -1 and $\pm(2+\eta_{s_k})$ are non-squares in k),
- $\mathrm{ord}_2(n) > s_k$ (so $8|n$),
- S contains the set S_k of all 2-adic places v of k at which $k_v(\mu_{2^{s_k+1}})/k_v$ is biquadratic (equivalently, -1 and $\pm(2+\eta_{s_k})$ are non-squares in k_v).

In this definition, note that: n only intervenes through the largeness of $\mathrm{ord}_2(n)$, the places of S not in S_k do not intervene, and when $S = \emptyset$ the final condition says exactly that $S_k = \emptyset$. Also, if 2 is unramified in k then $s_k = 2$ and S_k is the set of all 2-adic places of k.

The Grunwald–Wang theorem also says that in the special case $\text{Ш}_S^1(k, \mu_n)$ has order 2, with nontrivial element equal to the Kummer class of

$$a_{k,n} = (2 + \eta_{s_k})^{n/2} \in (k^\times)^{n/2}$$

(for any *fixed* choice of $\eta_{s_k} \in k$). Note that $a_{k,n}$ is totally positive at all real places and is a unit away from 2-adic places (since $2 + \eta_m = \eta_{m+1}^2$ and $\eta_{m+1} = \zeta_{2^{m+1}}^{-1}(1 + \zeta_{2^m})$, where $1 + \zeta_{2^m}$ has minimal polynomial $(X-1)^{2^{m-1}} + 1$ over $\mathbf{Q}$ with constant term 2 since $m \geq 2$).

Remark A.2. Consider the extension $k(\mu_{2^{s_k+1}})/k$ that is biquadratic or quadratic. In the biquadratic case its quadratic subfields are $k(\sqrt{2+\eta_{s_k}})$, $k(\mu_{2^{s_k}}) = k(i)$, and $k(\sqrt{-2-\eta_{s_k}})$, whereas in the quadratic case one of these three fields equals k (see [AT, Ch. X, §1]). This global extension induces an extension of each k_v, with local degree at most 2 when $v \notin S_k$: for $v|\infty$ it is obvious, for $v \nmid 2\infty$ it follows from unramifiedness at v, and for $v|2$ it follows from the hypothesis $v \notin S_k$. Thus, if $v \notin S_k$ then k_v contains one of $\sqrt{2+\eta_{s_k}}$, $1+\zeta_{2^{s_k}}$, or $\sqrt{-2-\eta_{s_k}}$, each of which has nth power $a_{k,n}$. This makes explicit that $a_{k,n}$ is locally an nth power at every $v \notin S_k$. It is *not* locally an nth power at any $v \in S_k$; see [AT, Ch. X, §1].

We need a standard application [AT, Ch. X, Thm. 5] of the Grunwald–Wang theorem:

Proposition A.3. *Let S be a finite set of places of k, and $\chi_v : G_{k_v}^{\times} \to \mathbf{C}^{\times}$ a character with finite order n_v for each $v \in S$. There exists a finite-order character $\chi : G_k \to \mathbf{C}^{\times}$ inducing each χ_v such that the order of χ is equal to the least common multiple n of the n_v* except *when (k, S, n) is in the special case with $\prod_{v\in S_k} \chi_v(a_{k,n}) \neq 1$. In these latter cases, the product is -1 and χ exists with order $2n$ but not with order n.*

Implicit in the finite product obstruction above is the convention that $\prod_{v\in S_k} \chi_v(a_{k,n}) = 1$ if S_k is empty. That is, if $S_k = \emptyset$ then χ can always be found with order n, even when $(k, \emptyset, n)$ is in the special case.

Appendix B. Local algebraicity and crystalline representations

Let L be a finite extension of $\mathbf{Q}_p$ and $\psi : G_L^{\mathrm{ab}} \to K^{\times}$ a continuous character, with K a finite extension of $\mathbf{Q}_p$. Upon composing with the local Artin map $r_L : L^{\times} \to G_L^{\mathrm{ab}}$ (using either normalization) we get a continuous composite map $L^{\times} \to K^{\times}$ whose source and target are respectively identified (as topological groups) with the groups of $\mathbf{Q}_p$-points of the $\mathbf{Q}_p$-tori $\underline{L}^{\times} = \mathrm{R}_{L/\mathbf{Q}_p}(\mathbf{G}_m)$ and $\underline{K}^{\times} = \mathrm{R}_{K/\mathbf{Q}_p}(\mathbf{G}_m)$ defined by Weil restriction of scalars.

Definition B.1. The representation ψ is *locally algebraic* if there exists a (necessarily unique) $\mathbf{Q}_p$-homomorphism $\underline{L}^{\times} \to \underline{K}^{\times}$ whose restriction to $\mathbf{Q}_p$-points agrees with $\psi \circ r_L$ near 1.

Since ψ is semisimple when viewed as a $\mathbf{Q}_p$-linear abelian representative of G_L (as we see by applying $(\cdot) \otimes_{\mathbf{Q}_p} \overline{\mathbf{Q}}_p$), it follows from a result of Tate (see [S6, III, A6, Cor. 2]) that ψ is locally algebraic if and only if its underlying $\mathbf{Q}_p$-linear representation is Hodge–Tate. In this appendix we address two useful refinements for which there do not seem to be suitable references in the literature: computing the Hodge–Tate weights for a Hodge–Tate ψ without requiring L to be "sufficiently large" ([S6, III, A.5] settles the case when L splits K over $\mathbf{Q}_p$), and relating the crystalline condition to local algebraicity.

Define the local Artin map $r_L : L^{\times} \to G_L^{\mathrm{ab}}$ using the arithmetic normalization. Assuming ψ is Hodge–Tate, we seek a formula for its Hodge–Tate weights in terms of K and L. Choose an auxiliary finite extension K'/K that splits L over $\mathbf{Q}_p$ (i.e., every factor field of the K'-algebra $K' \otimes_{\mathbf{Q}_p} L$ is equal to K'). By local algebaicity of ψ, near 1 the composite homomorphism

$$L^{\times} \xrightarrow{r_L} G_L^{\mathrm{ab}} \xrightarrow{\psi} K^{\times} \hookrightarrow K'^{\times}.$$

agrees with

$$x \mapsto \prod_{\tau \in \mathrm{Hom}_{\mathbf{Q}_p}(L,K')} \tau(x)^{-n_\tau}$$

for integers n_τ indexed by the set of $\mathbf{Q}_p$-embeddings of fields $\tau : L \to K'$. For the $\mathbf{Q}_p$-linear representation $V = K$ of G_L via ψ, the semi-linear representation $\mathbf{C}_L \otimes_{\mathbf{Q}_p} V$ of G_L naturally decomposes into G_L-stable $\mathbf{C}_L$-lines V_σ indexed by the $\mathbf{Q}_p$-embeddings $\sigma : K \to \mathbf{C}_L$, and $V_\sigma = \mathbf{C}_L(w_\sigma)$ for some $w_\sigma \in \mathbf{Z}$ (i.e., $\{w_\sigma\}$ is the set of Hodge–Tate weights of ψ). If $\psi' : G_L^{\mathrm{ab}} \to K^\times$ is a second Hodge–Tate homomorphism (with underlying $\mathbf{Q}_p$-linear representation V'), and its Hodge–Tate weights are $\{w'_\sigma\}$, then the product homomorphism $\psi\psi' : G_L^{\mathrm{ab}} \to K^\times$ is Hodge–Tate with weights $\{w_\sigma + w'_\sigma\}$ because $(V \otimes_K V')_\sigma = V_\sigma \otimes_{\mathbf{C}_L} V'_\sigma$ as $\mathbf{C}_L$-semilinear representations of G_L.

We shall compute the integers w_σ in terms of the factor fields of $K \otimes_{\mathbf{Q}_p} L$ viewed as both a K-algebra and an L-algebra. The first step is to encode in terms of the n_τ's the property that $\prod_\tau \tau(x)^{-n_\tau}$ is valued in the subfield $K \subset K'$ for $x \in L^\times$. (It is equivalent to require the same only for $x \in L^\times$ near 1). This is given by the following lemma.

Lemma B.2. *For each $\mathbf{Q}_p$-embedding $\tau : L \to K'$, let n_τ be an integer and let $1 \otimes \tau$ denote the unique K-algebra map $K \otimes_{\mathbf{Q}_p} L \to K'$ extending τ (i.e., $a \otimes b \mapsto a\tau(b)$). Let $\prod F_i$ denote the decomposition of $K \otimes_{\mathbf{Q}_p} L$ into a finite product of fields, and let $q_i : K \otimes_{\mathbf{Q}_p} L \to F_i$ denote the canonical quotient map.*

The product homomorphism $L^\times \to K'^\times$ defined by $x \mapsto \prod_\tau \tau(x)^{-n_\tau}$ is valued in $K^\times$ if and only if n_τ only depends on the index i for which $1 \otimes \tau$ factors through q_i.

Proof. We may assume that K' is finite Galois over $\mathbf{Q}_p$, and then invariance under $\mathrm{Gal}(K'/K)$ encodes the property of being valued in K. It is straightforward to check that such invariance is equivalent to the property that n_τ only depends on the factor field of $K \otimes_{\mathbf{Q}_p} L$ through which $1 \otimes \tau$ factors. ■

By Lemma B.2, for each factor field F_i of $K \otimes_{\mathbf{Q}_p} L$ we may define n_i to be the common value of the integers n_τ for those τ such that $1 \otimes \tau$ factors through q_i. In more intrinsic terms (i.e., without mentioning K'/K), $\psi \circ r_L : L^\times \to K^\times$ agrees near 1 with the homomorphism

$$x \mapsto \prod_i \mathrm{N}_{F_i/K}(q_i(1 \otimes x))^{-n_i}. \tag{B.1}$$

Here is a formula for the w_σ's in terms of the n_i's:

Proposition B.3. *With notation as above, for each $\mathbf{Q}_p$-embedding $\sigma : K \to \mathbf{C}_L$ the Hodge–Tate weight w_σ is equal to n_i for the unique i such that the L-algebra map $\sigma \otimes 1 : K \otimes_{\mathbf{Q}_p} L \to \mathbf{C}_L$ extending σ factors through q_i.*

Proof. For each i, we may certainly construct a continuous homomorphism $\psi_i : G_L^{\mathrm{ab}} \to K^\times$ such that $\psi_i \circ r_L$ agrees near 1 with $x \mapsto \mathrm{N}_{F_i/K}(q_i(1 \otimes x))^{-1}$. By (B.1) and multiplicativity considerations, $(\prod \psi_i^{n_i})/\psi$ has all Hodge–Tate weights equal to 0. Thus, it suffices to treat each ψ_i. In other words, we choose a finite extension $F/\mathbf{Q}_p$ containing K and L as subfields such that $KL = F$ (i.e., a factor field F of $K \otimes_{\mathbf{Q}_p} L$) and may assume that $\psi \circ r_L : L^\times \to K^\times$ is given near $1 \in L^\times$ by $x \mapsto \mathrm{N}_{F/K}(x)^{-1}$. In this case we wish to prove that $w_\sigma = 1$ when the L-algebra map $\sigma \otimes 1 : K \otimes_{\mathbf{Q}_p} L \to \mathbf{C}_L$ factors through the quotient map $K \otimes_{\mathbf{Q}_p} L \twoheadrightarrow F$ defined by multiplication inside F and that $w_\sigma = 0$ otherwise.

Since F and K' are both extensions of K, it makes sense to form $F \otimes_K K'$. Let F' be a finite Galois extension of a factor field of $F \otimes_K K'$, so F' contains K and L as subfields (inside F) with compositum F. In particular, $\mathrm{Gal}(F'/K) \cap \mathrm{Gal}(F'/L) = \mathrm{Gal}(F'/F)$. Since F' contains K', can compute the same integers n_τ using F' rather than K'.

To compute the w_σ's, we first pick an L-embedding $f : F' \to \mathbf{C}_L$ and use it to identify $\mathbf{C}_L$ with a completed algebraic closure $\mathbf{C}_{F'}$ of F'. (In particular, f "computes" the restriction map $G_{F'}^{\mathrm{ab}} \to G_L^{\mathrm{ab}}$.) Letting $j : K \to F'$ be the canonical inclusion and $\mathrm{res}_{F'/L} : G_{F'}^{\mathrm{ab}} \to G_L^{\mathrm{ab}}$ be the natural restriction map (which has no effect on Hodge–Tate weights), the homomorphism $\chi' = ((j \circ \psi) \circ \mathrm{res}_{F'/L}) \circ r_{F'} : F'^\times \to F'^\times$ is given near 1 by

$$x \mapsto \mathrm{N}_{F/K}(\mathrm{N}_{F'/L}(x))^{-1}.$$

It is convenient to raise this to the $[F' : F]$th power (at the cost of dividing Hodge–Tate weights by $[F' : F]$ at the end) because $\mathrm{N}_{F/K}(y)^{[F':F]}$ is given by a product $\prod_g g(y)$ over g varying through the entire Galois group $\mathrm{Gal}(F'/K)$ rather than through the coset space $\mathrm{Gal}(F'/K)/\mathrm{Gal}(F'/F) = \mathrm{Hom}_K(F, F')$ (as when computing $\mathrm{N}_{F/K}$).

The $[F' : F]$th power of χ' is given near $1 \in F'^\times$ by $x \mapsto \prod_{(g,h)} g(h(x))^{-1}$, where g varies through $\mathrm{Gal}(F'/K)$ and h varies through $\mathrm{Gal}(F'/L)$. This product expression motivates us to consider the problem of computing the Hodge–Tate weights for a locally algebraic representation $\rho : G_{F'}^{\mathrm{ab}} \to F'^\times$ such that $\rho \circ r_{F'} : F'^\times \to F'^\times$ agrees near 1 with $x \mapsto \tau(x)^{-1}$ for $\tau \in \mathrm{Gal}(F'/\mathbf{Q}_p)$. The semilinear extension $\rho \otimes_{\mathbf{Q}_p} \mathbf{C}_{F'}$ canonically decomposes into a direct product of lines indexed by $g \in \mathrm{Gal}(F'/\mathbf{Q}_p)$, and by [S6, III, A.5, Thm. 2] (especially Lemma 2 in its proof) the gth line has Hodge–Tate weight 0 except for $g = \tau^{-1}$, which has weight 1 (due to our arithmetic normalization of the local Artin map).

Note that every $\mathbf{Q}_p$-embedding $\iota : F' \to \mathbf{C}_L$ lands in the subfield $f(F')$ and so may be identified with an element $\tau_\iota \in \mathrm{Gal}(F'/\mathbf{Q}_p)$. Thus, for each $\mathbf{Q}_p$-embedding $\iota : F' \to \mathbf{C}_{F'} = \mathbf{C}_L$ the associated Hodge–Tate weight for $(j \circ \psi)^{[F':F]} : G_L^{\mathrm{ab}} \to F'^\times$ is equal to the number of pairs (g, h) such that $gh = \tau_\iota^{-1}$. There is a natural right action by $\mathrm{Gal}(F'/F)$ on the set of such pairs via $(g, h).\gamma = (g\gamma, \gamma^{-1}h)$, and this is simply transitive when the set is non-empty because $\mathrm{Gal}(F'/K) \cap \mathrm{Gal}(F'/L) = \mathrm{Gal}(F'/F)$ (as $F = LK$ inside F'). Hence, these Hodge–Tate weights are either 0 or $[F' : F]$.

We conclude that $j \circ \psi$ has all Hodge–Tate weights equal to 0 or 1, and 1 occurs precisely for the $\mathbf{Q}_p$-embeddings $\iota : F' \to \mathbf{C}_L$ whose associated $\mathbf{Q}_p$-automorphism $\tau = \tau_\iota$ of F' admits the form $\tau = (gh)^{-1} = h^{-1}g^{-1}$ for some $g \in \mathrm{Gal}(F'/K)$ and $h \in \mathrm{Gal}(F'/L)$. There are $\mathbf{Q}_p$-embeddings ι extending each $\mathbf{Q}_p$-embedding $\sigma : K \to \mathbf{C}_L$, and the condition "$\tau_\iota|_K = \sigma$" is invariant under right multiplication on τ_ι by $\mathrm{Gal}(F'/K)$. Thus, $w_\sigma \in \{0, 1\}$ for all σ, and $w_\sigma = 1$ if and only if $\sigma : K \to f(F') \subset \mathbf{C}_L$ lifts to an automorphism $h^{-1} \in \mathrm{Gal}(F'/L)$.

It remains to show that the image of $\mathrm{Gal}(F'/L)$ under the surjective restriction map $\mathrm{Gal}(F'/\mathbf{Q}_p) \twoheadrightarrow \mathrm{Hom}_{\mathbf{Q}_p}(K, F')$ consists of those σ for which the unique L-algebra extension $K \otimes_{\mathbf{Q}_p} L \to F'$ factors through the natural quotient map $q : K \otimes_{\mathbf{Q}_p} L \twoheadrightarrow F$ onto the composite subfield $F = KL \subset F'$. This is a general Galois theory fact, and it holds because the natural map $K \otimes_{\mathbf{Q}_p} L \to F'$ induced by multiplication inside F' factors through q. ■

Crystalline locally algebraic abelian semisimple representations are given by:

Proposition B.4. *Let L and K be finite extensions of $\mathbf{Q}_p$, and let $r_L : L^\times \to G_L^{\mathrm{ab}}$ be the local Artin map with arithmetic normalization. Let $\psi : G_L^{\mathrm{ab}} \to \mathscr{O}_K^\times \subset K^\times$ be a continuous homomorphism. Let V be the $\mathbf{Q}_p[G_L^{\mathrm{ab}}]$-module underlying a 1-dimensional K-vector space endowed with a K-linear action by G_L^{ab} via ψ.*

(i) *The representation space V is crystalline if and only if there exists a homomorphism of $\mathbf{Q}_p$-tori $\chi : \underline{L}^\times \to \underline{K}^\times$ such that $\psi \circ r_L$ and χ (on $\mathbf{Q}_p$-points) coincide on $\mathscr{O}_L^\times$.*

(ii) *Assume that the condition in* (i) *is satisfied. Let a denote the residual degree of L over $\mathbf{Q}_p$. The filtered ϕ-module $D_{\mathrm{cris}}(V)$ over L covariantly attached to the crystalline representation V is free of rank 1 over $K \otimes_{\mathbf{Q}_p} L_0$ and its L_0-linear endomorphism ϕ^a is given by the action of the product $\psi(r_L(\pi_L))^{-1} \cdot \chi(\pi_F) \in K^\times$, where $\pi_L \in \mathscr{O}_L$ is any uniformizer and χ is as in* (i).

The interested reader will easily check (akin to the proof of Lemma 4.1) that the equivalence in (i) implies a more general equivalence between the crystalline condition and algebraicity on $\mathscr{O}_L^\times$ when ψ is allowed to be any abelian semi-simple linear representation of G_L on a finite-dimensional $\mathbf{Q}_p$-vector space. This general equivalence goes back to [S4, §2.3, Cor. 2], whose proof rests on work of Fontaine using an early version of the formalism of p-adic Hodge theory. Since (ii) is not directly addressed in [S4], for the convenience of the reader we now give a proof of both parts of Proposition B.4.

Proof. First assume that V is crystalline, so it is Hodge–Tate. The 1-dimensionality over K implies that V is semi-simple as a $\mathbf{Q}_p$-representation space of G_L, so it is also semisimple on the normal inertia subgroup. Hence, by [S6, III, A.7] (and the initial hypotheses in [S6, III, A.3]), there is a finite extension L'/L in $\overline{L}$ splitting $K/\mathbf{Q}_p$ such that $\psi|_{\mathrm{Gal}(\overline{L}/L')}$ is locally algebraic. Let $\chi' : \underline{L'}^\times \to \underline{K}^\times$ be the unique map of $\mathbf{Q}_p$-tori such that on $\mathbf{Q}_p$-points it agrees with $\psi \circ r_L \circ \mathrm{N}_{L'/L}$ near $1 \in L'^\times$. The norm map $\mathrm{N}_{L'/L} : \underline{L'}^\times \to \underline{L}^\times$ is a surjection of $\mathbf{Q}_p$-tori with connected kernel, so we may use $\mathbf{Q}_p$-points near 1 to infer that χ' kills the torus $\ker(\mathrm{N}_{L'/L})$. Hence, $\chi' = \chi \circ \mathrm{N}_{L'/L}$ for a unique $\chi : \underline{L}^\times \to \underline{K}^\times$, so $\psi \circ r_L$ and χ agree near $1 \in L^\times$; in particular, ψ is locally algebraic.

Obviously $\chi|_{\mathscr{O}_L^\times}$ can be extended to an $\mathscr{O}_K^\times$-valued Galois character θ_χ of G_L, and we claim that such a character is crystalline. (The choice of extension θ_χ does not matter, since the crystalline property only depends on the inertial restriction.) Assuming this property holds, upon choosing θ_χ we get that $\psi \cdot \theta_\chi^{-1}$ is a crystalline representation of G_L with finite image on inertia. All such representations are unramified, since we can perform finite Galois descent on the filtered ϕ-module side (relative to a finite extension of $(L^{\mathrm{un}})^\wedge$ which splits the Galois representation) and then apply the Dieudonné–Manin classification of isocrystals for an algebraically closed residue field to infer that $D_{\mathrm{cris}}(\psi \cdot \theta_\chi^{-1}|_{I_L})$ as a filtered ϕ-module has trivial filtration and is isoclinic of slope 0. This would give that $\psi \circ r_L$ and χ coincide on $\mathscr{O}_L^\times$, assuming θ_χ is crystalline.

We see that to prove (i) it remains to show that for any $\mathbf{Q}_p$-homomorphism $\chi : \underline{L}^\times \to \underline{K}^\times$, if its $\mathscr{O}_L^\times$-restriction on $\mathbf{Q}_p$-points is extended to an $\mathscr{O}_K^\times$-valued Galois character θ_χ of L then θ_χ is crystalline. It is harmless to increase the scalar field K such that it splits $L/\mathbf{Q}_p$, so a basis of the $\mathbf{Z}$-module of such χ's consists of the maps $[\tau] : \underline{L}^\times \to \underline{K}^\times$ induced on A-points by $\tau \otimes 1 : (L \otimes_{\mathbf{Q}_p} A)^\times \to (K \otimes_{\mathbf{Q}_p} A)^\times$ for all $\mathbf{Q}_p$-algebras A, where τ varies through the $\mathbf{Q}_p$-embeddings of L into K. It is therefore enough to treat the case $\chi = [\tau]^{-1}$ for some τ, in which case the $\mathbf{Q}_p$-representation space on inertia is the scalar extension by $\tau : L \to K$ of the inertial restriction of any Galois character $\psi : G_L^{\rm ab} \to \mathscr{O}_L^\times$ such that $(\psi \circ r_L)|_{\mathscr{O}_L^\times}$ is inversion. By [S6, III, A.4] and our choice of the arithmetic normalization of local class field theory, examples of such Galois characters ψ arise from Lubin–Tate formal groups over $\mathscr{O}_L$, which are p-divisible groups and hence are crystalline. This proves (i).

For the proof of (ii), it is convenient first to compute $D_{\rm cris}(V)$ in a special case:

Example B.5. Let κ denote the finite residue field of L, with size q_L, and let $L_0 := W(\kappa)[1/p]$ be the maximal unramified subfield of L. Assume that ψ arises as the generic fiber of a p-divisible group Γ over $\mathscr{O}_L$, so ψ is crystalline and $D_{\rm cris}(\psi) := (\psi \otimes_{\mathbf{Q}_p} B_{\rm cris,L})^{{\rm Gal}(\overline{L}/L)}$ is a $K \otimes_{\mathbf{Q}_p} L_0$-module equipped with a structure of K-linear filtered ϕ-module over L. The compatibility of the $K \otimes_{\mathbf{Q}_p} L_0$-module structure with the ϕ-operator forces $D_{\rm cris}(\psi)$ to have a nonzero factor module over each factor field of $K \otimes_{\mathbf{Q}_p} L_0$ and hence be invertible as a $K \otimes_{\mathbf{Q}_p} L_0$-module for $\mathbf{Q}_p$-dimension reasons.

Let Γ_0 denote the special fiber over κ, and $\mathbf{D}(\Gamma_0)$ denote its (contravariant) Dieudonné module. The q_L-Frobenius on Γ_0 is the action of some $\lambda \in K^\times$ in the isogeny category of p-divisible groups over κ. By [F1, 6.6] there is a natural L_0-linear ϕ-compatible isomorphism

$$\eta_\Gamma : {\rm Hom}_{L_0}(\mathbf{D}(\Gamma_0)[1/p], L_0) \simeq D_{\rm cris}(\psi), \tag{B.2}$$

with ϕ acting Frobenius-semilinearly on the L_0-linear dual of $\mathbf{D}(\Gamma_0)[1/p]$ in the usual manner (sending a functional f to $\sigma \circ f \circ \phi_{\mathbf{D}(\Gamma_0)}^{-1}$, where σ is the absolute Frobenius automorphism of L_0). Naturality forces η_Γ to be $K \otimes_{\mathbf{Q}_p} L_0$-linear, so if $p^a = q_L$ then $\sigma^a = {\rm id}$ and the L_0-linear ϕ^a on $D_{\rm cris}(\psi)$ has to be multiplication by $1/\lambda$.

Returning to the proof of (ii), the invertibility of $D_{\rm cris}(V)$ as a $K \otimes_{\mathbf{Q}_p} L_0$-module follows from a consideration of the ϕ-operator, exactly as in Example B.5. Now we may increase K so that it splits $L/\mathbf{Q}_p$, and the proof of (i) shows that in such cases ψ is a product of $\mathscr{O}_L^\times$-valued Lubin–Tate characters of G_L (viewed with values in $\mathscr{O}_K^\times$ via $\mathbf{Q}_p$-embeddings $\tau : L \to K$). The tensor-compatibility of $D_{\rm cris}$ (using the coefficient field K) and the multiplicativity of the proposed formula for ϕ^a thereby reduces us to checking the special case when $K = L$ and ψ is the Lubin–Tate character $G_L^{\rm ab} \to \mathscr{O}_L^\times$ associated to a choice of uniformizer π_L. Let κ denote the residue field of L, and q_L its size.

We have $\psi(r_L(\pi_L)) = 1$, and the associated algebraic character $\chi : L^\times \to L^\times$ is inversion. Thus, by using the compatibility (B.2) of Dieudonné modules and $D_{\rm cris}$, the inversion at the end of Example B.5 cancels out and we are reduced to checking that if Γ_{π_L} is the Lubin–Tate p-divisible group associated to π_L then its reduction over κ has q_L-Frobenius endomorphism induced by π_L. This is the property that uniquely characterizes Γ_{π_L}. ∎

References

[AT] E. Artin, J. Tate, *Class field theory*, AMS-Chelsea, 2009.

[B] A. Borel, *Linear algebraic groups* (2nd ed.), Springer–Verlag, New York, 1991.

[KR] C. Khare, R. Ramakrishna, *Finiteness of Selmer groups and deformation rings*, Inv. Math. **154** (2003), no. 1, 179–198.

[F1] J-M. Fontaine, *Sur certains types de représentations p-adiques du groupe de Galois d'un corps local; construction d'un anneau de Barsotti–Tate*, Annals of Math. **115**, 1982, 529–577.

[F2] J-M. Fontaine, "Le corps des périodes p-adiques" in *Périodes p-adiques*, Astérisque 223 (1994), 59–111.

[F3] J-M. Fontaine, "Représentations p-adiques semi-stables" in *Périodes p-adiques*, Astérisque 223 (1994), 113–184.

[NSW] J. Neurkirch, A. Schmidt, K. Winberg, *Cohomology of number fields*, Springer-Verlag, 2000.

[S1] J-P. Serre, *Sur les corps locaux à corps résiduel algébriquement clos*, Bull. Soc. Math. de France **89** (1961), 105–154.

[S2] J-P. Serre, *Groupes proalgébriques*, Publ. Math. IHES **7** (1960), 5–68.

[S3] J-P. Serre, "Modular forms of weight one and Galois representations", in *Algebraic number theory: L-functions and Galois properties* (Prof. Symp., Univ. Durham, 1975), London, 1977, 193–268.

[S4] J-P. Serre, "Groupes algébriques associés aux modules du Hodge–Tate", in *Journées de Géométrie Algébrique de Rennes*, Astérisque **65** (1979), 155–188.

[S5] J-P. Serre, *Galois cohomology*, Springer–Verlag, New York, 1997.

[S6] J-P. Serre, *Abelian ℓ-adic representations and elliptic curves* (2nd ed.), A.K. Peters, 1998.

[T] J. Tate, "Number-theoretic background" in *Automorphic forms, representations, and L-functions*, Proc. of Symp. in Pure Math. AMS **33** (Vol. 2), AMS, 1977.

[W1] J-P. Wintenberger, *Relèvement selon une isogénie de systèmes p-adiques de représentations galoisiennes associés aux motifs*, Inv. Math **120** (1995), 215–240.

[W2] J-P. Wintenberger, *Propriétés du groupe Tannakien des structures de Hodge p-adiques et torseur entre cohomologies cristalline et étale*, Annales Fourier **47** (1997), 1289–1334.

Department of Mathematics, Stanford University, Building 380, Sloan Hall, Stanford, CA 94305 USA

E-mail address: conrad@math.stanford.edu