# Five Stories for Richard

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Richard Stanley writes with a clarity and originality that makes those of us in his orbit happy we can appreciate and apply his mathematics. One thing missing (for me) is the stories that gave rise to his questions, the stories that relate his discoveries to the rest of mathematics and its applications.

I mostly work on problems that start in an application. Here is an example, leading to my favorite story about stories. In studying the optimal strategy in a game, I needed many random permutations of 52 cards. The usual method of choosing a permutation on the computer starts with $n$ things in order, then one picks a random number from 1 to $n$ — say 17 — and transposes 1 and 17. Next one picks a random number from 2 to $n$ and transposes these, and so forth, finishing with a random number from $n-1$ to $n$. This generates all $n!$ permutations uniformly. When our simulations were done (comprising many hours of CPU time on a big machine), the numbers "looked funny." Something was wrong. After two days of checking thousands of lines of code, I asked "How did you choose the permutations?" The programmer said, "Oh yes, you told me that fussy thing, 'random with 1, etc' but I made it more random, with 100 transpositions of $(i, j), 1 \leq i, j \leq n$." I asked for the work to be rerun. She went to her boss and to her boss' boss who each told me, essentially, "You mathematicians are crazy; 100 random transpositions has to be enough to mix up 52 cards." So, I really wanted to know the answer to the question of how many transpositions will randomize 52 cards. Eventually, Mehrdad Shahshahani and I figured it out [11]: the answer is about $\frac{1}{2}n \log n$. More carefully, the refined estimates in [18] show that for $l = \frac{1}{2}(n \log n + c)$, $\|Q^{*l} - u\|_{\mathrm{TV}} \leq e^{-c}$. Set the right side $= 1/100$ and solve for $c$; this gives about 400 transpositions needed to randomize 52 cards.

Our method of proof used character theory in a novel way, and a small subject started [5]. I gave a talk on this at Harvard's math department and my friend Barry Mazur came up afterward and said, "That's great, and you know, you can use the same ideas to work things out for other groups: try $GL_n(F_q)$." I was puzzled. Why would I want to do that? Explaining my puzzlement I said, "Sure, but what's the story?" This puzzled Barry, who didn't understand my need for an application. When he did he said, "I see; you don't understand that in mathematics, somebody else makes up the story."

The author is supported in part by NSF grants DMS 1208775 and DMS 1208787.

I know that Richard appreciates motivation. He's even coined a term for my talents in this direction, occasionally asking me to try *persification* to find a story for one of his theorems; I think it makes both of us happy.

I've chosen five stories that relate Richard's theorems to an application. I'm sure there are hundreds more examples, at least one for every reader of this paper.

## 1. Story One: P-partitions and shuffling cards

P-partitions are a unifying machine that Richard built in his thesis [20]. He tells a cleaned-up version in [30, Chap. 3]. This is vintage Richard: "just the facts." It begins with a partially ordered set $\mathcal{P}$ on $[n]$. A $\mathcal{P}$-partition is a function $f : \mathcal{P} \to \{0, 1, 2, \dots\}$ such that if $i <_{\mathcal{P}} j$ then $f(i) \leq f(j)$ and if $i <_{\mathcal{P}} j$ with $i > j$, then $f(i) < f(j)$. Richard takes this and develops its enumerative theory, ties it with a rich variety of classical enumerative mathematics (particularly by MacMahon) and finds lots of new identities. The larger world of combinatorics haven't particularly warmed to this notion but see [13, 16] and their references.

His enumerative theory relates to descents: a permutation $\sigma$ in $S_n$ has a descent at $i$ if $\sigma(i) > \sigma(i+1)$, $1 \leq i \leq n-1$. Let $d(\sigma)$ equal the number of descents. A basic step in P-partition theory is:

Fact. Let $\mathcal{P}$ be the linear order on $[n]$ induced by $\sigma \in S_n$. Let $\Omega_{\mathcal{P}}(m)$ be the number of $\mathcal{P}$-partitions $f : \mathcal{P} \to [m]$. Then

$$\Omega_{\mathcal{P}}(m) = \binom{m+n-1-d(\sigma)}{n}. \tag{1.1}$$

This gives results for the enumerative theory of a general partial order $\mathcal{L}$ via the fundamental theorem of P-partitions: for any $\mathcal{P}$, let $\mathcal{L}(\mathcal{P})$ be all linear extensions, then

$$\Omega_{\mathcal{P}}(m) = \sum_{\sigma \in \mathcal{L}(\mathcal{P})} \binom{m+n-1-d(\sigma)}{n}. \tag{1.2}$$

There is a surprising connection to the ordinary method of riffle shuffling cards! Consider a deck of $n$ cards, originally in order $1, 2, \dots, n$. An $m$-shuffle results from cutting the deck into $m$ piles (by a multinomial distribution) and riffle shuffling the piles by dropping cards sequentially with probability proportional to packet size. The chance that this results in the permutation $\sigma$ is denoted $Q_m(\sigma)$. Repeated shuffles reduce to this case because $Q_m * Q_{m'} = Q_{mm'}$. In work with Dave Bayer [2], we showed

$$Q_m(\sigma) = \frac{\binom{m+n-r(\sigma)}{n}}{m^n} \tag{1.3}$$

with $r(\sigma) = d(\sigma^{-1}) + 1$, the number of rising sequences for inverse riffle shuffles. Thus $Q_m(\sigma^{-1}) = \Omega_\sigma(m)/m^n$. This means that much of the enumerative work of $\mathcal{P}$-partitions become theorems about shuffling, and vice versa. See [8] for an extensive development.

Richard himself has made a marvelous connection between riffle shuffling and algebraic combinatorics [27]. Here is one of his theorems. Let $x_1, x_2, \dots$ be nonnegative reals that sum to 1. Describe an $\boldsymbol{x}$-shuffle by starting with $n$ cards in order, and labeling them independently, with label $i$ chosen with probability $x_i$. Then remove all cards labeled 1, keeping them in the same relative order. Remove all the cards labeled 2 and place them under the 1s, and so on. If $x_1 = x_2 = \cdots = x_m = 1/m$ and $x_i = 0$ for $i > m$, this is an (inverse) $m$-shuffle. Let $Q_{\boldsymbol{x}}(\sigma)$ be the chance of $\sigma$. These satisfy a simple convolution formula. Moreover, for $\lambda$ a partition of $n$ let $Q_{\boldsymbol{x}}(\lambda)$ be the chance that an $x$-shuffle results in shape $\lambda$ under the Robinson–Schersted–Knuth algorithm. Richard proved that

$$Q_{\boldsymbol{x}}(\lambda) = f_\lambda s_\lambda(\boldsymbol{x}) \tag{1.4}$$

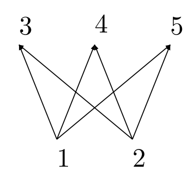
with $f_\lambda$ the dimension of the irreducible representation of $S_n$ and $s_\lambda$ the usual Schur function. This gives a shuffling interpretation of Schur functions that allows transferring algebraic combinatorics into card shuffling. See [14] for development and examples.

Here is a different effort to make up a story for P-partitions. Let $\mathcal{P}$ be a partial order on $[n]$ in a natural labeling, so $i <_{\mathcal{P}} j$ implies $i < j$. Consider dropping $n$ labeled balls into $m$ labeled boxes with probability $1/m$ (usual multinomial allocation). The chance that the allocation respects $\mathcal{P}$ so balls labeled $i < j$ are dropped into boxes with weakly increasing labels is

$$\Omega_{\mathcal{P}}(m)/m^n. \tag{1.5}$$

Thus, the many formulas for $\Omega_{\mathcal{P}}(m)$ yield answers to easy-to-interpret probability problems.

Example. Suppose there are $n_1$ boys and $n_2$ girls allocated to $m$ pay levels (with $n_1+n_2 = n$). If the allocation is random, the chance that all the girls are paid less than or equally much as all the boys is $\Omega_{\mathcal{P}}(m)/m^n$ with $\mathcal{P}$ corresponding to the complete bipartite graph with all the girls below all the boys, e.g., if $n_1 = 2,\ n_2 = 3$:




Alas, at this writing, for the boys/girls example, $\Omega_{\mathcal{P}}(m)$ is not so simple to write down. I'm happy to work backward, taking posets that Richard knows about and finding stories for them.

## 2. Story Two: Chromatic polynomials

Let $G = (V, E)$ be a graph, say undirected, simple (no loops or multiple edges) and connected. The chromatic polynomial of $G$, $p_G(r)$, is the number of proper colorings of $G$ with $r$ colors. A long time ago, I noticed that $p_G(r)$ has a simple probability interpretation: $p_G(r)/r^{|V|}$ is the answer to the birthday problem. What is the chance that if a graph is randomly $r$-colored there will be no monochromatic edges? The classical birthday problem has $G$ the complete graph on 23 vertices (the people) and $p_G(r) = r(r-1), \dots, (r-23+1)$ with $r = 365$, so the probability the birthday problem fails is $365 \cdot 364 \cdot \dots \cdot 343/365^{23} \doteq 0.50$. The graph version is a natural probability problem: what is the chance that two people who know each other have the same birthday? The probability community has many natural variations. Instead of choosing the colors uniformly (probability $1/r$) it is natural to choose the colors with probability $p_1, p_2, \dots, p_r$ with $p_i > 0$, $p_1 + \cdots + p_r = 1$. Then the probability of a proper coloring depends on $\boldsymbol{p}$, call it $p_G(\boldsymbol{p})$. I noticed that

$p_G(\boldsymbol{p})$ is a symmetric polynomial in $\boldsymbol{p}$ and it has a name: the Stanley chromatic polynomial!

Shortly after realizing this, I told Richard about it at lunch. His response was surprising: "That can't be true, otherwise I would know it." Then he got up and walked around for about five minutes, came back and said, "You're right." From there we began a fascinating game of mathematical ping-pong. "You know," he said, "I have a version enumerating by the number of monochromatic edges." This is the Stanley–Tutte polynomial [23, 25]. We do know. For a probabilist the natural question is "Fix a graph $G$, color the vertices with $r$ colors chosen from $p_1, \ldots, p_r$, what's the chance of having $r$ monochromatic edges?" We prove theorems showing that if $n$ and $\boldsymbol{p}$ have

$$\binom{n}{2} \sum_{i=1}^{n} p_i^2 = \lambda, \tag{2.1}$$

then

$$p\{k \text{ monochromatic edges}\} \doteq e^{-\lambda}\lambda^k/k!. \tag{2.2}$$

He responded, "What about the hypergraph version?" We have that: with 88 people it is even odds that three or more of them have the same birthday. For a survey of what probabilists know about birthdays, see [1, 4, 9].

The following beautiful work on the interfaces between combinatorics and probability has recently been done by my student Sukhada Fadnavis [12]. Fix a graph $G$ and consider coloring it with $(p_1, \ldots, p_r)$. What choice of $p_i$ makes a proper coloring most likely? It seems intuitive that $p_i = 1/r,\ 1 \leq i \leq r$, does best and this is true for the complete graph. Fadnavis shows it is true for claw-free graphs; this is a close relative of the Stanley–Stembridge conjecture [31]. It is false in general, e.g., for a $k$-star ($k \geq 5$) or a complete binary tree. Fadnavis shows it is always true if the number of colors is large enough: more than $400d^{3/2}$, where $d$ is the maximum degree of the graph, will do. These all translate into monotonicity of the Stanley chromatic polynomial.

Our ping-pong game continues with Bhattacharya and Mukherjee [3]. We recently determined the distribution of the number of monochromatic edges for quite general random graphs. What do you have to say to that Mr. Wise Guy? Have a look at these papers of Richard's. There is more in there to think about.

## 3. Story Three: The Jack polynomials

Many readers will know that enumerative combinatorics is much unified through its connection with symmetric polynomials. There are a number of standard bases for the homogeneous polynomials of degree $n$: monomial, elementary, homogeneous, power sum, and Schur functions are standard bases indexed by partitions $\lambda$ of $n$. The various change of basis matrices and inner products code up a huge swath of combinatorics. Much of this was originally discovered in Richard's thesis and is brilliantly brought together in [26, Chap. 7].

Statisticians use another basis, the zonal polynomials of Alan James. These are natural to statisticians because of their use in multivariate analyses such as principle coordinate analysis of covariance matrices. They were unknown to combinatorialists and during a visit to MIT in 1985-86, I gave a series of working seminars on them. Along the way, I mentioned the one-parameter family of bases $J_\lambda^\alpha$ of Jack polynomials. This included many standard bases: when $\alpha$ tends to infinity they are the monomials; when $\alpha = 0$ they are the elementaries; when $\alpha = 1$ they are Schur polynomials; when $\alpha = 2$ they are zonals. Richard got hooked at working out their properties as fully as he could [22]. Many remarkable identities and relationships were proved.

There was a nagging question: what did the Jacks "mean" and what were they good for? Aside from a few special values they weren't associated with natural groups or scientific problems. There was so much going on between them and the rest of symmetric function theory there must be a reason.

A first scientific application was found in joint work with Phil Hanlon [6]. There is a natural probability measure used by biologists and statisticians called "Ewens measure." This sets the probability of a permutation $\sigma \in S_n$ to

$$p_\theta(\sigma) = z^{-1}(\theta)\theta^{d(\sigma,\sigma_0)}. \tag{3.1}$$

In (3.1), $d$ is a metric on permutations, the Cayley distance: the minimum number of transpositions to bring $\sigma$ to $\sigma_0$. The permutation $\sigma_0$ is a fixed "center" and usually $0 < \theta \leq 1$ is a scale parameter determining how concentrated $p_\theta$ is about $\sigma_0$. Finally, $z(\theta)$ is a normalizing constant. In applied work, one needs a way of choosing random $\sigma$ from $p_\theta$, e.g., using Monte Carlo techniques to estimate things such as the distribution of the number of fixed points of $\sigma$ or the efficacy of the maximum likelihood estimate of $\theta$ and $\sigma_0$.

The standard way to sample from $p_\theta$ is the Metropolis algorithm: from $\sigma$, pick a transposition $(i,j)$ uniformly, let $\sigma' = (i,j)\sigma$. If $p_\theta(\sigma') \geq p_\theta(\sigma)$ move to $\sigma'$. If $p_\theta(\sigma') < p_\theta(\sigma)$, flip a coin with probability of heads $p_\theta(\sigma')/p_\theta(\sigma)$. If this comes up heads, move to $\sigma'$. If it comes up tails, stay at $\sigma$. The transition matrix $M(\sigma,\sigma')$ gives a Markov chain on $S_n$. If $\sigma_0 = \mathrm{id}$, everything is invariant under conjugation and there is an equivalent Markov chain on partitions of $n$, with transition matrix $M(\lambda,\lambda')$. This is a $\mathcal{P}(n) \times \mathcal{P}(n)$ matrix, and Hanlon and I found that the Jack polynomials $J_\lambda^\alpha$ gave an explicit diagonalization (here $\alpha = 1/\theta$). To explain, both the Jacks and power sums are bases for symmetric polynomials of degree $n$. Thus there is a change of basis matrix

$$J_\lambda^\alpha(\boldsymbol{x}) = \sum j_{\lambda,\mu}^\alpha p_\mu(\boldsymbol{x}). \tag{3.2}$$

For each fixed $\lambda$, $j_{\lambda,\mu}^\alpha$ as a function of $\mu$ is an eigenfunction of $M$ with a simple eigenvalue. Finding the appropriate normalizations needed Richard's formulas for inner products, his lovely hook-length formula for the deformed dimensions, and much else. The result is a deformation of the example in Story One ($\theta = 1$) and required much hard work by Hanlon and Stanley to bring to fruition. Further details are in [15] and [17]. My efforts to carry out a similar construction for the two-parameter Macdonald polynomials with Arun Ram [10] seemed forced to go in a different direction. *Why* these natural probabilistic deformations lead to algebraically natural deformations is a mystery. Richard? Where are you?

## 4. Story Four: Alternating permutations and poset polytopes

This story has a familiar ring. I was working on an applied probability problem; I asked Richard for help recognizing an object. It turned out to be his friend and that opened the problem up. I got stuck while inside and he helped with two amazing algorithms. I sent him a draft of the paper and he actually looked at it, giving a short proof of a theorem with a three-page argument. Wow. Anyone who looks will see that Richard loves alternating permutations [28, 29]: they go up, down, up, down, ..., like 4 5 1 3 2. He also loves polyhedral combinatorics: face numbers, $h$-vectors, $g$-conjectures. I never "got" these. Why does he care? What's it about? This story set me straight.

Working with Philip Matchett Wood [7], we were looking at the set $T_n$ of $(n+1) \times (n+1)$ tridiagonal doubly stochastic matrices. For a probabilist, these are "birth and death chains with a uniform stationary distribution." For the rest of you, they form a compact convex set. This inherits a uniform distribution and we were studying the following: Having selected $m \in T_n$ at random, what are its eigenvalues and mixing time? Drawing pictures and computing, we saw that our polytope has a Fibonacci number $F_{n+1}$ of extreme points because the rows and columns sum to 1; $m$ is determined by its super-diagonal $c_1, \ldots, c_n$. The polytope can be expressed as $0 \leq c_i$, $c_1 + c_{i+1} \leq 1$ for all $i$ (set $c_0 = c_{n+1} = 0$).

I asked Richard if he recognized this thing. He answered that he did, its volume is $E_n/n!$ where $E_n$ is the number of alternating permutations in $S_n$. In [21] he had given a triangulation of it into unit simplices indexed by alternating permutations. This gave us a fast, exact way to sample from $T_n$ if only we knew how to pick an alternating permutation. Richard outlined a method for this, and so it went. For Phil and I, there were two bottom lines: 1) for $m \in T_n$ chosen at random, order $n^2 \log n$ steps are necessary and sufficient to get close to random and 2) alternating permutations/poset polytopes are *really* interesting parts of mathematics. Our paper has some other nice stuff in it; we even managed to prove some new things about alternating permutations (divide one by $n$, the entries are asympotically distributed in the same way as $c_1, c_2, \ldots, c_n$ above). The story goes on, too: see [19]. Mostly, this is a workaday success story for getting Richard's attention.

## 5. Story Five: Stanley's problems and Selberg's integrals

Richard's enumerative combinatorics books are filled with interesting problems. Their available solutions make them practically a stand-alone tour of topics in enumerative combinatorics. I'm sure that most of the problems have stories attached to them; the most well-known is [24]. The purpose of the present entry is to bring one of these problems to life by telling its story.

The year was 1974. You have to know that I did my Ph.D. thesis in analytic number theory and had just taken a job at Stanford's Department of Statistics. Atle Selberg was giving a talk at Berkeley and I went up, just to look at one of the great living number theorists. Room 60 in Evans Hall was packed full, with people sitting on the floor. What was Selberg going to talk about? He came out and extracted a yellowed reprint and began copying. He was giving a talk on Selberg's integral, directly from his 1944 paper. The integral is

$$\int_0^1 \cdots \int_0^1 \prod_{i=1}^{n} x_i^{\alpha-1}(1-x_i)^{\beta-1} \prod_{i<j}(x_i - x_j)^{2\gamma}\, dx_1 \cdots dx_n = \prod_{j=0}^{n-1} \frac{\Gamma(\alpha + j\gamma)\Gamma(\beta + j\gamma)\Gamma(1 + (j+1)\gamma)}{\Gamma(\alpha+\beta) + (n + (j-1)\gamma)\Gamma(1+\gamma)}. \tag{5.1}$$

This was completely out of fashion and slow going for the modern audience. I began to think about Bayes' derivation of a simpler integral,

$$\int_0^1 \binom{n}{j} x^j (1-x)^{n-j}\, dx = 1/(n+1), \qquad 0 \le j \le n. \tag{5.2}$$

Bayes needed this for the first computation in Bayesian statistics. The left side represents the chance of $j$ heads out of $n$ tosses with an $x$-coin, when $x$ is given a uniform a priori distribution. It's still surprising that the answer doesn't depend on $j$. That was just Bayes' point, that using a uniform prior is equivalent to a uniform distribution of the outcome $j$. Nowadays, we prove Bayes' result using the beta integral. He didn't have this in 1789 and argued in *Bayes' billiard ball argument* as follows. Picture an interval: for us, [0,1]; for Bayes, the length of a billiard table. A red point (or ball) is dropped down at random. Following this, $n$ further black points are dropped down at random. Required to find the chance of $j$ black balls to the left of the red. On one hand, the answer is given by the left-hand integral in (5.2). On the other hand, Bayes argued, all the balls are the same. You may as well have dropped $n+1$ down at random, then picked a ball at random and colored it red. On this argument, the chance of $j$ black to the left of the red is clearly $1/(n+1)$, $0 \le j \le n$, and we are done. Now, if we move the binomial coefficient to the right, we have done the beta integral,

$$\int_0^1 x^j (1-x)^{n-j}\, dx = \frac{\Gamma(j+1)\Gamma(n-j+1)}{\Gamma(n+j+2)}. \tag{5.3}$$

This argument works for integer $j$ and $n$, but both sides are suitably analytic in these parameters and the result holds generally (Carlson's Lemma).

Back to Selberg's lecture. At the end, questions were asked for. No one had any, and after a minute or two, it became uncomfortable. Selberg's original integral was still on the board. He had written down a version in three variables with small integer choices for $\alpha$, $\beta$, and $\gamma$. Then the right-hand side is a simple combination of factorials and the whole has a probabilistic interpretation. I had figured out a Bayes-like interpretation of the left-hand side along the lines of "three points are dropped into the interval and then $n$ further points, what is the chance that $j$ points are to the left of the first, $k$ between, ...." I raised my hand and started to give my explanation of his example. He listened for a bit and then held up his hand to stop me. "I don't know what you are talking about and I don't know if you know what you are talking about. My proof is perfectly valid. Are there any other questions?" There were none and the talk ended with me devastated. I was enough in shock that I have been unable since to reconstruct my original argument.

This brings us to Richard's problem [30, Chap. 1, Prob. 11]. I told the story to Richard and he figured out the probabilistic interpretation of the left-hand side! No one has figured out a probabilistic proof.

For me, these stories are what make mathematics come alive. I have another dozen for Richard; perhaps this might make a good web site, telling the stories behind Richard's problems.

In fact, Richard is a pretty good storyteller himself: look at some of the extra stuff on his web site. He doesn't mix it with mathematics. I respect that, but hope that he finds a way to set some of them down. Perhaps the examples here will push him one way or the other.

## References

1. Andrew D. Barbour, Lars Holst, and Svante Janson, *Poisson Approximation*, Oxford Studies in Probability, vol. 2, The Clarendon Press Oxford University Press, New York, 1992, Oxford Science Publications. MR 1163825 (93g:60043)
2. Dave Bayer and Persi Diaconis, *Trailing the dovetail shuffle to its lair*, Ann. Appl. Probab. **2** (1992), no. 2, 294–313. MR MR1161056 (93d:60014)
3. Bhaswar B. Bhattacharya, Persi Diaconis, and Sumit Mukherjee, *Universal Poisson and normal limit theorems in graph coloring problems with connections to extremal combinatorics*, ArXiv e-prints (2013), arXiv:1310.2336.
4. Sourav Chatterjee, Persi Diaconis, and Elizabeth Meckes, *Exchangeable pairs and Poisson approximation*, Probab. Surv. **2** (2005), 64–106 (electronic). MR MR2121796 (2007b:60087)
5. Persi Diaconis, *Random walks on groups: Characters and geometry*, Groups St. Andrews 2001 in Oxford. Vol. I, London Math. Soc. Lecture Note Ser., vol. 304, Cambridge Univ. Press, Cambridge, 2003, pp. 120–142. MR MR2051523 (2005c:20109)
6. Persi Diaconis and Phil Hanlon, *Eigen-analysis for some examples of the Metropolis algorithm*, Hypergeometric Functions on Domains of Positivity, Jack Polynomials, and Applications (Tampa, FL, 1991), Contemp. Math., vol. 138, Amer. Math. Soc., Providence, RI, 1992, pp. 99–117. MR MR1199122 (94b:05205)
7. Persi Diaconis and Philip Matchett Wood, *Random doubly stochastic tridiagonal matrices*, Random Structures Algorithms **42** (2013), no. 4, 403–437. MR 3068032
8. Persi Diaconis, Michael McGrath, and Jim Pitman, *Riffle shuffles, cycles, and descents*, Combinatorica **15** (1995), no. 1, 11–29. MR MR1325269 (96g:05009)
9. Persi Diaconis and Frederick Mosteller, *Methods for studying coincidences*, J. Amer. Statist. Assoc. **84** (1989), no. 408, 853–861. MR MR1134485 (92e:62004)
10. Persi Diaconis and Arun Ram, *A probabilistic interpretation of the Macdonald polynomials*, Ann. Probab. **40** (2012), no. 5, 1861–1896. MR 3025704
11. Persi Diaconis and Mehrdad Shahshahani, *Generating a random permutation with random transpositions*, Z. Wahrsch. Verw. Gebiete **57** (1981), no. 2, 159–179. MR MR626813 (82h:60024)
12. Sukhada Fadnavis, *A generalization of the Birthday problem and the chromatic polynomial*, ArXiv e-prints (2011), arXiv:1105.0698.
13. Valentin Féray and Victor Reiner, *P-partitions revisited*, J. Commut. Algebra **4** (2012), no. 1, 101–152. MR 2913529
14. Jason Fulman, *Applications of symmetric functions to cycle and increasing subsequence structure after shuffles*, J. Algebraic Combin. **16** (2002), no. 2, 165–194. MR MR1943587 (2004b:05200)
15. ———, *Stein's method, Jack measure, and the Metropolis algorithm*, J. Combin. Theory Ser. A **108** (2004), no. 2, 275–296. MR 2098845 (2005h:60021)
16. Ira M. Gessel, *P-Partitions in Permutation Enumeration*, Presentation slides (2009), NIMS Workshop on Algebraic Combinatorics, KAIST, Daejeon, Korea. Available at `http://people.brandeis.edu/~gessel/homepage/slides/P-partitions-slides.pdf`.
17. Yunjiang Jiang, *Mixing time of Metropolis chain based on random transposition walk converging to multivariate Ewens distribution*, Ann. Appl. Probab. (2014),

to appear.

18. Laurent Saloff-Coste and Jessica Zuniga, *Refined estimates for some basic random walks on the symmetric and alternating groups*, ArXiv e-prints (2008), arXiv:0809.0688.
19. Aaron Smith, *The cutoff phenomenon for random birth and death chains*, ArXiv e-prints (2012), arXiv:1212.5614.
20. Richard P. Stanley, *Ordered Structures and Partitions*, American Mathematical Society, Providence, R.I., 1972, Memoirs of the American Mathematical Society, No. 119. MR 0332509 (48 #10836)
21. ———, *Two poset polytopes*, Discrete Comput. Geom. **1** (1986), no. 1, 9–23. MR 824105 (87e:52012)
22. ———, *Some combinatorial properties of Jack symmetric functions*, Adv. Math. **77** (1989), no. 1, 76–115. MR 1014073 (90g:05020)
23. ———, *A symmetric function generalization of the chromatic polynomial of a graph*, Adv. Math. **111** (1995), no. 1, 166–194. MR 1317387 (96b:05174)
24. ———, *Hipparchus, Plutarch, Schröder, and Hough*, Amer. Math. Monthly **104** (1997), no. 4, 344–350. MR 1450667 (98g:05008)
25. ———, *Graph colorings and related symmetric functions: Ideas and applications*, Discrete Math. **193** (1998), no. 1-3, 267–286, Selected papers in honor of Adriano Garsia (Taormina, 1994). MR 1661374 (2000c:05152)
26. ———, *Enumerative Combinatorics. Volume 2*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999, With a Foreword by Gian-Carlo Rota and Appendix 1 by Sergey Fomin. MR 1676282 (2000k:05026)
27. ———, *Generalized riffle shuffles and quasisymmetric functions*, Ann. Comb. **5** (2001), no. 3-4, 479–491, Dedicated to the memory of Gian-Carlo Rota (Tianjin, 1999). MR 1897637 (2003c:60016)
28. ———, *Alternating permutations and symmetric functions*, J. Combin. Theory Ser. A **114** (2007), no. 3, 436–460. MR 2310744 (2008a:05269)
29. ———, *A survey of alternating permutations*, Combinatorics and graphs, Contemp. Math., vol. 531, Amer. Math. Soc., Providence, RI, 2010, pp. 165–196. MR 2757798 (2012d:05015)
30. ———, *Enumerative Combinatorics. Volume 1*, second ed., Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 2012. MR 2868112
31. Richard P. Stanley and John R. Stembridge, *On immanants of Jacobi–Trudi matrices and permutations with restricted position*, J. Combin. Theory Ser. A **62** (1993), no. 2, 261–279. MR 1207737 (94d:05147)

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