RANDOM GRAPHS WITH A GIVEN DEGREE SEQUENCE

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ABSTRACT. Large graphs are sometimes studied through their degree sequences (power law or regular graphs). We study graphs that are uniformly chosen with a given degree sequence. Under mild conditons, it is shown that sequences of such graphs have graph limits in the sense of Lovász and Szegedy with identifiable limits. This allows simple determination of other features such as the number of triangles. The argument proceeds by studying a natural exponential model having the degree sequence as a sufficient statistic. The maximum likelihood estimate (MLE) of the parameters is shown to be unique and consistent with high probability. Thus n parameters can be consistently estimated based on a sample of size one. A fast, provably convergent, algorithm for the MLE is derived. These ingredients combine to prove the graph limit theorem. Along the way, a continuous version of the Erdős-Gallai characterization of degree sequences is derived.

1. INTRODUCTION

1.1. **Graphs with a given degree sequence.** Let G be an undirected simple graph on n vertices, and let $d_1, \ldots, d_n$ be the degrees of the vertices of G. The vector $\boldsymbol{d} := (d_1, \ldots, d_n)$ is usually called the *degree sequence* of G. Correspondingly, the *degree distribution* of G is the probability distribution function F supported on $[0, 1]$, defined as

$$F(x) := \frac{|\{i : d_i \leq nx\}|}{n}.$$

In other words, if a vertex is chosen uniformly at random, then the degree of that vertex, divided by n, is a random variable with probability distribution function F.

In recent years, the degree distributions of real world networks have received wide attention. The surveys [39, 40] contain many references, as does the detailed account in [10]. The enthusiasm of some authors for 'scale free' or 'power law graphs' has also generated much controversy [31, 51] which serves as additional motivation for the present paper.

Key words and phrases. Random graph, degree sequence, Erdős-Gallai criterion, threshold graphs, graph limit.

Sourav Chatterjee's research was partially supported by NSF grant DMS-0707054 and a Sloan Research Fellowship.

The interest in degree distributions stems from the fact that the degree sequences of real world networks sometimes appear to have power law behavior that is very different than those occurring in classical models of random graphs, like the Erdős-Rényi model [19]. Researchers have tried various ways of circumventing this problem. An obvious solution is to build random graph models that are forced to give us the degree distribution that we want, and then deduce other features by simulation or mathematics. A natural way to do this is to choose a graph uniformly at random from the set of all graphs with a give degree sequence. One frequent appearance of this model is for random regular graphs [52]. As explained in Section 13 of [10], the model also arises in testing if the exponential family with degree sequence as sufficient statistic fits a given data set. See [49] for applications where the number of triangles is wanted. The paper [10] has useful ways of simulating graphs with a given degree sequence and an extensive survey of the (mostly non-rigorous) literature for this model. Some rigorous results are also available in the 'sparse case', e.g. those in [37, 38].

In a recent series of papers [3, 4, 5, 6, 7, 8], Barvinok and Hartigan have looked at problems related to the structure of directed and undirected graphs with given degree sequence. The Barvinok-Hartigan work, especially [8], is related to the present paper. This is explained at the end of this introduction after we have stated our main theorems.

One of the objectives of this article is to give a rather precise description of the structure of random (dense) graphs with a given degree sequence, via the notion of graph limits introduced recently by Lovász and Szegedy [32] and developed by Borgs et. al. [12, 13]. See also the related work of Diaconis and Janson [17] which traces this back to work of Aldous [1] and Hoover [26]. This gives, in particular, a way to write down exact formulas for the expected number of subgraphs of a given type without simulation.

Before stating our result, we need to introduce the notion of graph limits. We quote the definition verbatim from [32] (see also [12, 13, 17]). Let G_n be a sequence of simple graphs whose number of nodes tends to infinity. For every fixed simple graph H, let $|\hom(H,G)|$ denote the number of homomorphisms of H into G (i.e. edge-preserving maps $V(H) \to V(G)$, where $V(H)$ and $V(G)$ are the vertex sets). This number is normalized to get the homomorphism density

$$t(H,G) := \frac{|\hom(H,G)|}{|V(G)|^{|V(H)|}}. \tag{1}$$

This gives the probability that a random mapping $V(H) \to V(G)$ is a homomorphism.

Suppose that the graphs G_n become more and more similar in the sense that $t(H,G_n)$ tends to a limit $t(H)$ for every H. One way to define a limit of the sequence $\{G_n\}$ is to define an appropriate limit object from which the values $t(H)$ can be read off.

The main result of [32] (following the earlier equivalent work of Aldous [1] and Hoover [26]) is that indeed there is a natural "limit object" in the form of a symmetric measurable function $W : [0,1]^2 \to [0,1]$ (we call W symmetric if $W(x,y) = W(y,x)$). Conversely, every such function arises as the limit of an appropriate graph sequence. This limit object determines all the limits of subgraph densities: if H is a simple graph with $V(H) = [k] = \{1, \ldots, k\}$, then

$$t(H,W) = \int_{[0,1]^k} \prod_{(i,j)\in E(H)} W(x_i, x_j)\; dx_1 \cdots dx_k.$$

Here $E(H)$ denotes the edge set of H.

Intuitively, the interval $[0,1]$ represents a 'continuum' of vertices, and $W(x,y)$ denotes the probability of putting an edge between x and y. For example, for the Erdős-Rényi graph $G_{n,p}$, if p is fixed and $n \to \infty$, then the limit graph is represented by the function that is identically equal to p on $[0,1]^2$.

Convergence of a sequence of graphs to a limit has many consequences. From the definition, the count of fixed size subgraphs converges to the right hand side of the expression for $t(H,W)$ given above. More global parameters also converge. For example, the degree distribution converges to the law of $\int_0^1 W(U,y)dy$ where U is a random variable distributed uniformly on $[0,1]$. Similarly, the distribution function of the eigenvalues of the adjacency matrix converges. More generally, a graph parameter is a function from the space of graphs into a space $\mathcal{X}$ which is invariant under isomorphisms. If $\mathcal{X}$ is a topological space, we may ask which graph parameters are continuous with respect to the topology induced by graph limits. This is called 'property testing' in the computer science theory literature which has identified many continuous graph parameters. See the survey in [12] for pointers to a healthy literature.

We are now ready to state our result about the limit of graphs with given degree sequences. Suppose that for each n, a degree sequence $\boldsymbol{d}^n = (d_1^n, \ldots, d_n^n)$ is given. Without loss of generality, assume that $d_1^n \ge d_2^n \ge \cdots \ge d_n^n$. Suppose that the sequence $\{\boldsymbol{d}^n\}$ has a *scaling limit*, in the sense that there is a non-increasing function f on $[0,1]$ such that

$$\lim_{n\to\infty} \left(\left|\frac{d_1^n}{n} - f(0)\right| + \left|\frac{d_n^n}{n} - f(1)\right| + \frac{1}{n}\sum_{i=1}^{n} \left|\frac{d_i^n}{n} - f\left(\frac{i}{n}\right)\right| \right) = 0. \tag{2}$$

It is not difficult to prove by a simple compactness argument that any sequence $\{\boldsymbol{d}^n\}$ of degree sequences has a subsequence that converges to a scaling limit in the above sense.

Define $D'[0,1]$ to be the set of non-increasing functions on $[0,1]$ which are left continuous on $(0,1)$. When the scaling limit of a degree sequence is discontinuous, it is not uniquely defined but there always exists a unique limit in $D'[0,1]$. So we shall restrict our attention to limits in $D'[0,1]$.

For each n, let G_n be a random graph chosen uniformly from the set of all simple graphs with degree sequence $\boldsymbol{d}^n$. Our objective is to compute the limit of the sequence $\{G_n\}$ in terms of the scaling limit of $\boldsymbol{d}^n$. We endow the set of scaling limits (i.e. $D'[0,1]$) with the topology induced by a modified L^1 norm $\|\cdot\|_{1'}$ given by

$$\|f\|_{1'} := |f(0)| + |f(1)| + \int_0^1 |f(x)|dx.$$

The choice of this norm is necessitated by the need to control the behavior of the largest and smallest degrees. In particular it will avoid the case that $d_1^n = n$ or $d_n^n = 0$ where the MLE described in the next section is not defined.

Not all functions can be scaling limits of degree sequences. Let $\mathcal{F}$ be the set of functions in $D'[0,1]$ that can be obtained as scaling limits of degree sequences in the sense stated above. By a simple diagonal argument, it is easy to see that $\mathcal{F}$ is a closed subset of $D'[0,1]$ under the topology of the modified L^1 norm. It is shown in Proposition 1.2 below that $\mathcal{F}$ has non-empty interior.

Theorem 1.1. *Let G_n and f be as in* (2) *above. Suppose that f belongs to the topological interior of the set $\mathcal{F}$ defined above. Then there exists a unique function $g : [0,1] \to \mathbb{R}$ in $D'[0,1]$ such that the function*

$$W(x,y) := \frac{e^{g(x)+g(y)}}{1+e^{g(x)+g(y)}},$$

satisfies, for all $x \in [0,1]$,

$$f(x) = \int_0^1 W(x,y)dy.$$

In this situation, the sequence $\{G_n\}$ converges to the limit graph represented by the function W.

The above theorem can be useful only if we can provide a simple way of checking whether f belongs to the interior of $\mathcal{F}$. (Being the limit of a sequence of degree sequences, it is clear that $f \in \mathcal{F}$. The nontrivial question is whether f is in the interior.) The following result gives an easily verifiable equivalent condition.

Proposition 1.2. *A function $f : [0,1] \to [0,1]$ in $D'[0,1]$ belongs to the interior of $\mathcal{F}$ if and only if*

(*i*) *there are two constants $c_1 > 0$ and $c_2 < 1$ such that $c_1 \le f(x) \le c_2$ for all $x \in [0,1]$, and*
(*ii*) *for each $x \in (0,1]$,*

$$\int_x^1 \min\{f(y), x\}dy + x^2 - \int_0^x f(y)dy > 0.$$

Remark 1: Condition (ii) in the above result is a continuum version of the well-known Erdős-Gallai criterion [20]. (See Mahadev and Peled [33] for extensive discussions and eight equivalent conditions.)

Remark 2: When the scaling limit f is continuous, convergence in the modified L^1 norm is same as supnorm convergence. In particular, for continuous scaling limits Theorem 1.1 and Proposition 1.2 both hold if we replace $D'[0,1]$ with $C[0,1]$ and redefine $\mathcal{F}$ analogously under the supnorm topology.

Remark 3: As an example, consider the limit of the Erdős-Rényi graph $G(n,p)$ as $n \to \infty$. Here $f(x) = p$ for all x. Condition (ii) becomes $(1-x)\min\{p,x\}+x^2-px > 0$, for all x. A simple case analysis shows this holds, so Erdős-Rényi graphs are in the interior of $\mathcal{F}$ for any fixed p, $0 < p < 1$.

Remark 4: In a recent article [35] (following up on the older work [36]), McKay has computed subgraph counts in random graphs with a given degree sequence. However, McKay's results hold only if either the graph is sparse, or if the graph is dense but all degrees are within $n^{1/2+\epsilon}$ of the average degree. Thus it may be possible to recover Theorem 1.1 from McKay's results when the limit shape is a constant function, but not in other cases.

The next natural question is whether one can feasibly compute the function g in Theorem 1.1 for a given f. It turns out that this is a central issue in the whole analysis. In fact, to prove Theorem 1.1 we analyze a related statistical model; computation of the maximum likelihood estimate in that model leads to an algorithm for computing g, which, in turn, yields a proof of Theorem 1.1. The statistical model is discussed next.

1.2. **Statistics with degree sequences.** Informally, if the degree sequence captures the information in a graph, different graphs with the same degree sequence are judged equally likely. This can be formalized by saying that the degree sequence is a sufficient statistic for a probability distribution on graphs. The Koopman-Pitman-Darmois theorem forces this distribution to be of exponential form. This approach to model building is explained and developed in Lauritzen [30]. Diaconis and Freedman [15] give a version of the Koopman-Pitman-Darmois theorem for discrete exponential families. The approach is also standard fare in statistical mechanics where the uniform distribution on graphs with fixed degree sequence is called 'micro-canonical' and the exponential distribution is called 'canonical' (see Park and Newman [41]). It turns out that the exponential model has a simple description in terms of independent Bernoulli random variables.

Given a vector $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n$, let $\mathbb{P}_{\boldsymbol{\beta}}$ be the law of the undirected random graph on n vertices defined as follows: for each $1 \le i \ne j \le n$, put an edge between the vertices i and j with probability

$$p_{ij} := \frac{e^{\beta_i+\beta_j}}{1+e^{\beta_i+\beta_j}},$$

independently of all other edges. Thus, if G is a graph with degree sequence $d_1, \ldots, d_n$, the probability of observing G under $\mathbb{P}_{\boldsymbol{\beta}}$ is

$$\frac{e^{\sum_i \beta_i d_i}}{\prod_{i<j}(1+e^{\beta_i+\beta_j})}.$$

Henceforth, this model of random graphs is called the '$\boldsymbol{\beta}$-model'. This model was considered by Holland and Lienhardt [25] in the directed case, by Park and Newman [41] and Blitztein and Diaconis [10] in the undirected case. It is a close cousin to the Bradley-Terry model for rankings (which itself goes back (at least) to Zermelo). See Hunter [27] for extensive references. The $\boldsymbol{\beta}$-model is also a simple version of a host of exponential models actively in use for analyzing network data. We will not try to survey this vast literature but recomend the extensive treatments in Newman [39], Jackson [28] and Robins et. al. [46]. The website for the International Network for Social Network Analysis contains further information.

Suppose a random graph G is generated from the $\boldsymbol{\beta}$-model, where $\boldsymbol{\beta} \in \mathbb{R}^n$ is unknown. Is it possible to estimate $\boldsymbol{\beta}$ from the observed G? It is not difficult to show that the maximum likelihood estimate (MLE) $\hat{\boldsymbol{\beta}}$ of $\boldsymbol{\beta}$ must satisfy the system of equations

$$d_i = \sum_{j \neq i} \frac{e^{\hat{\beta}_i+\hat{\beta}_j}}{1+e^{\hat{\beta}_i+\hat{\beta}_j}}, \quad i = 1, \ldots, n, \tag{3}$$

where $d_1, \ldots, d_n$ are the degrees of the vertices in the observed graph G. Questions may arise about the existence, uniqueness and accuracy of the MLE. Since the dimension of the parameter space grows with n, it is not clear if this is a 'good' estimate of $\boldsymbol{\beta}$ in the traditional sense of consistency in statistical estimation theory.

The following theorem shows that under certain mild assumptions on $\boldsymbol{\beta}$, there is a high chance that the MLE exists, is unique and estimates $\boldsymbol{\beta}$ with uniform accuracy in all coordinates.

Theorem 1.3. *Let G be drawn from the probability measure $\mathbb{P}_{\boldsymbol{\beta}}$ and let $d_1, \ldots, d_n$ be the degree sequence of G. Let $L := \max_{1 \le i \le n} |\beta_i|$. Then there is a constant $C(L)$ depending only on L such that with probability at least $1 - C(L)n^{-2}$, there exists a unique solution $\hat{\boldsymbol{\beta}}$ of the maximum likelihood equations* (3)*, that satisfies*

$$\max_{1 \le i \le n} |\hat{\beta}_i - \beta_i| \le C(L)\sqrt{\frac{\log n}{n}}.$$

It may seem surprising that all n parameters can be accurately estimated from a single realization of the graph. However, this is not surprising when one observes that there are, in fact, $n(n-1)/2$ independent random variables lurking in the background. Moreover, this is not the first result of this type in statistical theory; indeed, there is a well known heuristic that in a p-parameter model with n observations, 'the usual asymptotics' work provided that p^2/n tends to zero as n tends to infinity. See Portnoy [42, 43, 44, 45] for details (and counter examples). In work closer to the present paper, Simons and Yao [48] studied the Bradley-Terry model for comparing n contestants. Here a random orientation of the complete graph on n vertices is chosen based on 'player a beats player b with probability $\theta(a)/(\theta(a)+\theta(b))$'. They show that MLE is consistent here as well. Hunter [27] shows that the MM algorithm also behaves well in this problem.

The next theorem characterizes all possible expected degree sequences of the $\boldsymbol{\beta}$-model as $\boldsymbol{\beta}$ ranges over $\mathbb{R}^n$. The nice feature is that 'no degree sequence is left out'.

Theorem 1.4. *Let $\mathcal{R}$ denote the set of all expected degree sequences of random graphs following the law $\mathbb{P}_{\boldsymbol{\beta}}$ as $\boldsymbol{\beta}$ ranges over $\mathbb{R}^n$. Let $\mathcal{D}$ denote the set of all possible degree sequences of undirected graphs on n vertices. Then*

$$\operatorname{conv}(\mathcal{D}) = \overline{\mathcal{R}},$$

where $\operatorname{conv}(\mathcal{D})$ *denotes the convex hull of $\mathcal{D}$ and $\overline{\mathcal{R}}$ is the topological closure of $\mathcal{R}$.*

Incidentally, the convex hull of $\mathcal{D}$ is a well studied polytope. For example, its extreme points are the threshold graphs. See Mahadev and Peled [33] for much more on this.

A self-contained proof of Theorem 1.4 is given in Section 3. However, it is possible to derive it from classical results about the mean space of exponential families (see e.g. in Brown [14] or Barndorff-Nielsen [2]; in particular, see Theorem 3.3 in Wainwright and Jordan [50]).

Finally, let us describe a fast algorithm for computing the MLE if it exists. Recall that the L^∞ norm of a vector $\boldsymbol{x} = (x_1, \ldots, x_n)$ is defined as

$$|\boldsymbol{x}|_\infty := \max_{1\le i\le n} |x_i|.$$

For $1 \le i \ne j \le n$ and $\boldsymbol{x} \in \mathbb{R}^n$, let

$$r_{ij}(\boldsymbol{x}) := \frac{1}{e^{-x_j} + e^{x_i}}. \tag{4}$$

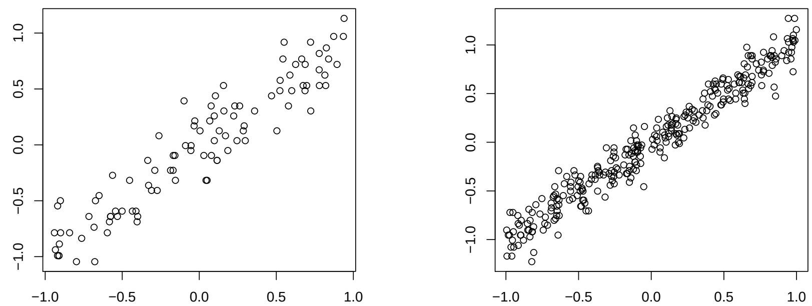
Given a realization of the random graph G with degree sequence $d_1, \ldots, d_n$, define for each i the function

$$\varphi_i(\boldsymbol{x}) := \log d_i - \log \sum_{j\ne i} r_{ij}(\boldsymbol{x}). \tag{5}$$

Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be the function whose ith component is φ_i.

Theorem 1.5. *Suppose the ML equations* (3) *have a solution $\hat{\boldsymbol{\beta}}$. Then $\hat{\boldsymbol{\beta}}$ is a fixed point of the function φ. Starting from any $\boldsymbol{x}_0 \in \mathbb{R}^n$, define $\boldsymbol{x}_{k+1} = \varphi(\boldsymbol{x}_k)$ for $k = 0, 1, 2, \ldots$. Then $\boldsymbol{x}_k$ converges to $\hat{\boldsymbol{\beta}}$ geometrically fast in the L^∞ norm, where the rate depends only on $(|\hat{\boldsymbol{\beta}}|_\infty, |\boldsymbol{x}_0|_\infty)$. In particular, $\hat{\boldsymbol{\beta}}$ must be the unique solution of* (3)*. Moreover,*

$$|\boldsymbol{x}_0 - \hat{\boldsymbol{\beta}}|_\infty \le C|\boldsymbol{x}_0 - \boldsymbol{x}_1|_\infty,$$



(a) $n = 100$, $\beta_i \sim \mathrm{Unif}[-1, 1]$. (b) $n = 300$, $\beta_i \sim \mathrm{Unif}[-1, 1]$.

FIGURE 1. Simulation results: plot of $\hat{\beta}_i$ vs. β_i.

where C is a continuous function of the pair $(|\hat{\boldsymbol{\beta}}|_\infty, |\boldsymbol{x}_0|_\infty)$. Conversely, if the ML equations (3) *do not have a solution, then the sequence $\{\boldsymbol{x}_k\}$ must have a divergent subsequence.*

There are many other algorithms available for calculating the MLE. For example Holland and Leinhardt [25] use an iterative scaling algorithm and discuss the method of scoring and weighted least squares. Hunter [27] develops the MM algorithm for a similar task. Markov chain Monte Carlo algorithms and the Robbins-Monro stochastic approximation approach are also used for computing the MLE in exponential random graph models. See Section 6.5.2 in [29] for examples and literature. The iterative algorithm we use is a hybrid of standard algorithms which works well in practice and allows the strong conclusions of Theorem 1.5. We hope that variants can be developed for related high dimensional problems.

Let us now look at the results of some simulations. The left panel in Figure 1 shows the plot of $\hat{\beta}_i$ versus β_i for a graph with 100 vertices, where $\beta_1, \ldots, \beta_n$ were chosen independently at uniform from the interval $[-1, 1]$. The right panel is the same, except that n has been increased to 300. The increased accuracy for larger n is clearly visible.

We have also compared our results with the simulation results from the importance sampling algorithm of Blitzstein and Diaconis [10] for a variety of other examples. The results of Figure 1 are typical. This convinces us that the procedures developed in this paper are useful for practical problems.

Comparison to the Barvinok-Hartigan work. As mentioned before, the present work is closely related to a recent series of papers by Barvinok and Hartigan [3, 4, 5, 6, 7, 8]. The work was initiated by Barvinok, who looked at directed and bipartite graphs in [4]. In their most recent article [8] (uploaded to arXiv when our paper was near completion), they study uniform random (undirected) graphs on n vertices with a given degree sequence $\boldsymbol{d} = (d_1, \ldots, d_n)$ and work with an exponential model as in Subsection 1.2 with β_i chosen so that the expected degree at i under the $\boldsymbol{\beta}$-model is d_i. Let $G_{\boldsymbol{d}}$ be a uniformly chosen random graph with the degree sequence $\boldsymbol{d}$ and $G_{\boldsymbol{\beta}}$ be a random graph chosen from the $\boldsymbol{\beta}$-model. One of their main results shows that (under hypothesis), these two graphs are close together in the following sense: Fix a set of edges S in the complete graph on n vertices. Let $X_{\boldsymbol{d}}$ be the number of edges of $G_{\boldsymbol{d}}$ in S. Let $X_{\boldsymbol{\beta}}$ be the number of edges of $G_{\boldsymbol{\beta}}$ in S. They prove that $X_{\boldsymbol{d}}/n^2$ and $X_{\boldsymbol{\beta}}/n^2$ are each concentrated about their means (using results from the earlier work [3]) and that these means are approximately equal. Their theorem is proved under a condition on the degree sequences that they call 'delta tame'.

While the two sets of results (i.e. ours and those of Barvinok and Hartigan) were proved independently and the methods of proof are quite different in certain parts (but similar in others), the possible connections are tantalizing. We believe that their mode of convergence ($G_{\boldsymbol{d}}$ and $G_{\boldsymbol{\beta}}$ contain about the same number of edges in a given set) is equivalent to the graph limit convergence used here. Perhaps this can be established using the 'cut-metric' of Frieze and Kannan, as expounded in Borgs et. al. [12]. We further conjecture, based on Lemma 4.1 in this paper, that their delta tame condition is equivalent to our condition that the limiting degree sequence f is in the interior of $\mathcal{F}$. If this is so, then Proposition 1.2 (or more accurately, Lemma 4.1) gives a necessary and sufficient condition for a degree sequence to be 'delta tame', showing that essentially all degree sequences except the ones close to the Erdős-Gallai boundary are delta tame.

In summary, the Barvinok-Hartigan work [8] contains elegant estimates of the number of graphs with a given degree sequence and extensions to bipartite graphs under a condition called 'delta tameness'; we work in the emerging language of graph limits and prove a limit theorem under a continuum version of the easily verifiable Erdős-Gallai criterion. Our work contains an efficient algorithm for computing the maximum likelihood estimates of $\boldsymbol{\beta}$ for a given degree sequence with proofs of convergence of the algorithm and consistency of the estimates.

The rest of the paper is organized as follows. In Section 2 we prove Theorem 1.5. This is followed by the proof of Theorem 1.4 in Section 3. Both of these theorems are required for the proof of Theorem 1.3, which is given in Section 4. Proposition 1.2 is proved in Section 5. Finally, the proof of Theorem 1.1, which uses all the other theorems, is given in Section 6.

Acknowledgments. The authors are indebted to Joe Blitzstein for many helpful tips and pointers to the literature, and Martin Wainwright for the references to [14], [2] and [50]. We particularly thank Alexander Barvinok for calling our attention to [8] and suggesting possible connections to our work.

2. Proof of Theorem 1.5

For a matrix $A = (a_{ij})_{1\le i,j\le n}$, the L^∞ operator norm is defined as

$$|A|_\infty := \max_{|x|_\infty \le 1} |A\boldsymbol{x}|_\infty$$

It is a simple exercise to verify that

$$|A|_\infty = \max_{1\le i\le n} \sum_{j=1}^n |a_{ij}|.$$

Given $\delta > 0$, let us say the matrix A belongs to the class $\mathcal{L}_n(\delta)$ if $|A|_\infty \le 1$, and for each $1 \le i \ne j \le n$,

$$a_{ii} \ge \delta \quad \text{and} \quad a_{ij} \le -\frac{\delta}{n-1}.$$

The following lemma is our key tool.

Lemma 2.1. *Let $\mathcal{L}_n(\delta)$ be defined as above. If $A, B \in \mathcal{L}_n(\delta)$, then*

$$|AB|_\infty \le 1 - \frac{2(n-2)\delta^2}{(n-1)}.$$

Proof. Fix $1 \le i \ne k \le n$. By the definition of $\mathcal{L}_n(\delta)$,

$$\sum_{j\notin\{i,k\}} a_{ij}b_{jk} \ge \frac{(n-2)\delta^2}{(n-1)^2} \quad \text{and } a_{ii}b_{ik} + a_{ik}b_{kk} \le -\frac{2\delta^2}{n-1}.$$

Now, if x, y are two positive real numbers, then $|x-y| = |x|+|y|-2\min\{x,y\}$. Taking $x = \sum_{j\notin\{i,k\}} a_{ij}b_{jk}$ and $y = -(a_{ii}b_{ik} + a_{ik}b_{kk})$, we get

$$\begin{aligned}
\left|\sum_{j=1}^n a_{ij}b_{jk}\right| &\le \sum_{j=1}^n |a_{ij}b_{jk}| - 2\min\left\{\sum_{j\notin\{i,k\}} a_{ij}b_{jk}, -(a_{ii}b_{ik} + a_{ik}b_{kk})\right\} \\
&\le \sum_{j=1}^n |a_{ij}b_{jk}| - \frac{2(n-2)\delta^2}{(n-1)^2}.
\end{aligned}$$

Combining this with the hypothesis that $|A|_\infty \le 1$ and $|B|_\infty \le 1$, we get

$$\begin{aligned}
|AB|_\infty &= \max_{1\le i\le n} \sum_{k=1}^n \left|\sum_{j=1}^n a_{ij}b_{jk}\right| \\
&\le \max_{1\le i\le n} \sum_{j=1}^n \sum_{k=1}^n |a_{ij}b_{jk}| - \frac{2(n-2)\delta^2}{(n-1)} \\
&\le 1 - \frac{2(n-2)\delta^2}{(n-1)}.
\end{aligned}$$

This completes the proof of the lemma. $\square$

Now recall the functions r_{ij} defined in (4). Let

$$q_{ij}(\boldsymbol{x}) := \frac{r_{ij}(\boldsymbol{x})}{\sum_{k\neq i} r_{ik}(\boldsymbol{x})}.$$

Note that for each i and $\boldsymbol{x}$, $\sum_{j\neq i} q_{ij}(\boldsymbol{x}) = 1$. Again, for each i

$$\frac{\partial \varphi_i}{\partial x_i} = -\frac{\sum_{j\neq i} \frac{\partial r_{ij}}{\partial x_i}}{\sum_{j\neq i} r_{ij}} = \sum_{j\neq i} \frac{e^{x_i}}{e^{-x_j} + e^{x_i}} q_{ij},$$

and similarly for each distinct i and j,

$$\frac{\partial \varphi_i}{\partial x_j} = -\frac{e^{-x_j}}{e^{-x_j} + e^{x_i}} q_{ij}.$$

Now, if $|\boldsymbol{x}|_\infty \le K$, then clearly

$$\tfrac{1}{2}e^{-K} \le r_{ij}(\boldsymbol{x}) \le \tfrac{1}{2}e^{K} \quad \text{for all } 1 \le i \neq j \le n.$$

Thus,

$$\frac{e^{-2K}}{n-1} \le q_{ij}(\boldsymbol{x}) = \frac{r_{ij}(\boldsymbol{x})}{\sum_{k\neq i} r_{ik}(\boldsymbol{x})} \le \frac{e^{2K}}{n-1}.$$

It follows that for every $1 \le i \neq j \le n$

$$-\frac{e^{2K}}{n-1} \le \frac{\partial \varphi_i}{\partial x_j} \le -\frac{e^{-4K}}{2(n-1)}. \tag{6}$$

and also, for every $1 \le i \le n$,

$$\frac{1}{2}e^{-2K} \le \frac{\partial \varphi_i}{\partial x_i} \le e^{2K}. \tag{7}$$

Now take any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and let K be the maximum of the L^∞ norms of $\boldsymbol{x}$, $\boldsymbol{y}$, $\varphi(\boldsymbol{x})$, and $\varphi(\boldsymbol{y})$. Let $J(\boldsymbol{x}, \boldsymbol{y})$ be the matrix whose $(i,j)^{\text{th}}$ element is

$$J_{ij}(\boldsymbol{x}, \boldsymbol{y}) = \int_0^1 \frac{\partial \varphi_i}{\partial x_j}(t\boldsymbol{x} + (1-t)\boldsymbol{y})dt.$$

It is a simple calculus exercise to verify that

$$\varphi(\boldsymbol{x}) - \varphi(\boldsymbol{y}) = J(\boldsymbol{x}, \boldsymbol{y})(\boldsymbol{x} - \boldsymbol{y}). \tag{8}$$

From (6) and (7) and the fact that $|J(\boldsymbol{x}, \boldsymbol{y})|_\infty = 1$, we see that $J(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{L}_n(\delta)$ for $\delta = \frac{1}{2}e^{-4K}$. Similarly,

$$\begin{aligned}\varphi(\varphi(\boldsymbol{x})) - \varphi(\varphi(\boldsymbol{y})) &= J(\varphi(\boldsymbol{x}), \varphi(\boldsymbol{y}))(\varphi(\boldsymbol{x}) - \varphi(\boldsymbol{y}))\\ &= J(\varphi(\boldsymbol{x}), \varphi(\boldsymbol{y}))J(\boldsymbol{x}, \boldsymbol{y})(\boldsymbol{x} - \boldsymbol{y}),\end{aligned}$$

and $J(\varphi(\boldsymbol{x}), \varphi(\boldsymbol{y})) \in \mathcal{L}_n(\delta)$ also. Applying Lemma 2.1, we get

$$|J(\varphi(\boldsymbol{x}), \varphi(\boldsymbol{y}))J(\boldsymbol{x}, \boldsymbol{y})|_\infty \le 1 - \frac{2(n-2)\delta^2}{n-1}.$$

Thus,

$$|\varphi(\varphi(\boldsymbol{x})) - \varphi(\varphi(\boldsymbol{y}))|_\infty \le \left(1 - \frac{2(n-2)\delta^2}{n-1}\right)|\boldsymbol{x} - \boldsymbol{y}|_\infty. \tag{9}$$

The quantity inside the brackets will henceforth be denoted by $\theta(\boldsymbol{x}, \boldsymbol{y})$. Note that $0 \le \theta(\boldsymbol{x}, \boldsymbol{y}) < 1$, and θ is uniformly bounded on bounded subsets of $\mathbb{R}^n \times \mathbb{R}^n$. Moreover, since $|J(\boldsymbol{x}, \boldsymbol{y})|_\infty = 1$, we also have the trivial but useful bound

$$|\varphi(\boldsymbol{x}) - \varphi(\boldsymbol{y})|_\infty \le |\boldsymbol{x} - \boldsymbol{y}|_\infty.$$

Now suppose φ has a fixed point $\hat{\boldsymbol{\beta}}$. If we start with arbitrary $\boldsymbol{x}_0$ and define $\boldsymbol{x}_{k+1} = \varphi(\boldsymbol{x}_k)$ for each $k \ge 0$, then for each k, we have

$$\begin{aligned} |\boldsymbol{x}_{k+1} - \hat{\boldsymbol{\beta}}|_\infty &= |\varphi(\boldsymbol{x}_k) - \varphi(\hat{\boldsymbol{\beta}})|_\infty \\ &\le |\boldsymbol{x}_k - \hat{\boldsymbol{\beta}}|_\infty. \end{aligned}$$

In particular, the sequence $\{\boldsymbol{x}_k\}_{k\ge 0}$ remains bounded. Therefore by (9), there is a single $\theta \in [0, 1)$, depending only on $|\hat{\boldsymbol{\beta}}|_\infty$ and $|\boldsymbol{x}_0|_\infty$ in a continuous manner, such that for all $k \ge 0$, we have

$$|\boldsymbol{x}_{k+3} - \boldsymbol{x}_{k+2}|_\infty \le \theta |\boldsymbol{x}_{k+1} - \boldsymbol{x}_k|_\infty \tag{10}$$

and

$$|\boldsymbol{x}_{k+2} - \hat{\boldsymbol{\beta}}|_\infty \le \theta |\boldsymbol{x}_k - \hat{\boldsymbol{\beta}}|_\infty.$$

The second inequality shows that $\boldsymbol{x}_k$ converges to $\hat{\boldsymbol{\beta}}$ geometrically fast, and the first inequality gives

$$\begin{aligned} |\boldsymbol{x}_0 - \hat{\boldsymbol{\beta}}|_\infty &\le \sum_{k=0}^{\infty} |\boldsymbol{x}_k - \boldsymbol{x}_{k+1}|_\infty \\ &\le \frac{1}{1-\theta}(|\boldsymbol{x}_0 - \boldsymbol{x}_1|_\infty + |\boldsymbol{x}_1 - \boldsymbol{x}_2|_\infty) \\ &\le \frac{2}{1-\theta}|\boldsymbol{x}_0 - \boldsymbol{x}_1|_\infty. \end{aligned}$$

Finally, note that if $\hat{\boldsymbol{\beta}}$ does not exist, then the sequence $\{\boldsymbol{x}_k\}$ must have a divergent subsequence. For, otherwise, (9) would imply that (10) must hold for all k for some $\theta \in [0, 1)$. And this, in turn, would imply that $\boldsymbol{x}_k$ must converge to a limit as $k \to \infty$, which would then be a fixed point of φ and therefore a solution of the ML equations. This completes the proof.

Before moving to the next section we will prove a technical lemma which will be of use in the proof of Theorem 1.1 based on the above calculations.

Lemma 2.2. *Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ such that $\max\{|\boldsymbol{x}|_\infty, |\boldsymbol{y}|_\infty\} \le K$. Then*

$$|\varphi(\boldsymbol{x}) - \varphi(\boldsymbol{y})|_1 \le 2e^{2K} |\boldsymbol{x} - \boldsymbol{y}|_1,$$

where $|\cdot|$ is the usual L^1 norm on $\mathbb{R}^n$.

Proof. By equation (8),

$$\begin{aligned}
|\varphi(\boldsymbol{x}) - \varphi(\boldsymbol{y})|_1 &= |J(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})|_1 \\
&= \sum_{i=1}^{n} \left| \sum_{j=1}^{n} (x_j - y_j) \cdot \int_0^1 \frac{\partial \varphi_i}{\partial x_j}(t\boldsymbol{x} + (1-t)\boldsymbol{y})dt \right| \\
&\le \sum_{j=1}^{n} |x_j - y_j| \cdot \left(\sum_{i=1}^{n} \sup_{t\in[0,1]} \left| \frac{\partial \varphi_i}{\partial x_j}(t\boldsymbol{x} + (1-t)\boldsymbol{y}) \right| \right) \\
&\le \sum_{j=1}^{n} 2e^{2K} |x_j - y_j| = e^{2K}|\boldsymbol{x}-\boldsymbol{y}|_1,
\end{aligned}$$

where the second inequality follows from equations (6) and (7).

□

3. Proof of Theorem 1.4

We need the following simple technical lemma.

Lemma 3.1. *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a twice differentiable function such that $M := \sup_{\boldsymbol{x}\in\mathbb{R}^n} f(\boldsymbol{x}) < \infty$. Let ∇f and $\nabla^2 f$ denote the gradient vector and the Hessian matrix of f, and suppose there is a finite constant C such that the L^2 operator norm of $\nabla^2 f$ is uniformly bounded by C. Then for any $\boldsymbol{x} \in \mathbb{R}^n$,*

$$|\nabla f(\boldsymbol{x})|^2 \le 2C(M - f(\boldsymbol{x}))$$

where $|\cdot|$ denotes the Euclidean norm. In particular, there exists a sequence $\{\boldsymbol{x}_k\}_{k\ge1}$ such that $\lim_{k\to\infty} \nabla f(\boldsymbol{x}_k) = 0$.

Proof. Fix a point $\boldsymbol{x} \in \mathbb{R}^n$ and let $\boldsymbol{y} = \nabla f(\boldsymbol{x})$. Suppose C is a uniform bound on the L^2 operator norm of $\nabla^2 f$. Then for any $t \ge 0$,

$$|\nabla f(\boldsymbol{x} + t\boldsymbol{y}) - \nabla f(\boldsymbol{x})| \le Ct|\boldsymbol{y}|. \tag{11}$$

Now let $g(t) = f(\boldsymbol{x} + t\boldsymbol{y})$. Then for all t,

$$g(t) - g(0) \le M - f(\boldsymbol{x}).$$

Again, note that

$$\begin{aligned}
g'(t) &= \langle \boldsymbol{y}, \nabla f(\boldsymbol{x} + t\boldsymbol{y}) \rangle \\
&= \langle \boldsymbol{y}, \nabla f(\boldsymbol{x} + t\boldsymbol{y}) - \nabla f(\boldsymbol{x}) \rangle + \langle \boldsymbol{y}, \nabla f(\boldsymbol{x}) \rangle \\
&\ge -Ct|\boldsymbol{y}|^2 + |\boldsymbol{y}|^2.
\end{aligned}$$

(The last step follows by (11) and Cauchy-Schwarz.) Thus, for any $t \ge 0$,

$$M - f(\boldsymbol{x}) \ge g(t) - g(0) = \int_0^t g'(s)ds \ge |\boldsymbol{y}|^2 \int_0^t (1 - Cs)ds.$$

Taking $t = 1/C$ gives the desired result. □

Proof of Theorem 1.4. Let $g = (g_1, \ldots, g_n) : \mathbb{R}^n \to \mathbb{R}^n$ be the function defined as

$$g_i(\boldsymbol{x}) = \sum_{j \neq i} \frac{e^{x_i + x_j}}{1 + e^{x_i + x_j}}, \quad i = 1, \ldots, n.$$

Then $\mathcal{R}$ is the range of g. This is because the expected degree of vertex i of a random graph following the law $\mathbb{P}_{\boldsymbol{x}}$ is $g_i(\boldsymbol{x})$. In particular, the vector $g(\boldsymbol{x})$ is a weighted average of degree sequences, and hence

$$\operatorname{conv}(\mathcal{D}) \supseteq \overline{\mathcal{R}}.$$

Now, for every $\boldsymbol{y} \in \mathbb{R}^n$, let $f_{\boldsymbol{y}} : \mathbb{R}^n \to \mathbb{R}$ be the function

$$f_{\boldsymbol{y}}(\boldsymbol{x}) = \sum_{i=1}^n x_i y_i - \log \sum_{1 \le i < j \le n} (1 + e^{x_i + x_j}).$$

Note that under $\mathbb{P}_{\boldsymbol{x}}$, the probability of obtaining a given graph with degree sequence $d = (d_1, \ldots, d_n)$ is exactly

$$\frac{e^{\sum_i x_i d_i}}{\prod_{i<j} (1 + e^{x_i + x_j})}.$$

Thus, the above quantity must be bounded by 1, and hence taking logs, we get $f_d(\boldsymbol{x}) \le 0$. Since $f_{\boldsymbol{y}}(\boldsymbol{x})$ depends linearly on $\boldsymbol{y}$, this implies that

$$f_{\boldsymbol{y}}(\boldsymbol{x}) \le 0 \ \text{ for all } \boldsymbol{y} \in \operatorname{conv}(\mathcal{D}), \boldsymbol{x} \in \mathbb{R}^n.$$

Now fix $\boldsymbol{y} \in \operatorname{conv}(\mathcal{D})$. Then $f_{\boldsymbol{y}}(\boldsymbol{x}) \le 0$ for all $\boldsymbol{x} \in \mathbb{R}^n$. Moreover,

$$\frac{\partial^2 f_{\boldsymbol{y}}}{\partial x_i \partial x_j} = \frac{-e^{x_i + x_j}}{(1 + e^{x_i + x_j})^2} \in [-1, 0],$$

and therefore $\nabla^2 f$ is uniformly bounded. Hence it follows from Lemma 11 that there exists a sequence $\{\boldsymbol{x}_k\}_{k \ge 1}$ such that $\lim_{k \to \infty} \nabla f_{\boldsymbol{y}}(\boldsymbol{x}_k) = 0$. But

$$\nabla f_{\boldsymbol{y}}(\boldsymbol{x}) = -\boldsymbol{y} + g(\boldsymbol{x}).$$

Thus, $\boldsymbol{y} = \lim_{k \to \infty} g(\boldsymbol{x}_k)$. This shows that

$$\operatorname{conv}(\mathcal{D}) \subseteq \overline{\mathcal{R}}$$

and hence completes the proof of the claim that $\operatorname{conv}(\mathcal{D}) = \overline{\mathcal{R}}$. □

4. Proof of Theorem 1.3 (Existence and consistency of the MLE)

This section uses the notation of Section 1 without explicit reference. The proof consists of two lemmas. The first lemma gives a condition for the 'tightness' of the MLE. This result is closely related to the Erdős-Gallai characterization of degree sequences. The second lemma shows that the conditions needed for the first lemma are satisfied with high probability. An addenda at the end of the section contains some results about existence of the MLE and the closely related topic of conjugate Bayesian analysis.

Lemma 4.1. *Let $(d_1, \ldots, d_n)$ be a point in the set $\overline{\mathcal{R}}$ of Theorem 1.4. Suppose there exist $c_1, c_2 \in (0,1)$ such that $c_2(n-1) \le d_i \le c_1(n-1)$ for all i. Then there exists $b \in (0,1)$ depending only on c_1, c_2, such that if the quantity*

$$c_3 := \frac{1}{n^2} \inf_{B \subseteq \{1,\ldots,n\},\ |B| \ge bn} \biggl\{ \sum_{j \notin B} \min\{d_j, |B|\} + |B|(|B|-1) - \sum_{i \in B} d_i \biggr\}$$

is positive, then a solution $\hat{\boldsymbol{\beta}}$ of (3) exists and satisfies $|\hat{\boldsymbol{\beta}}|_\infty \le c_4$, where c_4 is a finite constant that depends only on c_1, c_2, c_3.

Proof. In this proof, $C(c_1, c_2, c_3)$ denotes positive constants that depend only on c_1, c_2, c_3. The argument repeatedly uses the monotonicity of $e^{x+y}/(1+e^{x+y})$ in x for each y.

Assume first that $\hat{\boldsymbol{\beta}}$ exists, in the sense that there exists $\hat{\boldsymbol{\beta}} \in \mathbb{R}^n$ such that (3) is satisfied. It is proved below that $|\hat{\boldsymbol{\beta}}|_\infty$ is bounded above by $C(c_1, c_2, c_3)$.

Let $d_{\max} := \max_i d_i$ and $d_{\min} := \min_i d_i$. Similarly let $\hat{\beta}_{\max} := \max_i \hat{\beta}_i$ and $\hat{\beta}_{\min} := \min_i \hat{\beta}_i$. The first step is to prove that $\hat{\beta}_{\max} \le C(c_1, c_2, c_3)$. If $\hat{\beta}_{\max} \le 0$, there is nothing to prove. So assume that $\hat{\beta}_{\max} > 0$. Let

$$m := |\{i : \hat{\beta}_i > -\tfrac{1}{2}\hat{\beta}_{\max}\}|.$$

Clearly by the assumption that $\hat{\beta}_{\max} > 0$, it is guaranteed that $m \ge 1$. Let i^* be an index that maximizes $\hat{\beta}_i$. Then by (3) of Section 1, we see that

$$d_{\max} \ge d_{i^*} > (m-1)\frac{e^{\frac{1}{2}\hat{\beta}_{\max}}}{1 + e^{\frac{1}{2}\hat{\beta}_{\max}}}.$$

This implies

$$\begin{aligned} n - m &> n - 1 - d_{\max}(1 + e^{-\frac{1}{2}\hat{\beta}_{\max}}) \\ &\ge n - 1 - c_1(n-1)(1 + e^{-\frac{1}{2}\hat{\beta}_{\max}}). \end{aligned}$$

In particular, this shows that if $\hat{\beta}_{\max} > C(c_1)$ then $m < n$ and hence there exists i such that $\hat{\beta}_i \le -\frac{1}{2}\hat{\beta}_{\max}$. Suppose this is true and fix any such i. (In particular note that $\hat{\beta}_i < 0$.) Let

$$m_i := |\{j : j \ne i,\ \hat{\beta}_j < -\tfrac{1}{2}\hat{\beta}_i\}|.$$

Then by (3) in Section 1,

$$d_{\min} \le d_i < m_i \frac{e^{\frac{1}{2}\hat{\beta}_i}}{1 + e^{\frac{1}{2}\hat{\beta}_i}} + n - 1 - m_i,$$

which gives

$$\begin{aligned} m_i &< (n - 1 - d_{\min})(1 + e^{\frac{1}{2}\hat{\beta}_i}) \\ &\le (n-1)(1 - c_2)(1 + e^{-\frac{1}{4}\hat{\beta}_{\max}}). \end{aligned}$$

Note that there are at least $n - m_i$ indices j such that $\hat{\beta}_j \ge -\frac{1}{2}\hat{\beta}_i \ge \frac{1}{4}\hat{\beta}_{\max}$. The last display implies that if $\hat{\beta}_{\max} > C(c_1, c_2)$, then there exists i such that $n - m_i \ge bn$, where $b \in (0,1)$ is a constant that depends only on c_1, c_2. Consequently, if $\hat{\beta}_{\max} > C(c_1, c_2)$, there is a set $A \subseteq \{1, \ldots, n\}$ of size at least bn such that $\hat{\beta}_j \ge \frac{1}{4}\hat{\beta}_{\max}$ for all $j \in A$, where $b \in (0,1)$ is a constant that depends only on c_1, c_2. Henceforth, assume that $\hat{\beta}_{\max}$ is so large that such a set exists. Let

$$h := \sqrt{\hat{\beta}_{\max}}.$$

For each integer r between 0 and $\frac{1}{16}h - 1$, let

$$D_r := \{i : -\tfrac{\hat{\beta}_{\max}}{8} + rh \le \hat{\beta}_i < -\tfrac{\hat{\beta}_{\max}}{8} + (r+1)h\}.$$

Since $D_0, D_1, \ldots$ are disjoint, there exists r such that

$$|D_r| \le \frac{n}{\frac{1}{16}h - 1},$$

provided $h > 16$. By assumption, $\hat{\beta}_{\max} > C(c_1, c_2)$. Since we are free to choose $C(c_1, c_2)$ as large as we like, it can be assumed without loss of generality that $h > 16$.

Fix such an r between 0 and $\frac{1}{16}h - 1$. Let

$$B := \{i : \hat{\beta}_i \ge \tfrac{\hat{\beta}_{\max}}{8} - (r + \tfrac{1}{2})h\}.$$

Clearly, the set B contains the previously defined set A, and hence

$$|B| \ge bn. \tag{12}$$

Now, for each $i \neq j$, define

$$\hat{p}_{ij} := \frac{e^{\hat{\beta}_i + \hat{\beta}_j}}{1 + e^{\hat{\beta}_i + \hat{\beta}_j}}.$$

For each i, let

$$d_i^B := \sum_{j \in B \backslash \{i\}} \hat{p}_{ij}.$$

Since $\hat{\beta}_i \ge \frac{\hat{\beta}_{\max}}{16}$ for each $i \in B$, it follows that

$$\begin{aligned}
|B|(|B|-1) - \sum_{i \in B} d_i^B &= |B|(|B|-1) - \sum_{i,j \in B,\ i \neq j} \hat{p}_{ij} \\
&= \sum_{i,j \in B,\ i \neq j} (1 - \hat{p}_{ij}) \\
&\le \frac{|B|(|B|-1)}{1 + e^{\frac{1}{8}\hat{\beta}_{\max}}}.
\end{aligned} \tag{13}$$

The above inequality is the first step of a two-step argument. For the second step, take any $j \notin B$. Consider three cases. First, suppose $\hat{\beta}_j \ge -\frac{\hat{\beta}_{\max}}{8} +$

$(r+1)h$. Then for each $i \in B$, $\hat{\beta}_i + \hat{\beta}_j \geq \frac{h}{2}$ and therefore

$$\begin{aligned}\min\{d_j, |B|\} - d_j^B &\leq |B| - \sum_{i \in B} \hat{p}_{ij} \\ &= \sum_{i \in B} (1 - \hat{p}_{ij}) \leq \frac{|B|}{1 + e^{\frac{h}{2}}}.\end{aligned}$$

Next, suppose $\hat{\beta}_j \leq -\frac{\hat{\beta}_{\max}}{8} + rh$. Then for any $i \notin B$, $\hat{\beta}_i + \hat{\beta}_j \leq -\frac{h}{2}$. Thus,

$$\begin{aligned}\min\{d_j, |B|\} - d_j^B &\leq d_j - d_j^B \\ &= \sum_{i \notin B,\ i \neq j} \hat{p}_{ij} \leq n e^{-\frac{h}{2}}.\end{aligned}$$

Finally, the third case covers all $j \notin B$ that do not fall in either of the previous two cases. This is a subset of the set of all j comprising the set D_r. Combining the three cases gives

$$\sum_{j \notin B} (\min\{d_j, |B|\} - d_j^B) \leq \frac{n^2}{1 + e^{\frac{h}{2}}} + n^2 e^{-\frac{h}{2}} + \frac{16n^2}{h - 16}. \tag{14}$$

But

$$\sum_{j \notin B} d_j^B = \sum_{i \in B,\ j \notin B} \hat{p}_{ij} = \sum_{i \in B} (d_i - d_i^B).$$

Thus, adding (13) and (14),

$$\begin{aligned}&\sum_{j \notin B} \min\{d_j, |B|\} + |B|(|B| - 1) - \sum_{i \in B} d_i \\ &\leq \frac{n^2}{1 + e^{\frac{1}{8}\hat{\beta}_{\max}}} + \frac{n^2}{1 + e^{\frac{h}{2}}} + n^2 e^{-\frac{h}{2}} + \frac{16n^2}{h - 16}.\end{aligned} \tag{15}$$

The left hand side of the above inequality is bounded below by $c_3 n^2$, by the definition of c_3 in the statement of the theorem. The coefficient of n^2 on the right hand side tends to zero as $\hat{\beta}_{\max} \to \infty$. This shows that $\hat{\beta}_{\max} \leq C(c_1, c_2, c_3)$, where the bound is finite since $c_3 > 0$. Next, note that for any i,

$$d_i \leq \frac{n e^{\hat{\beta}_i + \hat{\beta}_{\max}}}{1 + e^{\hat{\beta}_i + \hat{\beta}_{\max}}},$$

and therefore if i^{**} is a vertex that minimizes $\hat{\beta}_i$, then

$$d_{\min} \leq d_{i^{**}} \leq \frac{n e^{\hat{\beta}_{\min} + \hat{\beta}_{\max}}}{1 + e^{\hat{\beta}_{\min} + \hat{\beta}_{\max}}}.$$

Combined with the upper bound on $\hat{\beta}_{\max}$ and the lower bound on $d_{\min}$, this shows that $\hat{\beta}_{\min} \geq -C(c_1, c_2, c_3)$.

To complete the proof of the lemma, it must be proved that $\hat{\boldsymbol{\beta}}$ exists. Since $(d_1, \ldots, d_n) \in \overline{\mathcal{R}}$, by Theorem 1.4 there is a sequence of points $\{\boldsymbol{x}_k\}_{k \geq 0}$ in $\mathbb{R}^n$

that converge to $(d_1, \ldots, d_n)$, for which solutions to (3) exist. Let $\{\hat{\boldsymbol{\beta}}_k\}_{k\geq 0}$ denote a sequence of solutions. The steps above prove that $|\hat{\boldsymbol{\beta}}_k|_\infty \leq C$ for all large enough k where C is some constant depending only on c_1, c_2, c_3. Therefore the sequence $\{\hat{\boldsymbol{\beta}}_k\}_{k\geq 0}$ must have a limit point. This limit point is clearly a solution to (3) for the original sequence $d_1, \ldots, d_n$. □

The next lemma shows that the degree sequence in a typical realization of our random graph satisfies the conditions of Lemma 4.1.

Lemma 4.2. *Let G be drawn from the probability measure $\mathbb{P}_{\boldsymbol{\beta}}$ and let $d_1, \ldots, d_n$ be the degree sequence of G. Let $L := \max_{1\leq i\leq n} |\beta_i|$, and let $c \in (0,1)$ be any constant. Then there are constants $C > 0$ and $c_1, c_2 \in (0,1)$ depending only on L and a constant $c_3 \in (0,1)$ depending only on L and c such that if $n > C$, then with probability at least $1 - 2n^{-2}$, $c_2(n-1) \leq d_i \leq c_1(n-1)$ for all i, and*

$$\frac{1}{n^2} \inf_{B\subseteq\{1,\ldots,n\},\ |B|\geq cn} \left\{\sum_{j\notin B} \min\{d_j, |B|\} + |B|(|B|-1) - \sum_{i\in B} d_i\right\} \geq c_3.$$

Proof. Let

$$\bar{d}_i := \sum_{j\neq i} \frac{e^{\beta_i+\beta_j}}{1+e^{\beta_i+\beta_j}}.$$

Note that for each i, $d_i \sim \mathit{Binomial}(n, \bar{d}_i)$. Therefore by Hoeffding's inequality [24],

$$\mathbb{P}(|d_i - \bar{d}_i| > x) \leq 2e^{-x^2/2n}.$$

Thus, if we let E be the event

$$\{\max_i |d_i - \bar{d}_i| > \sqrt{6n \log n}\},$$

then by a union bound,

$$\mathbb{P}(E) \leq \frac{2}{n^2}.$$

Now clearly, there are constant c_1' and c_2' depending only on L such that $c_2'(n-1) \leq \bar{d}_i \leq c_1'(n-1)$ for all i. Therefore under E^c, if n is sufficiently large (depending on L), we get constants c_1, c_2 depending only on L such that $c_2(n-1) \leq d_i \leq c_1(n-1)$ for all i.

Next, define

$$g(d_1, \ldots, d_n, B) := \sum_{j\notin B} \min\{d_j, |B|\} + |B|(|B|-1) - \sum_{i\in B} d_i.$$

Note that

$$|g(d_1, \ldots, d_n, B) - g(\bar{d}_1, \ldots, \bar{d}_n, B)| \leq \sum_{i=1}^n |d_i - \bar{d}_i| \leq n \max_i |d_i - \bar{d}_i|.$$

Moreover, following the notation introduced in the proof of Lemma 4.1, we have

$$\begin{aligned}
&g(\bar{d}_1, \ldots, \bar{d}_n, B) \\
&= \sum_{j \notin B} (\min\{\bar{d}_j, |B|\} - \bar{d}_j^B) + |B|(|B| - 1) - \sum_{i \in B} \bar{d}_i^B \\
&\geq |B|(|B| - 1) - \sum_{i \in B} \bar{d}_i^B \\
&= \sum_{i,j \in B,\ i \neq j} (1 - p_{ij}) \geq c_4 |B|(|B| - 1),
\end{aligned}$$

where $c_4 \in (0,1)$ is a constant depending only on L. Thus, under E^c and n sufficiently large, we have

$$g(d_1, \ldots, d_n, B) \geq c_3 n^2$$

where $c_3 \in (0,1)$ is a constant depending only on L and c. This completes the proof of the lemma. □

Proof of Theorem 1.3. Let E be the event defined in the proof of Lemma 4.2. Let C, c_1, c_2 be as in Lemma 4.2. By Lemma 4.2 and Lemma 4.1, if E^c happens and $n > C$, then a solution $\hat{\boldsymbol{\beta}}$ of (3) exists and satisfies $|\hat{\boldsymbol{\beta}}|_\infty \leq C(L)$, where $C(L)$ generically denotes a constant that depends only on L. This proves the existence of the MLE. The uniqueness follows from Theorem 1.5.

The proof of the error bound uses Theorem 1.5. Let $\boldsymbol{x}_0 = \boldsymbol{\beta}$, and define $\{\boldsymbol{x}_k\}_{k \geq 1}$ as in Theorem 1.5. A simple computation shows that the ith component of $\boldsymbol{x}_0 - \boldsymbol{x}_1$ is simply $\log(\bar{d}_i/d_i)$. Under E^c and $n > C$, this is bounded by $C(L)\sqrt{n^{-1}\log n}$. The error bound now follows directly from Theorem 1.5.

Finally, the remove the condition $n > C$, we simply increase $C(L)$ in Theorem 1.3 so that $1 - C(L)n^{-2} < 0$ for $n \leq C$. This completes the proof of Theorem 1.3. □

Addenda. (A) Practical remarks on the MLE. Theorem 1.3 shows that with high probability, under the $\mathbb{P}_{\boldsymbol{\beta}}$ measure, for large n, the MLE exists and is unique. In applications, a graph is given and Theorem 1.3 may be used to test the $\mathbb{P}_{\boldsymbol{\beta}}$ model. The MLE may fail to exist because the maximum is taken on at $\beta_i = \pm\infty$ for one or more values of i. For example with $n = 2$ vertices, an observed graph will either have zero edges or one edge. In the first case, the likelihood is $1/(1 + e^{\beta_1 + \beta_2})$, maximized at $\beta_1 = \beta_2 = -\infty$. In the second case the likelihood is $e^{\beta_1 + \beta_2}/(1 + e^{\beta_1 + \beta_2})$, maximized at $\beta_1 = \beta_2 = \infty$. Here, the MLE fails to exist with probability one.

Similar considerations holds when the observed graph has any isolated vertices and for a star graph. We conjecture: Let G be a graph on n vertices. The MLE for the $\boldsymbol{\beta}$-model exists if and only if the degree sequence lies in the interior of the convex polytope $\mathrm{conv}(\mathcal{D})$ defined in Theorem 1.4.

In cases where the MLE does not exist, it is customary to add a small amount to each degree (see the discussion in Bishop-Fienberg-Holland [9]). This is often done in a convenient and principled way by using a Bayesian argument.

(B) Conjugate prior analysis for the $\boldsymbol{\beta}$-model. Background on conjugate priors for exponential families is in [18] and [22, 23]. The $\boldsymbol{\beta}$-model

$$\mathbb{P}_{\boldsymbol{\beta}}(G) = Z(\boldsymbol{\beta})^{-1} e^{\sum_{i=1}^{n} d_i(G)\beta_i}, \ \boldsymbol{\beta} \in \mathbb{R}^n, \ Z(\boldsymbol{\beta}) = \prod_{1\le i<j\le n} (1 + e^{\beta_i+\beta_j})$$

has sufficient statistic $\boldsymbol{d} = (d_1, \ldots, d_n)$. Here $\boldsymbol{d}$ takes values in $\mathcal{D}$, the set of degree sequences for graphs on n vertices. Thus $\mathbb{P}_{\boldsymbol{\beta}}$ induces a natural exponential family on $\mathcal{D}$ with a base measure μ that does not depend on $\boldsymbol{\beta}$. Following notation in [18], write

$$\mathbb{P}_{\boldsymbol{\beta}}(\boldsymbol{d}) = \mu(\boldsymbol{d}) e^{\boldsymbol{\beta}\cdot\boldsymbol{d} - m(\boldsymbol{\beta})} \text{ with } m(\boldsymbol{\beta}) = \log Z(\boldsymbol{\beta}) = \sum_{1\le i<j\le n} \log(1 + e^{\beta_i+\beta_j}).$$

Following [18], for $\boldsymbol{d}_0$ in the interior of $\mathrm{conv}(\mathcal{D})$ and $n_0 > 0$, define the conjugate prior of $\mathbb{R}^n$ by

$$\pi_{n_0,\boldsymbol{d}_0}(\boldsymbol{\beta}) = Z(n_0, \boldsymbol{d}_0)^{-1} e^{n_0\boldsymbol{d}_0\cdot\boldsymbol{\beta} - n_0 m(\boldsymbol{\beta})}.$$

Here $Z(n_0, \boldsymbol{d}_0)$ is the normalizing constant, shown to be positive and finite in [18]. By the theory in [18], $\nabla m(\boldsymbol{\beta}) = \mathbb{E}_{\boldsymbol{\beta}}(\boldsymbol{d})$ and

$$\mathbb{E}_{\pi_{n_0,\boldsymbol{d}_0}}(\nabla m(\boldsymbol{\beta})) = \mathbb{E}_{\pi_{n_0,\boldsymbol{d}_0}}(\mathbb{E}_{\boldsymbol{\beta}}(\boldsymbol{d})) = \boldsymbol{d}_0.$$

This identity characterizes the prior $\pi_{n_0,\boldsymbol{d}_0}$. The posterior, given an observed degree sequence $\boldsymbol{d}(G)$, is

$$\pi_{n_0+1,(\boldsymbol{d}(G)+n_0\boldsymbol{d}_0)/(n_0+1)}.$$

Clearly, the mode of the posterior can be found by using the iteration of Theorem 1.5. The proof of Theorem 1.5 shows that the mode exists uniquely for any observed $\boldsymbol{d}(G)$. The posterior mean must be found using standard Markov chain Monte Carlo techniques.

A natural way to obtain feasible prior mean parameters (i.e. values of $\boldsymbol{d}_0$ that lie within the interior of $\mathrm{conv}(\mathcal{D})$) is to consider a model of random graphs that puts positive mass on every possible graph on n vertices, and take its expected degree sequence. For example, the Erdős-Rényi graph $G(n,p)$, for $0 < p < 1$, is one such model. Its expected degree sequence is $(c, c, \ldots, c)$ where $c = (n-1)p$. Thus, $(c, c, \ldots, c)$ is a feasible mean parameter for every $c \in (0, n-1)$. Similarly, the expected degree sequence in any of the standard models of power law graphs is a feasible value of $\boldsymbol{d}_0$ that has power law behavior.

5. Proof of Proposition 1.2 (Characterization of the interior)

Suppose $d_1 \ge d_2 \ge \cdots \ge d_n$ are nonnegative integers. The Erdős-Gallai criterion says that $d_1, \ldots, d_n$ can be the degree sequence of a simple graph on n vertices if and only if $\sum_{i=1}^n d_i$ is even and for each $1 \le k \le n$,

$$\sum_{i=1}^k d_i \le k(k-1) + \sum_{i=k+1}^n \min\{d_i, k\}.$$

Now take any function $f \in D'[0,1]$ and let

$$G_f(x) := \int_x^1 \min\{f(y), x\}dy + x^2 - \int_0^x f(y)dy.$$

Clearly $G_f(x)$ is continuous as a function of x. If $f \in \mathcal{F}$, the E-G criterion clearly shows that G_f must be a nonnegative function. We claim that this implies that if f belongs to the interior of $\mathcal{F}$, then $G_f(x)$ must be strictly positive for every $x \in (0,1]$. For, otherwise, there exists $x \in (0,1]$ such that $G_f(x) = 0$. If we show that there exists a sequence $f_n \to f$ in the modified L^1 topology such that $G_{f_n}(x) < 0$ for each n, then we get a contradiction which proves the claim. This is quite easily done by producing f_n that is strictly bigger than f in $[0,x)$ and equal to f elsewhere, all the while maintaining left-continuity.

Similarly, it is clear that any $f \in \mathcal{F}$ must take values in $[0,1]$. If f attains 0 or 1, then we can produce a sequence $f_n \to f$ whose ranges are not contained in $[0,1]$, and therefore f cannot belong to the interior of $\mathcal{F}$.

Thus, we have proved that if f belongs to the interior of $\mathcal{F}$, then f must satisfy the two conditions of Proposition 1.2. Let us now prove the converse. Suppose $f \in D'[0,1]$ such that $0 < c_1 < f(x) < c_2 < 1$ for all $x \in [0,1]$ and $G_f(x) > 0$ for all $x \in (0,1]$. We have to show that any function that is sufficiently close to f in the modified L^1 norm must belong to $\mathcal{F}$.

To do that let us first prove that $f \in \mathcal{F}$. Take any n. Let $d_i^n = \lfloor nf(i/n) \rfloor$, $i = 1, \ldots n$. Since f is non-increasing, we have $d_1^n \ge d_2^n \ge \cdots \ge d_n^n$. Increase some of the d_i^n's by 1, if necessary, so that $\sum d_i^n$ is even (and monotonicity is maintained). With this construction, it is clear that

$$\max_{1\le i\le n} |d_i^n/n - f(i/n)| \le 1/n.$$

Thus, if $\boldsymbol{d}^n$ denotes the vector $(d_1^n, \ldots, d_n^n)$, then $\boldsymbol{d}^n$ converges to the scaling limit f. We need to show that for all large enough n, $\boldsymbol{d}^n$ is a valid degree sequence.

Since f is bounded and non-increasing

$$\lim_{n\to\infty} \int_0^1 |f(x) - f(\lceil nx \rceil/n)|\, dx = 0$$

and so uniformly in $1 \le k \le n$,

$$\left| \frac{\sum_{i=k+1}^n \min\{d_i^n, k\} + k(k-1) - \sum_{i=1}^k d_i^n}{n^2} - G_f(k/n) \right| \le \epsilon(n),$$

where $\epsilon(n) \to 0$ as $n \to \infty$. Thus, there exists a sequence of integers $\{k_0(n)\}$, where $k_0(n)/n \to 0$ as $n \to \infty$, such that whenever $k \geq k_0(n)$, we have

$$\sum_{i=k+1}^{n} \min\{d_i^n, k\} + k(k-1) - \sum_{i=1}^{k} d_i^n > 0.$$

Again, there exists $c_1' > 0$ and $c_2' < 1$ such that if n is sufficiently large, we have $c_1' \leq d_i^n/n \leq c_2'$ for all i. Suppose n is so large that $k_0(n)/n < c_1'$ and $(1-c_2')n - k_0(n) > 0$. Then, if $k \leq k_0(n)$, we have

$$\begin{aligned}
&\sum_{i=k+1}^{n} \min\{d_i^n, k\} + k(k-1) - \sum_{i=1}^{k} d_i^n \\
&\geq \sum_{i=k+1}^{n} \min\{c_1'n, k\} + k(k-1) - \sum_{i=1}^{k} nc_2' \\
&= (n-k)k + k(k-1) - c_2'nk = ((1-c_1')n - k)k + k(k-1) > 0.
\end{aligned}$$

Thus, for n so large, we have that for all $1 \leq k \leq n$,

$$\sum_{i=k+1}^{n} \min\{d_i^n, k\} + k(k-1) - \sum_{i=1}^{k} d_i^n > 0.$$

By the Erdős-Gallai criterion, this shows that $(d_1^n, \ldots, d_n^n)$ is a valid degree sequence.

Thus, we have shown that any f that satisfies the two conditions of Proposition 1.2 must belong to $\mathcal{F}$. Now we only have to show that if f satisfies the two criteria, then any h sufficiently close to f in the modified L^1 norm must also satisfy them.

Note that G_f is a continuous function that is positive in $(0,1]$. Moreover, for all $0 \leq x \leq 1$,

$$|G_f(x) - G_{f'}(x)| \leq \|f - f'\|_{1'}$$

so if $f_n \to f$ in the modified L^1 norm, then $G_{f_n} \to G_f$ in the supnorm. Thus, for any $\epsilon > 0$ there exists $\delta > 0$ such that whenever $\|h - f\|_{1'} < \delta$, we have $G_h(x) > 0$ for all $x \in [\epsilon, 1]$. We also have that $c_1 - \delta \leq h(x) \leq c_2 + \delta$ for all $0 \leq x \leq 1$. Choosing $\delta, \epsilon > 0$ small as necessary, we can ensure that $c_1 - \delta > \epsilon$ and $1 - \epsilon - \delta - c_2 > 0$. Fix such ϵ, δ and h. Then, for $x \in (0, \epsilon)$, we have

$$\begin{aligned}
G_h(x) &\geq \int_x^1 \min\{c_1 - \delta, x\} dy + x^2 - \int_0^x (c_2 + \delta) dy \\
&= (1-x)x + x^2 - (c_2+\delta)x = (1 - \epsilon - \delta - c_2)x + x^2 > 0.
\end{aligned}$$

But we also have $G_h(x) > 0$ for $x \in [\epsilon, 1]$ by the choice of δ. Thus, we have proved that there exists $\delta > 0$ such that whenever $\|h - f\|_{1'} < \delta$, we have $G_h(x) > 0$ for all $x \in (0,1]$. Choosing δ sufficiently small, we can ensure that the range of h does not contain 0 or 1. This completes the proof of Proposition 1.2. $\square$

Proposition 1.2 can be extended into a complete version of the Erdős-Gallai criterion for graph limits. Suppose that $W(x, y)$ is a symmetric function from $[0, 1]^2$ into $[0, 1]$. In Section 4 of [16] it is shown that the correct analog of the degree distribution for the graph limit W is the distribution of the random variable

$$X = \int_0^1 W(U, y)dy, \tag{16}$$

where U is a random variable distributed uniformly in $[0, 1]$. If a sequence of graphs converges to W then the distribution of the random variable d_i/n (where i is chosen uniformly from n vertices and d_i is the degree of i) converges to X in distribution. The following result characterizes limiting degree variates.

Proposition 5.1. *Let X be a random variable with values in $[0, 1]$. Let $D(x) = \sup\{y : P(X > y) \geq x\}$. Then X has the representation (16) if and only if for all $x \in (0, 1]$*

$$\int_0^x D(y)dy \leq x^2 + \int_x^1 \min\{D(y), x\}dy.$$

The proof is essentially as given above, approximating W by a sequence of finite graphs and using the Erdős-Gallai criterion. We omit further details.

6. Proof of Theorem 1.1 (Convergence to graph limit)

6.1. **Preliminary lemmas.** We need a couple of probabilistic results before we can embark on the proof of Theorem 1.1. The first one is a simple application of the method of bounded differences for concentration inequalities.

Lemma 6.1. *Let H be a finite simple graph of size $\leq n$. Let G be a random graph on n vertices with independent edges. Let $t(H, G)$ be the homomorphism density of H in G, defined in (1). Then for any $\epsilon > 0$,*

$$\mathbb{P}(|t(H, G) - \mathbb{E}t(H, G)| > \epsilon) \leq 2e^{-C\epsilon^2 n^2},$$

where C is a constant that depends only on H.

Proof. The proof is a simple consequence of the bounded difference inequality [34]. Note that the quantity $t(H, G)$ is a function of the edges of G, considered as independent Bernoulli random variables. When a particular edge is added or removed (i.e. the corresponding Bernoulli variable is set equal to 1 or 0), $\hom(H, G)$ is altered by at most $Cn^{|V(H)|-2}$, where C is a constant that depends only on H. This is because when we fix an edge, we are fixing its two endpoints, which leaves us the freedom of choosing the remaining $|V(H)|-2$ vertices arbitrarily when constructing a homomorphism.

Thus, alteration of the status of an edge changes $t(H, G)$ by at most Cn^{-2}. The bounded difference inequality completes the proof. $\square$

The second preliminary result that we need is a kind of local limit theorem that we need to pass from the $\boldsymbol{\beta}$-model to graphs with given degree sequence.

Let $\boldsymbol{d} = (d_1, ..., d_n)$ be a valid degree sequence on a graph of size n. Let $G = (V, E)$ be a random graph on n vertices labeled $1, \ldots, n$ so that edges i, j are connected with probability p_{ij} satisfying $d_i = \sum_{j \neq i} p_{ij}$ and so that $\delta \leq p_{ij} \leq 1 - \delta$ for some fixed $0 < \delta < \frac{1}{2}$. Let w_{ij} denote the indicator that there is an edge from i to j. We obtain a lower bound on the probability that G has degree sequence $\boldsymbol{d}$.

Lemma 6.2. *For any $\epsilon > 0$ and large enough n the random graph G has degree sequence $\boldsymbol{d}$ with probability at least $\frac{1}{2}\exp(-\log(\delta) n^{\frac{3}{2}+\epsilon})$.*

We first prove the following claim about the existence of 0-1 contingency tables. An $m \times n$ 0-1 contingency table with integer row and column sums $r_1, \ldots, r_m$ and $c_1, \ldots, c_n$ is an $m \times n$ matrix whose entries are 0 or 1 whose ith row and jth column sum to r_i and c_j respectively. Denote the conjugate sequences as $r_i^* = \#\{r_j : r_j \geq i\}$ and $c_i^* = \#\{c_j : c_j \geq i\}$. Let $(r_{[i]}), (c_{[i]})$ denote the order statistics of (r_i) and (c_i), that is the permutation of the sequences such that $r_{[1]} \geq r_{[2]} \geq \ldots \geq r_{[m]}$.

A condition of Gale and Ryser [21, 47] says that there exists a 0-1 contingency table for row and column sums $r_1, \ldots, r_m$ and $c_1, \ldots, c_n$ if and only if $\sum_{i=1}^m r_i = \sum_{i=1}^n c_i$ and

$$\sum_{i=1}^{k} r_{[i]} \leq \sum_{i=1}^{k} c_i^*, \quad 1 \leq k \leq m, \tag{17}$$

$$\sum_{i=1}^{k} c_{[i]} \leq \sum_{i=1}^{k} r_i^*, \quad 1 \leq k \leq n. \tag{18}$$

Claim 6.1. *Let $0 < \delta < \frac{1}{2}$ and let (p_{ij}) be an $m \times n$ matrix such that $\delta \leq p_{ij} \leq 1 - \delta$. Suppose that (r_i) and (c_i) are integer sequences satisfying the following:*

- $\sum_{i=1}^m r_i = \sum_{i=1}^n c_i$
- $|r_i - \sum_{j=1}^n p_{ij}| \leq \frac{1}{4}\delta^2 n$ *for* $1 \leq i \leq m$
- $|c_j - \sum_{i=1}^m p_{ij}| \leq \frac{1}{4}\delta^2 m$ *for* $1 \leq j \leq n$

Then there exists a 0-1 contingency table with row and column sums (r_i) and (c_i).

Proof. We establish that the Gale-Ryser conditions hold. Without loss of generality we may assume that $r_1 \geq r_2 \geq \ldots \geq r_m$. Then condition (17) is equivalent to

$$\sum_{i=1}^{k} r_i \leq \sum_{i=1}^{k} c_i^* = \sum_{i=1}^{k} \sum_{j=1}^{n} \mathbb{1}_{\{c_j \geq i\}} = \sum_{j=1}^{n} \min\{k, c_j\}. \tag{19}$$

Now

$$\sum_{i=1}^{k} r_i \leq \sum_{i=1}^{k}\sum_{j=1}^{n} p_{ij} + \frac{1}{4}\delta^2 kn$$

and hence

$$\begin{aligned}\sum_{j=1}^{n} \min\{k, c_j\} &\geq \sum_{j=1}^{n} \min\{k, \sum_{i=1}^{m} p_{ij} - \frac{1}{4}\delta^2 m\} \\ &\geq \sum_{i=1}^{k}\sum_{j=1}^{n} p_{ij} + \sum_{j=1}^{n} \min\{k - \sum_{i=1}^{k} p_{ij}, \sum_{i=k+1}^{m} p_{ij} - \frac{1}{4}\delta^2 m\} \\ &\geq \sum_{i=1}^{k}\sum_{j=1}^{n} p_{ij} + \sum_{j=1}^{n} \min\{\delta k, (m-k)\delta - \frac{1}{4}\delta^2 m\} \\ &\geq \sum_{i=1}^{k} r_i + n(\min\{\delta k, (m-k)\delta - \frac{1}{4}\delta^2 m\} - \frac{1}{4}\delta^2 k),\end{aligned}$$

where we used the fact that $\delta \leq p_{ij} \leq 1-\delta$. Now $\delta k \geq \frac{1}{4}\delta^2 k$ and when $1 \leq k \leq m(1-\delta+\frac{1}{4}\delta^2)$,

$$(m-k)\delta - \frac{1}{4}\delta^2 m \geq m(\delta - \frac{1}{4}\delta^2 m) - \frac{1}{4}\delta^2 m \geq \frac{1}{4}\delta^2 k$$

and hence $n(\min\{\delta k, (m-k)\delta - \frac{1}{4}\delta^2 m\} - \frac{1}{4}\delta^2 k) \geq 0$. To establish equation (19) it then suffices to consider $m(1-\delta+\frac{1}{4}\delta^2) \leq k \leq m$. In this case,

$$c_j \leq \sum_{i=1}^{m} p_{ij} + \frac{1}{4}\delta^2 m \leq (1-\delta)m + \frac{1}{4}\delta^2 m \leq k$$

and so

$$\sum_{j=1}^{n} \min\{k, c_j\} = \sum_{j=1}^{n} c_j = \sum_{i=1}^{m} r_i \geq \sum_{i=1}^{k} r_{[i]}$$

establishing (17). Condition (18) follows similarly and hence there exists a 0-1 contingency table with the prescribed row and column sums. □

We are now ready to prove Lemma 6.2.

Proof of Lemma 6.2. We split the n vertices into subsets $A = 1, 2, ..., n - n^a$ and $B = n - n^a + 1, ..., n$ where $a = \frac{1}{2} + \epsilon$. For $1 \leq i < j \leq |A|$, choose w_{ij} according to p_{ij}. Let $\mathcal{G}$ denote the event that the following conditions hold:

- For all $i \in A$

$$\left| \sum_{j \in A-i} w_{ij} - \sum_{j \in A-i} p_{ij} \right| < n^{\frac{1+\epsilon}{2}}, \tag{20}$$

- That the total number of edges in the subgraph induced by A satisfies

$$\sum_{i\in A}\Big(d_i - \sum_{j\in A-i} w_{ij}\Big) < \sum_{i\in B} d_i < \sum_{i\in A}\Big(d_i - \sum_{j\in A-i} w_{ij}\Big) + |B|(|B|-1). \tag{21}$$

Both conditions hold with high probability by simple applications of Azuma's inequality. For example the first follows from Azuma's inequality as

$$P\Big(\Big|\sum_{j\in A-i} w_{ij} - Ew_{ij}\Big| \geq n^{\frac{1+\epsilon}{2}}\Big) \leq 2e^{-\frac{1}{2}n^{\epsilon}}$$

and taking a union bound over $i \in A$.

We will show that given $\mathcal{G}$ there is always a way to add edges between vertices in $B \times V$ so that the graph has degree sequence $\boldsymbol{d}$. First we chose any assignment of the edges $(w_{ij})_{i,j\in B}$ in $B \times B$ so that the total number of edges satisfies

$$\frac{1}{2}\Big(\sum_{i\in B} d_i - \sum_{i\in A}\Big(d_i - \sum_{j\in A-i} w_{ij}\Big)\Big)$$

which is an integer because the sum of the degrees is even and is between 0 and $\frac{1}{2}|B|(|B|-1)$ by equation (21).

It remains to assign edges between A and B so that the graph has degree sequence $\boldsymbol{d}$. This is exactly equivalent to the question of finding a 0-1 contingency table with dimensions $|A| \times |B|$, row sums $r_i = d_i - \sum_{j\in A-i} w_{ij}$ for $i \in A$ and column sums $c_i = d_i - \sum_{j\in B-i} w_{ij}$ for $i \in B$.

Condition (20) guarantees that $r_i = (1+o(1))\sum_{j\in B} p_{ij}$ and since $|B| = o(|A|))$ we have that $c_i = (1+o(1))\sum_{j\in A} p_{ij}$ uniformly in n. Hence by Claim 6.1 a 0-1 contingency table with row and column sums (r_i) and (c_j) exists.

Hence whenever the edges $(w_{ij})_{i,j\in A}$ satisfy $\mathcal{G}$ there exists at least one way to assign the other edges so that the graph has degree sequence $\boldsymbol{d}$. Since any configuration $(w_{ij})_{i\in V,j\in B}$ has probability at least $\delta^{|V||B|}$ and is independent of $\mathcal{G}$ the probability that G has the degree sequence $\boldsymbol{d}$ is at least $P(\mathcal{G})\exp(-\log(\delta)n^{1+a})$ and the result follows since $\mathcal{G}$ holds with high probability. $\square$

An alternative approach in the above lower bound could be through the enumeration of the number of graphs of a particular degree sequence, as carried out in [8]. In fact, this approach would give a better lower bound than the one we obtain. This was brought to our notice recently by Alexander Barvinok.

6.2. **Proof of Theorem 1.1.** Let $\boldsymbol{d}^n$, G_n and f be as in the statement of the theorem. By Proposition 1.2 we know that f has the following two properties.

A. There are two constants $c_1 > 0$ and $c_2 < 1$ such that $c_1 \leq f(x) \leq c_2$ for all $x \in [0,1]$.

B. For each $0 < b \le 1$,

$$\inf_{x\ge b}\left\{\int_x^1 \min\{f(y),x\}dy + x^2 - \int_0^x f(y)dy\right\} > 0.$$

(The infimum is positive because the term within the brackets is a positive continuous function of x.) Now fix n, and for each $B \subseteq \{1,\ldots,n\}$, consider the quantity

$$\mathcal{E}(B) := \sum_{j\notin B}\min\{d_j^n, |B|\} + |B|(|B|-1) - \sum_{i\in B} d_i^n.$$

Under the assumption that $d_1^n \ge d_2^n \ge \cdots \ge d_n^n$, we claim that for each $1 \le k \le n$, $\mathcal{E}(B)$ is minimized over all subsets B of size k when $B = \{1,\ldots,k\}$. To prove this, take any B of size k. Suppose there is $a \in B$ and $b \notin B$ such that $b < a$. Let $B' = (B\backslash\{a\}) \cup \{b\}$. Then clearly, since $d_b^n \ge d_a^n$, we have

$$\sum_{j\notin B}\min\{d_j^n, k\} \ge \sum_{j\notin B'}\min\{d_j^n, k\},$$

and

$$\sum_{i\in B} d_i^n \le \sum_{i\in B'} d_i^n.$$

Thus, $\mathcal{E}(B) \ge \mathcal{E}(B')$, which proves the claim. Now by the definition of convergence of degree sequences and the fact that f is bounded and non-increasing,

$$\begin{aligned}\left|\sum_{i=1}^k \frac{1}{n}\cdot\frac{d_i^n}{n} - \int_0^{k/n} f(y)dy\right| &\le \sum_{i=1}^n \left|\frac{1}{n}\cdot\frac{d_i^n}{n} - \int_{(i-1)/n}^{i/n} f(y)dy\right| \\ &\le \frac{1}{n}\sum_{i=1}^n\left|\frac{d_i^n}{n} - f\left(\frac{i}{n}\right)\right| + \int_0^1 \Big|f(x) - f(\lceil nx\rceil/n)\Big|dx \to 0 \end{aligned} \tag{22}$$

for $1 \le k \le n$. Similarly

$$\sum_{j=k+1}^n \frac{1}{n}\min\left\{\frac{d_j^n}{n}, \frac{k}{n}\right\} - \int_x^1 \min\{f(y),x\}dy \to 0$$

uniformly in $1 \le k \le n$ as $n \to \infty$. Hence we have that for any $b \in (0,1)$,

$$\begin{aligned}&\frac{1}{n^2}\min_{B\subseteq\{1,\ldots,n\},\,|B|\ge bn}\mathcal{E}(B)\\ &= \min_{k\ge bn}\left\{\sum_{j=k+1}^n \frac{1}{n}\min\left\{\frac{d_j^n}{n},\frac{k}{n}\right\} + \frac{k(k-1)}{n^2} - \sum_{i=1}^k \frac{1}{n}\cdot\frac{d_i^n}{n}\right\}\\ &\to \inf_{x\ge b}\left\{\int_x^1 \min\{f(y),x\}dy + x^2 - \int_0^x f(y)dy\right\} \quad \text{as } n\to\infty.\end{aligned}$$

Thus, we can apply Properties A and B of the function f, the definition of scaling limit of degree sequences, and Lemma 4.1 to conclude that for all large n, a solution $\boldsymbol{\beta}^n = (\beta_1^n \ldots, \beta_n^n)$ to (3) for $\boldsymbol{d}^n$ exists and $|\boldsymbol{\beta}^n|_\infty$ is uniformly bounded.

For each n, define a function $g_n : [0,1] \to \mathbb{R}$ as

$$g_n(x) := \beta_i^n \ \text{ if } \ \frac{i-1}{n} < x \le \frac{i}{n},$$

and let $g_n(0) := \beta_1^n$. Now fix two positive integers m, n, and let

$$N := mn.$$

Define a vector $\boldsymbol{x}_0 = (x_{0,1}, \ldots, x_{0,N}) \in \mathbb{R}^N$ as follows:

$$x_{0,i} = \beta_k^n \ \text{ if } \ m(k-1) + 1 \le i \le mk.$$

In other words,

$$\boldsymbol{x}_0 = (\beta_1^n, \beta_1^n, \ldots, \beta_1^n, \beta_2^n, \beta_2^n, \ldots, \beta_2^n, \ldots, \beta_n^n, \beta_n^n, \ldots, \beta_n^n),$$

where each β_k^n is repeated m times. For $\ell \ge 1$ define $\boldsymbol{x}_\ell = \varphi(\boldsymbol{x}_{\ell-1})$ as in Theorem 1.5 (with N in place of n). Equivalently,

$$x_{\ell,i} - x_{\ell-1,i} = \log d_i^N - \log y_{\ell-1,i} = \log \frac{d_i^N/N}{y_{\ell-1,i}/N}. \tag{23}$$

where

$$y_{\ell,i} := \sum_{j \ne i} \frac{e^{x_{\ell,i}+x_{\ell,j}}}{1 + e^{x_{\ell,i}+x_{\ell,j}}}.$$

Note that by definition of $y_{0,1}$ and $x_{0,1}$ if $m(k-1)+1 \le i \le mk$,

$$y_{0,i} - m d_k^n = (m-1) \frac{e^{2\beta_k^n}}{1 + e^{2\beta_k^n}} \le m.$$

Consequently, if $m(k-1)+1 \le i \le mk$,

$$|y_{0,i}/N - d_k^n/n| \le 1/n. \tag{24}$$

Hence by equation (22) it follows that

$$\frac{1}{N} \sum_{k=1}^{N} \left| y_{0,i}/N - d_k^N/N \right| \le \epsilon_1(n)$$

uniformly in N where $\epsilon_1(n) \to 0$ as $n \to \infty$. From (2), (23), (24) (and implicitly using the continuity of log, Property A of the function f and the uniform boundedness of $|\boldsymbol{\beta}^n|_\infty$), we see that

$$|\boldsymbol{x}_0 - \boldsymbol{x}_1|_1 \le N \epsilon_2(n)$$

uniformly in m where $\epsilon_2(n) \to 0$ as $n \to \infty$. Since $|\boldsymbol{\beta}^n|_\infty$ is uniformly bounded in n by Theorem 1.5 it follows that for large enough n, m,

$$|\boldsymbol{x}_\ell - \boldsymbol{\beta}^N|_\infty \le K\theta^\ell, \tag{25}$$

for some K and $0 < \theta < 1$ independent of n and m. Hence for some K' also independent of n, m,

$$\sup |\boldsymbol{x}_\ell|_\infty \le K'.$$

Consequently by Lemma 2.2 we have that

$$|\boldsymbol{x}_0 - \boldsymbol{x}_\ell|_1 \le \left(\sum_{i=1}^{\ell} (2e^{2K'})^i\right) |\boldsymbol{x}_0 - \boldsymbol{x}_1|_1 \tag{26}$$

Combining equations (25) and (26) and using the fact that $|\boldsymbol{x}|_1 \le N|\boldsymbol{x}|_\infty$ we have that

$$\begin{aligned}|\boldsymbol{x}_0 - \boldsymbol{\beta}^N|_1 &\le |\boldsymbol{x}_0 - \boldsymbol{x}_\ell|_1 + |\boldsymbol{x}_\ell - \boldsymbol{\beta}^N|_1 \\ &\le \left(\sum_{i=1}^{\ell} (2e^{2K'})^i\right) N\epsilon_2(n) + K\theta^\ell N.\end{aligned}$$

Now taking $\ell = \ell(n)$ to infinity slowly enough so that

$$\left(\sum_{i=1}^{\ell} (2e^{2K'})^i\right) \epsilon_2(n) \to 0$$

it follows that

$$|\boldsymbol{x}_0 - \boldsymbol{\beta}^N|_1 \le N\epsilon_3(n)$$

uniformly in m where $\epsilon_1(n) \to 0$ as $n \to \infty$. But

$$|\boldsymbol{x}_0 - \boldsymbol{\beta}^N|_1 = N\|g_n - g_N\|_1.$$

where $\|\cdot\|_1$ is the usual L^1 norm on functions on $[0,1]$. Thus,

$$\|g_n - g_m\|_1 \le \|g_n - g_N\|_\infty + \|g_m - g_N\|_\infty \le \epsilon_3(n) + \epsilon_3(m).$$

This shows that the sequence $\{g_n\}$ is Cauchy under the L^1 norm, and thus there exists a uniformly bounded function g^* such that $\|g_n - g^*\|_1 \to 0$. Now, for each n define a function f_n as

$$f_n(x) := \int_0^1 \frac{e^{g_n(x)+g_n(y)}}{1+e^{g_n(x)+g_n(y)}} dy.$$

Now by the uniform boundedness of the $|g_n|_\infty$,

$$\int_0^1 \left| f_n(x) - \frac{e^{g^*(x)+g^*(y)}}{1+e^{g^*(x)+g^*(y)}} dy \right| dx \to 0$$

as $n \to \infty$. But from the relation between $\boldsymbol{\beta}^n$ and $\boldsymbol{d}^n$, it is easy to see that for $x \in (0,1]$ that $f_n(x) = d^n_{\lceil nx \rceil}/n + O(1/n)$ and hence

$$\lim_n \|f - f_n\|_1 \to 0.$$

It follows that

$$f(x) = \int_0^1 W^*(x,y)dy \quad \text{a.e.} \tag{27}$$

where

$$W^*(x,y) = \frac{e^{g^*(x)+g^*(y)}}{1+e^{g^*(x)+g^*(y)}}.$$

We now adjust g^* on a set of measure 0 so that equation (27) holds for all x. Set $\psi : \mathbb{R} \to (0,1)$ as

$$\psi(z) = \int_0^1 \frac{e^{z+g^*(y)}}{1+e^{z+g^*(y)}} dy.$$

By construction and since g^* is uniformly bounded it follows that $\psi(z)$ is continuous, strictly increasing and bijective. By equation (27) we have that

$$f(x) = \psi(g^*(x)) \ a.e.$$

and hence if we set

$$g(x) = \psi^{-1}(f(x))$$

then $g(x) = g^*(x)$ almost everywhere. Then for all $x \in [0,1]$,

$$f(x) = \int_0^1 W(x,y)dy$$

where

$$W(x,y) = \frac{e^{g(x)+g(y)}}{1+e^{g(x)+g(y)}}.$$

Moreover, by the properties of ψ and f, we have that $g \in D'[0,1]$ and its points of discontinuity are same as f.

Let us now prove that g is the only function in $D'[0,1]$ with the above relationship with f. Suppose h is another such function. Fix any n. Define a vector $\boldsymbol{x}_0 = (x_{0,1}, \ldots, x_{0,n}) \in \mathbb{R}^n$ as

$$x_{0,i} := h(i/n), \quad i = 1, \ldots, n.$$

For each $1 \le i \le n$, define

$$y_i := \sum_{j \ne i} \frac{e^{x_{0,i}+x_{0,j}}}{1+e^{x_{0,i}+x_{0,j}}}.$$

Then since $h \in D'[0,1]$,

$$\sup_i |y_i/n - f(i/n)| \le \sup_i \left| y_i/n - \int_0^1 \frac{e^{h(i/n)+h(y)}}{1+e^{h(i/n)+h(y)}} dy \right| \le \epsilon_4(n), \tag{28}$$

where $\epsilon_4(n) \to 0$ as $n \to \infty$. Define $\boldsymbol{x}_1$ in terms of $\boldsymbol{x}_0$ and $\boldsymbol{d}^n$ as in Theorem 1.5. Then for each i,

$$x_{1,i} - x_{0,i} = \log d_i^n - \log y_i = \log \frac{d_i^n/n}{y_i/n}.$$

From (2), (28) and the above identity (and implicitly using the Property A of f), we see that

$$|\boldsymbol{x}_1 - \boldsymbol{x}_0|_\infty \le \epsilon_5(n),$$

where $\epsilon_5(n) \to 0$ as $n \to \infty$. Thus by Theorem 1.5 we get

$$|\boldsymbol{x}_0 - \boldsymbol{\beta}^n|_\infty \le \epsilon_6(n)$$

where $\epsilon_6(n) \to 0$ as $n \to \infty$. This implies that $\|h - g_n\|_1 \to 0$ and hence that $h = g$ a.e. Since we assumed both h and g are in $D'[0,1]$ this implies that $g = h$.

Now fix a finite simple graph H. Let $\boldsymbol{\beta}^n$ be as above. Let G'_n denote a random graph from the $\boldsymbol{\beta}^n$-model. Let $\boldsymbol{d}'_n$ be the degree sequence of G'. Then it is easy to see that conditional on the event $\{\boldsymbol{d}'_n = \boldsymbol{d}_n\}$ the law of G'_n is the same as that of G_n.

By Lemma 6.1, given any $\epsilon > 0$, we have that

$$\mathbb{P}(|t(H, G'_n) - \mathbb{E}t(H, G'_n)| > \epsilon) \leq e^{-C_1 n^2},$$

where C_1 is a constant that depends only of H and ϵ. By Lemma 6.2, we know that

$$\mathbb{P}(\boldsymbol{d}'_n = \boldsymbol{d}_n) \geq e^{-C_2 n^{7/4}},$$

where C_2 is another constant that depends only on $|\boldsymbol{\beta}|_\infty$. Thus,

$$\begin{aligned}
\mathbb{P}(|t(H, G_n) - \mathbb{E}t(H, G'_n)| > \epsilon) &= \mathbb{P}(|t(H, G'_n) - \mathbb{E}t(H, G'_n)| > \epsilon \mid \boldsymbol{d}'_n = \boldsymbol{d}_n) \\
&\leq \frac{\mathbb{P}(|t(H, G'_n) - \mathbb{E}t(H, G'_n)| > \epsilon)}{\mathbb{P}(\boldsymbol{d}'_n = \boldsymbol{d}_n)} \\
&\leq e^{-C_3 n^2},
\end{aligned}$$

where C_3 is a constant depending on H, ϵ and $|\boldsymbol{\beta}|_\infty$. Since $g_n \to g$, it is easy to prove that G'_n converges to W. From the above inequality, it follows that G'_n and G_n must have the same limit. This completes the proof of the theorem.

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