ON CONFIDENCE INTERVALS FOR CYCLIC REGENERATIVE PROCESSES *

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We study precise conditions under which the cyclic regenerative confidence intervals of Sargent and Shanthikumar are asymptotically valid. We also obtain an optimal way of implementing the cyclic regenerative variance reduction technique, and obtain a sufficient condition under which the procedure yields a lower variance than that of the standard regenerative method.

Regenerative processes, variance reduction techniques

1. Introduction

Recently, Sargent and Shanthikumar [5] developed an interesting new variance reduction technique designed to exploit the stochastic structure associated with a cyclic regenerative process. Our purpose here is to study precise conditions under which the confidence intervals proposed in [5] are asymptotically valid. This analysis will provide us with the side benefit of obtaining an optimal way of implementing the variance reduction procedures introduced there. To be precise, we will obtain the minimum variance estimate in the class of estimates proposed in [5]. We will also determine conditions under which the minimum variance estimate achieves a lower variance than that of the standard regenerative method (see Crane and Lemoine [4] for a description of the standard procedure).

We will use the convention that assumptions in force throughout the entire paper will be prefixed by A (e.g. A1) whereas all others will be prefixed by B. We can now state our basic assumptions for the problem:

A1. $\{X_n: n \geq 0\}$ is a regenerative process with regenerative times $0 = T_0 < T_1 < \cdots$ satisfying $E\tau_1 < \infty$, where $\tau_n = T_n - T_{n-1}$.

A2. f is a real-valued function such that $EY_n(|f|) \triangleq E(|f(X_{T_{n-1}})| + \cdots + |f(X_{T_n - 1})|) < \infty$.

A3. There exist random times $\{\alpha_{n,i}: n \geq 0, 0 \leq i \leq t\}$ such that $T_{n-1} = \alpha_{n,0} < \alpha_{n,1} < \cdots < \alpha_{n,t} = T_n$ and for which $\{(Y_{n,i}, \tau_{n,i}): n \geq 1\}$ are independent and identically distributed (i.i.d.) random vectors (r.v.'s) for $1 \leq i \leq t$, where $\tau_{n,i} = \alpha_{n,i} - \alpha_{n,i-1}$ and $Y_{n,i} = f(X_{\alpha_{n,i-1}}) + \cdots + f(X_{\alpha_{n,i}-1})$.

Assumptions A1 and A3 basically define the notion of a t-phase cyclic regenerative process. We will also suppose that the simulator possesses the following knowledge:

A4. $EY_{1,i}$, $E\tau_{1,i}$ are known for $i \in D$.

A5. The simulator can sample *independently* from each of the distributions

$$(Y_{1,i}, \tau_{1,i}), \quad i \in F \triangleq \{1, \ldots, t\} \setminus D.$$

Under A1 and A2, $\sum_{k=0}^{n} f(X_k)/n \to r = EY_1(f)/E\tau_1$ a.s. (see [4] for a proof). The goal of the simulator is to obtain confidence intervals for r.

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2. A central limit theorem

In the setting of a cyclic regenerative process, the practitioner must decide on a sampling order before initiating the simulation. To be precise, the simulator must assign, for each $n \geqslant 1$, an integer m_n from $G = F \cup \{0\}$. The practitioner then simulates the sequence of independent r.v.'s $\{(W_n, x_n): n \geqslant 1\}$, where (W_n, x_n) is sampled from the distribution of $(Y_{1,m_n}, \tau_{1,m_n})$ if $m_n \in F$ and from that of $(Y_1(f), \tau_1)$ if $m_n = 0$ (independent sampling is possible on account of A5). Put $\omega_{n,i} = \{j \leqslant n: m_j = i\}$ and let $k_{n,i}$ be the cardinality of $\omega_{n,i}$ for $i \in G$. The natural point estimate for r is given by

$$r_n = \Big(\sum_{i \in G} \bar{Y}_{n,i} + \sum_{i \in D} EY_{1,i} \Big) \Big/ \Big(\sum_{i \in G} \bar{\tau}_{n,i} + \sum_{i \in D} E\tau_{1,i} \Big)$$

where

$$\bar{Y}_{n,i} = \sum_{j \in \omega_{n,i}} W_j / k_{n,i}, \qquad \bar{\tau}_{n,i} = \sum_{j \in \omega_{n,i}} x_j / k_{n,i}.$$

The a.s. convergence of r_n to r is ensured by

A6. either (i) $k_{n,i} \to \infty$ if $i \in F$, $k_{n,0} \equiv 0$, or (ii) $k_{n,i} \to \infty$ if $i \in G$.

To obtain a confidence interval for r, we need a central limit theorem (CLT); such behaviour is guaranteed by

B1. $0 < \sigma_i^2 \triangleq \sigma^2(Y_{1,i} - r\tau_{1,i}) < \infty$ for $i \in F$,
$0 < \sigma^2 \triangleq \sigma^2(Y_1(f) - r\tau_1) < \infty$.

Theorem 1. *Under* B1, *There exist constants* a_n *such that* $a_n(r_n - r) \Rightarrow N(0, 1)$, *where* $N(0, 1)$ *is a unit normal r.v.*

Proof. We shall prove the result under A6(i), the proof under A6(ii) being similar. We view the problem in terms of a triangular array of r.v.'s by setting

$$U_{n,j} = (W_j - rx_j + \beta_i)/k_{n,i} \quad \text{if } m_j = i,$$

where $\beta_i = rE\tau_{1,i} - EY_{1,i}$. Set $U_n = \sum_{j \leqslant n} U_{n,j}$ and observe that

$$s_n^2 \triangleq \sigma^2(U_n) = \sum_{i \in F} \sigma_i^2 / k_{n,i}.$$

Then the triangular array $\{U_{n,j}/s_n\}$ satisfies Lindeberg's condition since for any $\epsilon > 0$,

$$\sum_{i=1}^{n} E\{U_{n,i}^2/s_n^2;\ U_{n,i}^2 \geqslant \epsilon^2 s_n^2\} = \sum_{i \in F} E\{Z_{1,i}^2/k_{n,i}s_n^2;\ Z_{1,i}^2 \geqslant \epsilon^2 s_n^2 k_{n,i}^2\}$$

$$\leqslant \sum_{i \in F} E\{Z_{1,i}^2/\sigma_i^2;\ Z_{1,i}^2 \geqslant \epsilon^2 k_{n,i} \sigma_i^2\} \to 0 \quad \text{as } n \to \infty;$$

here $Z_{1,i} = Y_{1,i} - r\tau_{1,i} + \beta_i$ (in the inequality, we used $s_n^2 \geqslant \sigma_i^2/k_{n,i}$). Also, we used the notational convention $E\{X; A\} \triangleq E\{XI_A\}$, for any r.v. X and event A, where I_A is 1 or 0 depending on whether or not A occurs. Since $EU_{n,j} = 0$, it follows by Lindeberg's Theorem (see Chung [3, p. 205]) that $U_n/s_n \Rightarrow N(0, 1)$. Hence,

$$\Big(\sum_{i \in F} \bar{Y}_{n,i} - r\bar{\tau}_{n,i} + \beta_i \Big) / s_n \Rightarrow N(0, 1).$$

But $\sum_{i \in F} \beta_i = \sum_{i \in D} EY_{1,i} - rE\tau_{1,i}$. Thus, using the fact that $\hat{\tau}_n \triangleq \sum_{i \in F} \bar{\tau}_{n,i} + \sum_{i \in D} E\tau_{1,i} \to E\tau_1$ a.s. and the converging-together lemma (Billingsley [2, p. 25]), we have

$$a_n(r_n - r) \Rightarrow N(0, 1)$$

where $a_n = E\tau_1/s_n$. □

In a simulation application, one needs to estimate the constants a_n. For the estimation, we need to add an additional hypothesis:

B2. $E(Y_{1,i}^2 + \tau_{1,i}^2) < \infty$ for $i \in F$, $E(Y_1^2(f) + \tau_1^2) < \infty$.

Notice that if $Y_{1,i} = r\tau_{1,i} + \theta_i$ where $E\tau_{1,i}^2 = \infty$ and $0 < E\theta_i^2 < \infty$, then B1 is satisfied but not B2.

Corollary 1. *Under* B1–B2, *there exist estimates* $\hat{a}_n$ *such that* $\hat{a}_n(r_n - r) \Rightarrow \mathrm{N}(0, 1)$.

Proof. By the converging-together lemma, this follows from Theorem 1 if we obtain estimates $\hat{a}_n$ such that $\hat{a}_n/a_n \to 1$ a.s. Under A6(i) an appropriate candidate for $\hat{a}_n$ is $\hat{\tau}_n/\hat{s}_n$ where $\hat{s}_n^2 = \Sigma_{i\in F}\hat{\sigma}_{n,i}^2/k_{n,i}$ and

$$\hat{\sigma}_{n,i}^2 = \sum_{j\in\omega_{n,i}} (W_j - r_n x_j)^2/k_{n,i} - \left(\sum_{j\in\omega_{n,i}} W_j - r_n x_j/k_{n,i}\right)^2.$$

But $|1 - a_n^2/\hat{a}_n^2| \leqslant \Sigma_{i\in F}|1 - E^2\tau_1\hat{\sigma}_{n,i}^2/\hat{\tau}_n^2\sigma_i^2| \to 0$ a.s. A similar proof works under A6(ii). □

The CLT of Corollary 1 can be used to construct confidence intervals for r. The half-width of a $100(1-\delta)\%$ confidence interval for r, based on a sample of size n, will be $z_\delta/\hat{a}_n$, where z_δ solves $P\{\mathrm{N}(0, 1) \leqslant z_\delta\} = 1 - \frac{1}{2}\delta$.

3. Another central limit theorem

To analyze the degree of variance reduction of a method, one needs to compare the half-width of competing intervals generated in a given amount of simulation time. In our context, this is accomplished by constructing intervals based on $(W_1, x_1), \ldots, (W_{l(N)}, x_{l(N)})$, where $l(N) = \max\{k: x_1 + \cdots + x_k \leqslant N\}$.

To base a CLT on a random number $l(N)$ of independent r.v.'s requires control on the growth of the $k_{n,i}$'s:

B3. if $k_{n,i} \to \infty$, then $k_{n,i}/n \to c_i$. If c_i, c_j are both zero, then $k_{n,j}/k_{n,i} \to \gamma_{ij} > 0$.

Theorem 2. *Under* B1 *and* B3, $a_{l(N)}(r_{l(N)} - r) \Rightarrow \mathrm{N}(0, 1)$ *where the* a_n*'s are the constants of Theorem* 1.

Proof. We assume we are dealing with A6(i), the proof for A6(ii) being similar. Suppose then that c_i is minimal for $i = s$. Then, by B1 and B3,

$$k_{n,s}^{1/2}(\bar{Z}_{n,i}: i \in F) \Rightarrow N \tag{3.1}$$

where N is a multivariate normal r.v. with (possibly) singular components ($\bar{Z}_{n,i} = \Sigma_{j\in\omega_{n,i}} Z_{n,j}/k_{n,i}$); in fact, it is easy to obtain a weak invariance principle version of (3.1).

Put $S_n = x_1 + \cdots + x_n$ and observe that

$$S_n/n = \sum_{i\in F}\sum_{j\in\omega_{n,i}} x_j/n = \sum_{i\in F}\left(\sum_{j\in\omega_{n,i}} x_j/k_{n,i}\right)\cdot k_{n,i}/n \to \sum_{i\in F} c_i E\tau_{1,i} \quad \text{a.s.}$$

by B3. But $S_{l(N)} \leqslant N < S_{l(N)+1}$ so

$$S_{l(N)}/l(N) \leqslant N/l(N) < S_{l(N)+1}/l(N)$$

and thus, by 'squeezing' $N/l(N)$, we obtain the result $N/l(N) \to \Sigma_{i\in F}c_i E\tau_{1,i}$ a.s. Then, using the weak invariance version of (3.1) and the random time change results of [2, p. 146], we have that

$$k_{l(N),s}^{1/2}\sum_{i\in F}\bar{Z}_{l(N),i}/\underline{\sigma} \Rightarrow \mathrm{N}(0, 1)$$

where $\underline{\sigma}^2 = \Sigma_{i\in F}c_s\sigma_i^2/c_i$ (if $c_s = c_i = 0$, set $c_s/c_i = \gamma_{is}$). Another application of the converging-together lemma shows that

$$\underline{a}_{l(N)}(r_{l(N)} - r) \Rightarrow \mathrm{N}(0, 1)$$

where $\underline{a}_n^2 = k_{n,s} E^2\tau_1/\underline{\sigma}^2$. But B3 guarantees that $\underline{a}_n/a_n \to 1$, yielding the theorem. □

Again, in terms of the confidence interval problem, one needs to estimate $a_{l(N)}$. The following corollary follows immediately from Theorem 2 and the fact that $\hat{a}_n/a_n \to 1$ a.s.

Corollary 2. *Under* B1–B3, $\hat{a}_{l(N)}(r_{l(N)} - r) \Rightarrow \mathrm{N}(0, 1)$ *where the* $\hat{a}_n$*'s are the estimators of Corollary* 1.

Finally, we can often rewrite the CLT of Theorem 2 in another form. If c_s is positive, then $k_{n,s}/nc_s \to 1$, so that we obtain the following result.

Corollary 3. *Assume* B3 *holds with all* c_i*'s positive. Then, under* B1, *there exists* $\bar{\sigma}$ *such that* $\sqrt{N}(r_{l(N)} - r)/\bar{\sigma} \Rightarrow \mathrm{N}(0, 1)$. *Also, under* B1–B2, *there exist estimators* $\bar{\sigma}_N$ *such that* $\sqrt{N}(r_{l(N)} - r)/\bar{\sigma}_N \Rightarrow \mathrm{N}(0, 1)$.

Proof. The result is obvious, upon identifying $\bar{\sigma}^2$, $\bar{\sigma}_N^2$. Under A6(i),

$$\bar{\sigma}^2 = (\underline{\sigma}^2/E^2\tau_1) \cdot \Big(\sum_{i \in F} c_i E\tau_{1,i}/c_s \Big)$$

and

$$\bar{\sigma}_N^2 = \Big(\sum_{i \in F} c_s \hat{\sigma}^2_{l(N),i}/c_i \hat{\tau}^2_{l(N)} \Big) \cdot \Big(\sum_{i \in F} c_i \bar{\tau}_{l(N),i}/c_s \Big). \quad \square$$

4. Optimal confidence intervals

We now wish to investigate the amount of variance reduction over the standard regenerative method that is accomplished by using the intervals proposed in Section 3. Let $\nu(N)$, $\nu(c, N)$ be the half-widths of $100(1-\delta)\%$ confidence intervals based on simulating N time units and using the standard regenerative interval and the interval of Corollary 2, respectively (we write $\nu(c, N)$ to reflect dependence on $c = (c_i)$). The following result may be found in [4].

Lemma 1. *Under* B1–B2, $N^{1/2}\nu(N) \to z_\delta \sigma/(E\tau_1)^{1/2} \triangleq z_\delta \tilde{\sigma}$ *a.s.*

In view of Lemma 1, the next lemma shows that it is never optimal to allow $k_{n,i}$ to tend to ∞ in such a way that $k_{n,i}/n \to 0$.

Lemma 2. *Suppose* B1–B3 *hold and* $k_{n,s} \to \infty$ *with* $c_s = 0$. *Then* $N^{1/2}\nu(N, c) \to \infty$ *a.s.*

Proof. The assertion is equivalent to proving that $N/\hat{a}^2_{l(N)} \to \infty$. But

$$\begin{aligned} N/\hat{a}^2_{l(N)} &= N\hat{s}^2_{l(N)}/\hat{\tau}^2_{l(N)} \geq N\hat{\sigma}^2_{l(N),s}/k_{l(N),s}\hat{\tau}^2_{l(N)} \\ &= (N/l(N)) \cdot (l(N)/k_{l(N),s}) \cdot (\hat{\sigma}^2_{l(N),s}/\hat{\tau}^2_{l(N)}) \to \infty \quad \text{a.s.} \quad \square \end{aligned}$$

Thus, in our search for optimal intervals, we need only consider the case where all c_i's are positive. This allows Corollary 3 to be applied to obtain a second cyclic regenerative interval with half-length $\kappa(N, c)$ (say). The following result follows from the proofs of Theorem 2 and Corollary 3.

Lemma 3. *Suppose* B1–B3 *hold with all* c_i*'s positive. Then* $\nu(N, c)/\kappa(N, c) \to 1$ *a.s. Furthermore, under*

A6(i),

$$N^{1/2}\nu(N,c) \to z_\delta\Big(\sum_{i\in F}\sigma_i^2/c_i\Big)^{1/2}\Big(\sum_{i\in F}c_iE\tau_{1,i}\Big)^{1/2}/E\tau_1 \quad a.s.$$

and under A6(ii)

$$N^{1/2}\nu(N,c) \to z_\delta\Big(\sigma^2/c_0+\sum_{i\in F}\sigma_i^2/c_i\Big)^{1/2}\Big(c_0E\tau_1+\sum_{i\in F}c_iE\tau_{1,i}\Big)^{1/2}/2E\tau_1 \quad a.s.$$

We are now in a position to determine the optimal constants $\underset{\sim}{c}$.

Theorem 3. *Assume* B1–B2 *hold. If $\sigma^2 \leqslant (\sum_{i\in F}\sigma_i(E\tau_{1,i})^{1/2})^2$, then no variance reduction is possible via the cyclic intervals of Section* 3. *Otherwise, the maximal reduction is obtained via the cyclic interval of Section* 3 *in which $k_{n,i}/n \to \underset{\sim}{c}_i$ for $i \in F$, where*

$$\underset{\sim}{c}_i = \alpha\sigma_i/(E\tau_{1,i})^{1/2}, \qquad \alpha = \Big(\sum_{i\in F}\sigma_i/(E\tau_{1,i})^{1/2}\Big)^{-1}. \tag{4.1}$$

The percentage variance reduction achieved is then

$$100\Big(1-\Big(\sum_{i\in F}\sigma_i(E\tau_{1,i})^{1/2}/\sigma\Big)^2\Big)\%.$$

Proof. By Lemma 3, it is clear that the optimal interval possible via a cyclic method of type A6(i) is obtained by choosing $k_{n,i}/n \to \underset{\sim}{c}_i$ for $i \in F$, where $\underset{\sim}{c}_i$ solves the optimization problem

$$\begin{aligned}&\text{minimize} \quad \Big(\sum_{i\in F}\sigma_i^2/c_i\Big)\Big(\sum_{i\in F}c_iE\tau_{1,i}\Big)/E^2\tau_1,\\ &\text{subject to} \quad \sum_{i\in F}c_i = 1, \quad c_i > 0.\end{aligned}$$

Application of the method of Lagrange multipliers to this problem (see Avriel [1, p. 48]) show that a minimal $\underset{\sim}{c}$ must satisfy

$$-\sigma_i^2\Big(\sum_{i\in F}c_iE\tau_{1,i}\Big)/c_i^2 + E\tau_{1,i}\Big(\sum_{i\in F}\sigma_i^2/c_i\Big)+\lambda = 0 \tag{4.3}$$

for each $i \in F$ and some constant λ. Multipliying the ith equation of (4.3) by c_i and adding all the resulting equations proves that $\lambda = 0$. Equation (4.3) shows that

$$\underset{\sim}{c}_i^2 = \eta\sigma_i^2/E\tau_{1,i}$$

for some η. The proportionality factor η is determined by $\sum_{i\in F}\underset{\sim}{c}_i = 1$. It is easily checked that $\underset{\sim}{c}$, as given, is the minimum desired, with minimal value

$$\underset{\sim}{\sigma}^2 = \Big(\sum_{i\in F}\sigma_i(E\tau_{1,i})^{1/2}\Big)^2/E^2\tau_1.$$

A similar analysis for cyclic intervals of type A6(ii) shows that the minimal possible value for the analog of (4.2) is given by

$$\Big(\sigma(E\tau_1)^{1/2}+\sum_{i\in F}\sigma_i(E\tau_{1,i})^{1/2}\Big)^2/4E^2\tau_1 = (\tfrac{1}{2}\tilde{\sigma}+\tfrac{1}{2}\underset{\sim}{\sigma})^2 \geqslant \min(\tilde{\sigma}^2, \underset{\sim}{\sigma}^2)$$

which shows that intervals of type A6(ii) can never achieve lower asymptotic half-width than the better of the standard or cyclic (of type A6(i)) regenerative intervals. The other assertions of the theorem are trivial. □

This theorem suggests that the practitioner should execute a small 'pilot run' to obtain approximate values for $\underset{\sim}{c}_i$. If the 'pilot run' suggests a variance reduction over the standard method, the simulator should construct a sampling order which ensures that $k_{n,i}/n \to \underset{\sim}{c}_i$ for $i \in F$, and then employ the cyclic regenerative method.

We conclude with a sufficient condition that guarantees that the cyclic regenerative method achieves a variance reduction over the standard procedure.

Lemma 4. *If* B1–B3 *holds, then* $\underset{\sim}{\sigma}^2 \leqslant \tilde{\sigma}^2$ *if*

$$\operatorname{cov}(Y_{1,i} - r\tau_{1,i},\ Y_{1,j} - r\tau_{1,j}) \geqslant 0 \quad \textit{for } 1 \leqslant i, j \leqslant t.$$

Proof. Since $\underset{\sim}{\sigma}^2$ is minimal for (4.2),

$$\underset{\sim}{\sigma}^2 \leqslant \Big(\sum_{i\in F} \sigma_i^2\Big)\Big(\sum_{i\in F} E\tau_{1,i}\Big)/E^2\tau_1 \leqslant \Big(\sum_{i\in F} \sigma_i^2\Big)/E\tau_1 \leqslant \sigma^2\Big(\sum_{i\in F} Y_{1,i} - r\tau_{1,i}\Big)/E\tau_1 \leqslant \tilde{\sigma}^2,$$

the last two inequalities by the covariance condition. □

We caution that $\underset{\sim}{\sigma}^2 > \tilde{\sigma}^2$ is possible if the $Y_{1,i} - r\tau_{1,i}$ are negatively correlated.

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