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Structural characterization of taboo-stationarity for general processes in two-sided time

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Abstract

This note considers the taboo counterpart of stationarity. A general stochastic process in two-sided time is defined to be *taboo-stationary* if its global distribution does not change by shifting the origin to an arbitrary non-random time in the future *under taboo*, that is, conditionally on some taboo-event not having occurred up to the new time origin. The main result is the following basic structural characterization: a process is taboo-stationary *if and only if* it can be represented as a stochastic process with origin shifted backward in time by an independent exponential random variable. An application to reflected Brownian motion is given.

Keywords: Quasi-stationarity

1. Introduction

The aim of this note is to define the concept of "taboo-stationarity" for general stochastic processes in two-sided time (Definition 1) and present a basic but amazingly simple structural characterization of this property (Theorem 2, see also Example 1). Taboo-stationarity is the characterizing property of a "taboo-limit" process (Theorem 1) in the same way as stationarity is the characterizing property of an ordinary limit process. An application of this general theory to Markov processes can be found in Glynn and Thorisson (2001, the proof of Proposition 3), and an application to reflected Brownian motion is given at the end of this paper.

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This paper is a companion paper to Glynn and Thorisson (2001) and Glynn and Thorisson (2002) where taboo-limits are investigated in two important special cases: the Markov case and the regenerative case. Unlike these papers the present one is not concerned with establishing conditions for the existence of taboo-limits, but rather the focus is on the properties of the limit process *when it exists*. The paper is self-contained, but because of its general character the reader might find it helpful to consult the companion papers before embarking on this one (unless the survey in Section 3 is sufficient).

These three papers continue a research that has a long tradition in the Markov process context, including the work on rarity and exponentiality by Keilson (1979) and the substantial body of literature on R-recurrence for non-negative kernels and the associated quasi-stationary distribution theory; see, for example, Seneta and Vere-Jones (1966), Tweedie (1974), Nummelin and Arjas (1976), Nummelin and Tweedie (1978), Nummelin (1984), and Jacka and Roberts (1995). For a coupling approach to taboo-stationarity, see Thorisson (2000).

In Section 2 we establish notation, and in Section 3 we put taboo-stationarity briefly into the Markov context. In Section 4 we define taboo-stationarity for general processes (Definition 1) and show that it is the characterizing property of a taboo-limit (Theorem 1). In Section 5 we give a simple general example of a taboo-stationary process (Example 1) and then establish the main result of this note (Theorem 2) which states that a process is taboo-stationary if and only if it has the structure in that example. In Section 6 we establish the result needed in Glynn and Thorisson (2001), and in Section 7 we present the application to reflected Brownian motion. In Section 8 we conclude with a comment on a 1969 paper by Vere-Jones, the only paper that we are aware of which, like the present paper, considers conditional limits of general non-Markovian processes.

2. Notation

We shall consider a pair (X^*, Γ^*) where Γ^* is a non-negative finite random time and

$$X^* = (X^*(s) : s \in \mathbb{R})$$

is a general stochastic process in two-sided time taking values in a Polish space E and with $D_E(\mathbb{R})$ valued paths. We use θ_t to denote the two-sided shift:

$$\theta_t X^* = (X^*(t+s) : s \in \mathbb{R}), \quad t \in \mathbb{R} \qquad \text{(two-sided shift)}$$

while we shall denote the one-sided shift by

$$X_t^* = (X^*(t+s) : s \in [0,\infty)), \quad t \in \mathbb{R} \qquad \text{(one-sided shift).}$$

We also consider a pair (X, Γ), where Γ is a non-negative finite random time and

$$X = (X(s) : s \in [0,\infty))$$

is a one-sided process with paths in $D_E([0,\infty))$. Denote the one-sided shift by

$$X_t = (X(t+s) : s \in [0,\infty)), \quad t \in [0,\infty) \qquad \text{(one-sided shift).}$$

Note that $\theta_t X^*$ is two-sided like X^* while both X_t^* and X_t are one-sided like X. Let $\stackrel{\text{t.v.}}{\to}$ denote convergence in total variation.

3. The Markov case

Taboo-stationarity is the "taboo" counterpart of stationarity and extends the well-known property of quasi-stationarity from the Markov case. Before giving the definition of taboo-stationarity we shall briefly consider the Markov case and contrast taboo-stationarity to stationarity.

Let X be a Markov process. Recall that if π is a total variation limit distribution:

$$\mathbf{P}_\pi(X(t) \in \cdot) \stackrel{\text{t.v.}}{\to} \pi, \quad t \to \infty \qquad (\pi \text{ is a t.v. limit distribution}),$$

then π is *stationary*:

$$\mathbf{P}_\pi(X(t) \in \cdot) = \pi, \quad t \in [0, \infty) \qquad (\pi \text{ is stationary}).$$

Moreover, if there exists a total variation limit distribution π then there actually exists a two-sided total variation limit process X^* such that for all $h \geqslant 0$

$$\mathbf{P}(X_{t-h} \in \cdot) \stackrel{\text{t.v.}}{\to} \mathbf{P}(X^*_{-h} \in \cdot), \quad t \to \infty \qquad (X^* \text{ is a limit process}).$$

Clearly, the distribution of the limit does not depend on h and thus X^* must be *stationary*:

$$\mathbf{P}(\theta_t X^* \in \cdot) = \mathbf{P}(X^* \in \cdot), \quad t \in [0, \infty) \qquad (X^* \text{ is stationary}).$$

Analogously, if Γ is a hitting time of a set A and λ is a total variation *taboo-limit* distribution:

$$\mathbf{P}_\pi(X(t) \in \cdot \,|\Gamma > t) \stackrel{\text{t.v.}}{\to} \lambda, \quad t \to \infty \qquad (\lambda \text{ is a t.v. taboo-limit distribution}),$$

then λ is *quasi-stationary* with respect to A, that is,

$$\mathbf{P}_\lambda(X(t) \in \cdot \,|\Gamma > t) = \lambda, \quad t \in [0, \infty) \qquad (\lambda \text{ is quasi-stationary}).$$

Now, a natural question is whether the existence of a quasi-stationary taboo-limit distribution λ implies the existence of a full two-sided taboo-limit process. The answer is yes. In our Markov paper (Glynn and Thorisson, 2001), we study both discrete- and continuous-time processes and give conditions (in terms of eigenvalues and eigenvectors) for the existence of a quasi-stationary total variation taboo-limit distribution λ, and further show that this yields the existence of a two-sided total variation limit process, that is, a process X^* such that for all $h \geqslant 0$

$$\mathbf{P}(X_{t-h} \in \cdot \,|\Gamma > t) \stackrel{\text{t.v.}}{\to} \mathbf{P}(X^*_{-h} \in \cdot), \quad t \to \infty \qquad (X^* \text{ is a taboo-limit}). \tag{1}$$

In Glynn and Thorisson (2001) we show that this limit process X^* is a Markov process, but time inhomogenous. Clearly, $X_0^* = (X^*(s) : s \in [0, \infty))$ must be time homogenous with the same transition probabilities as X, and it turns out that the time-reversed process $(X^*(-s) : s \in (0, \infty))$ is also time homogeneous with certain transition probabilities that in particular do not allow entrance into A. Moreover, we show in the discrete time Markov chain case that Γ^*—the hitting time of A—is geometric and that $\theta_{\Gamma^*}X^*$ and Γ^* are independent. This hints at Example 1 and Theorem 2 below.

In our regenerative paper (Glynn and Thorisson, 2002), we give quite different conditions (in terms of cycle-length moments) for the existence of a two-sided taboo-limit process X^* satisfying (1). In the classical regenerative case we show that X^* consists of independent (but not i.i.d.) cycles. Clearly, $X_0^* = (X^*(s) : s \in [0,\infty))$ must have i.i.d. cycles with the same distribution as those of X, and it turns out that the time-reversed process $(X^*(-s) : s \in (0,\infty))$ has also i.i.d. cycles with a certain distribution that, in particular, does not allow entrance into A.

Clearly, the taboo-limit process at (1) is not stationary. It has another characterizing property as we shall see in the next section.

4. Taboo-stationarity for general processes in two-sided time

It turns out that the characterizing property of a two-sided taboo-limit process is "taboo-stationarity" defined as follows.

Definition 1. Let X^* be a general stochastic process in two-sided time taking values in a Polish space E and with $D_E(\mathbb{R})$ valued paths. Let Γ^* be a non-negative finite random time. Call X^* *taboo-stationary* with *taboo-time* Γ^* if shift under taboo does not change the distribution of the pair (X^*, Γ^*), that is, if

$$\mathbf{P}((\theta_t X^*, \Gamma^* - t) \in \cdot \mid \Gamma^* > t) = \mathbf{P}((X^*, \Gamma^*) \in \cdot), \quad t \in [0,\infty). \tag{2}$$

Call the pair (X^*, Γ^*) *taboo-stationary* if this holds. Think of Γ^* as the time when some "taboo" event occurs, for instance the time when X^* hits a "taboo" region of its state space.

The taboo-time Γ^* has to be explicitly included in the definition because it is not necessarily defined by the process, it is not necessarily a hitting time for instance.

We shall now show that taboo-stationarity is, in fact, the characterizing property of a total variation *taboo-limit*.

Theorem 1. *A pair (X^*, Γ^*) is taboo-stationary if and only if there is a pair (X, Γ), where $X = (X(s) : s \in [0,\infty))$ is a one-sided process with paths in $D_E([0,\infty))$ and Γ is a non-negative finite random time, such that*

$$\mathbf{P}((X_{t-h}, \Gamma - t) \in \cdot \mid \Gamma > t) \stackrel{\text{t.v.}}{\to} \mathbf{P}((X^*_{-h}, \Gamma^*) \in \cdot), \quad t \to \infty, \tag{3}$$

for all $h \in [0,\infty)$.

Proof. If (2) holds then clearly so does (3) with $(X, \Gamma) := (X_0^*, \Gamma^*)$. In order to establish the converse [that (3) implies (2)], assume that (3) holds. Take $x \in [0,\infty)$ and $h \in [x,\infty)$ and note that (3) implies (with h replaced by $h - x$) that

$$\mathbf{P}((X_{t-(h-x)}, \Gamma - t - x) \in \cdot, \Gamma - t > x \mid \Gamma > t)$$

$$\stackrel{\text{t.v.}}{\to} \mathbf{P}((X^*_{-(h-x)}, \Gamma^* - x) \in \cdot, \Gamma^* > x), \quad t \to \infty.$$

Divide by $\mathbf{P}(\Gamma - t > x | \Gamma > t)$ on the left and by the limit $\mathbf{P}(\Gamma^* > x)$ on the right (and note that $X_{t-(h-x)} = X_{(t+x)-h}$ and $X^*_{-(h-x)} = X^*_{x-h}$) to obtain: as $t \to \infty$,

$$\mathbf{P}((X_{(t+x)-h}, \Gamma - t - x) \in \cdot \, | \Gamma > t + x) \xrightarrow{\text{t.v.}} \mathbf{P}((X^*_{x-h}, \Gamma^* - x) \in \cdot \, | \Gamma^* > x).$$

But, according to (3), the left-hand side tends also to $\mathbf{P}((X^*_{-h}, \Gamma^*) \in \cdot)$. Since the two limits must be identical we have (replace x by t) that

$$\mathbf{P}((X^*_{t-h}, \Gamma^* - t) \in \cdot \, | \Gamma^* > t) = \mathbf{P}((X^*_{-h}, \Gamma^*) \in \cdot), \quad 0 \leqslant t \leqslant h.$$

Since h is arbitrary this yields (2). □

5. The basic structural characterization

Consider the following amazingly simple and general example of a taboo-stationary process.

Example 1. Let $Y = (Y(s) : s \in \mathbb{R})$ be any stochastic process in two-sided time taking values in a Polish space E and with $D_E(\mathbb{R})$ valued paths. Let V be exponential and independent of Y. Then $(X^*, \Gamma^*) := (\theta_{-V} Y, V)$ is always taboo-stationary. This can be seen as follows. Since V is exponential we have, for all paths $x \in D_E(\mathbb{R})$,

$$\mathbf{P}((\theta_{t-V} x, V - t) \in \cdot \, | V > t) = \mathbf{P}((\theta_{-V} x, V) \in \cdot), \quad t \in [0, \infty).$$

Since V and Y are independent we may replace x by Y to obtain (since $\theta_{t-V} Y = \theta_t X^*$ and $\theta_{-V} Y = X^*$)

$$\mathbf{P}((\theta_t X^*, \Gamma^* - t) \in \cdot \, | \Gamma^* > t) = \mathbf{P}((X^*, \Gamma^*) \in \cdot), \quad t \in [0, \infty),$$

that is, (X^*, Γ^*) is taboo-stationary.

We shall now prove that this example is really not an example but a complete characterization of taboo-stationarity: all taboo-stationary processes are of this form.

Theorem 2. *The pair (X^*, Γ^*) is taboo-stationary if and only if Γ^* is exponential and independent of $\theta_{\Gamma^*} X^*$.*

Proof. If Γ^* is exponential and independent of $\theta_{\Gamma^*} X^*$ take $V = \Gamma^*$ and $Y = \theta_{\Gamma^*} X^*$ in Example 1 to obtain that (X^*, Γ^*) is taboo-stationary. Conversely, suppose (X^*, Γ^*) is taboo-stationary. From (2) we obtain

$$\mathbf{P}(\Gamma^* - t \in \cdot \, | \Gamma^* > t) = \mathbf{P}(\Gamma^* \in \cdot), \quad t \in [0, \infty),$$

which is the standard characterization of exponentiality. Moreover,

$$\theta_{\Gamma^*} X^* = \theta_{\Gamma^* - t} \theta_t X^*, \quad t \in [0, \infty),$$

that is, $\theta_{\Gamma^*} X^* = g(\theta_t X^*, \Gamma^* - t)$ for all $t \in [0, \infty)$ where g is the mapping defined by $g(X^*, \Gamma^*) = \theta_{\Gamma^*} X^*$. Applying g on both sides in (2) yields

$$\mathbf{P}(\theta_{\Gamma^*} X^* \in \cdot \, | \Gamma^* > t) = \mathbf{P}(\theta_{\Gamma^*} X^* \in \cdot), \quad t \in [0, \infty).$$

Multiply by $\mathbf{P}(\Gamma^* > t)$ to obtain

$$\mathbf{P}(\theta_{\Gamma^*}X^* \in \cdot, \Gamma^* > t) = \mathbf{P}(\theta_{\Gamma^*}X^* \in \cdot)\mathbf{P}(\Gamma^* > t), \quad t \in [0, \infty)$$

that is, $\theta_{\Gamma^*}X^*$ and Γ^* are independent. □

In addition to its immediate theoretical value and to applications like those in Sections 6 and 7 below, Theorem 2 is useful in simulation. It is of the same importance for simulating taboo-stationary processes as the independent uniformity of the origin is for simulating stationary regenerative processes; see Asmussen et al. (1992).

6. Application to Markov processes

The corollary below is the key result needed in the proof of Proposition 3 in our Markov paper (Glynn and Thorisson, 2001). That proposition gives the behaviour of the different taboo-limit process obtained when the taboo is broken *at* time t. It turns out that that process behaves like $\theta_{\Gamma^*}X^*$, where X^* is the taboo-limit process obtained when the taboo is broken *after* time t as in the present paper.

Corollary 1. *If* (X^*, Γ^*) *is taboo-stationary then* $\mathbf{P}(X^* \in \cdot \,|\Gamma^* < h)$ *goes weakly* (*in the Skorohod topology*) *to* $\mathbf{P}(\theta_{\Gamma^*}X^* \in \cdot)$ *as* h *decreases to zero.*

Proof. Let f be a bounded continuous function defined on $D(\mathbb{R})$ and let V_h be independent of (X^*, Γ^*) and have the distribution $\mathbf{P}(\Gamma^* \in \cdot \,|\Gamma^* < h)$, that is, an exponential distribution truncated by h. Then the independence of $\theta_{\Gamma^*}X^*$ and Γ^* yields the second identity in

$$\begin{aligned}\mathbf{E}[f(X^*)|\Gamma^* < h] &= \mathbf{E}[f(\theta_{-\Gamma^*}\theta_{\Gamma^*}X^*)|\Gamma^* < h] \\ &= \mathbf{E}[f(\theta_{V_h}\theta_{\Gamma^*}X^*)]. \end{aligned} \tag{4}$$

Since $V_h \leqslant h$ and f is continuous and θ_t is continuous in t, we have that $f(\theta_{V_h}\theta_{\Gamma^*}X^*)$ goes pointwise to $f(\theta_{\Gamma^*}X^*)$ as h decreases to zero. Since f is bounded this implies that $\mathbf{E}[f(\theta_{V_h}\theta_{\Gamma^*}X^*)]$ goes to $\mathbf{E}[f(\theta_{\Gamma^*}X^*)]$ and a reference to (4) completes the proof. □

7. Application to reflected Brownian motion

In the following concrete example, much can be worked out explicitly.

Take standard Brownian motion on the real line and put a reflecting boundary at the origin, so that we now have reflecting Brownian motion on the positive half-line with zero drift and variance parameter one. Let the taboo-time be the hitting time of the interval $[1, \infty)$.

In our Markov paper (Glynn and Thorisson, 2001), the key is to solve a certain eigenvalue problem. In this setting, we need to find a positive eigenfunction u and an eigenvalue $-\lambda$ (with λ positive) so that $Au = -\lambda u$ with $u'(0) = 0$ (this is the reflecting boundary condition) and $u(1) = 0$. Since the eigenfunction is only determined up to a constant, we can choose to require that $u(0)=1$. Here, A is the second-order differential operator $\frac{1}{2}\mathrm{d}^2/\mathrm{d}x^2$ (that is, one half the second derivative).

When solving this problem, one notes that there are three boundary conditions. In order that we satisfy all three, λ will need to be chosen appropriately (that is, this determines the eigenvalue that we need). Here, the linearly independent solutions of the second-order ODE are $\cos((2\lambda)^{1/2}x)$ and $\sin((2\lambda)^{1/2}x)$. Let a and b be the coefficients of the two linearly independent solutions. The boundary conditions $u'(0)=0$ and $u(0)=1$ require setting $a=b=1$. In order that we satisfy $u(1)=0$, we must set $(2\lambda)^{1/2}=3\pi/4$. In other words, $\lambda = 9\pi^2/32$. This gives us the required positive eigenfunction and eigenvalue.

8. Remark

Unlike in the standard quasi-stationary literature, the process X considered here is not necessarily Markovian, and the taboo-time Γ need not be a stopping time. It could, for instance, be a last exit time. Taboo-limit results for such times seem not to have been worked out even in the Markov case. But according to Theorems 1 and 2, the taboo-time Γ^* of the two-sided limit process X^* will still be exponential and independent of the limit process seen from that time, $\theta_{\Gamma^*}X^*$.

The only paper that we are aware of, where conditional limits are considered for general non-Markovian processes, is Vere-Jones (1969). According to Theorem 1 in that paper it holds for an integer-valued process X, that $X(t)$ goes in probability to 0 and the conditional distribution of $X(t)$ given $X(t)\neq 0$ converges to a proper distribution, as t goes to infinity, if and only if there are non-random integers $C(t)$ such that the sum of $C(t)$ independent copies of $X(t)$ converges in distribution to a proper non-degenerate random variable.

This differs in several ways from our results. Firstly, it deals with convergence in the state space to a limit variable rather than in a two-sided path space to a limit process. Secondly, the conditioning is on $X(t)\neq 0$ rather than on $\Gamma > t$. Finally and most importantly, Vere-Jones's Theorem 1 focuses on a characterization of the *existence* of a limit variable while the present paper is not concerned with establishing conditions for the existence of taboo-limits. Theorems 1 and 2 above focus on the characterizing properties of the limit process *when it exists*.

References

Asmussen, S., Glynn, P.W., Thorisson, H., 1992. Stationarity detection in the initial transient problem. ACM Trans. Modelling Comput. Simulation 2, 130–157.

Glynn, P.W., Thorisson, H., 2001. Two-sided taboo limits for Markov processes and associated perfect simulation. Stochastic Process Appl. 91, 1–20.

Glynn, P.W., Thorisson, H., 2002. Limit theory for taboo-regenerative processes, to appear.

Jacka, S.D., Roberts, G.O., 1995. Weak convergence of conditioned processes on a countable state space. J. Appl. Probab. 32, 902–916.

Keilson, J., 1979. Markov Chain Models—Rarity and Exponentiality. Springer, New York.

Nummelin, E., 1984. General Irreducible Markov Chains and Non-negative Operators. Cambridge University Press, Cambridge.

Nummelin, E., Arjas, E., 1976. A direct construction of the *R*-invariant measure for a Markov chain on a general state space. Ann. Probab. 4, 674–679.

Nummelin, E., Tweedie, R., 1978. Geometric ergodicity and *R*-positivity for general Markov chains. Ann. Probab. 6, 404–420.

Seneta, E., Vere-Jones, D., 1966. On quasi-stationary distributions in discrete-time Markov chains with a denumerable infinity of states. J. Appl. Probab. 3, 403–434.

Thorisson, H., 2000. Coupling, Stationarity, and Regeneration. Springer, New York.

Tweedie, R., 1974. Quasi-stationary distributions for Markov chains on a general state space. J. Appl. Probab. 11, 726–741.

Vere-Jones, J., 1969. Some limit theorems for evanescent processes. Austral. J. Statist. 11, 67–78.