THE ALEPH-ZERO OR ZERO DICHOTOMY

(New and extended version with new arguments)

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ABSTRACT. This paper examines Zeno's Dichotomies from the perspective of w-order. The conclusion is a contradictory result that compromises the Axiom of Infinity from which w-order derives.

1. INTRODUCTION: ZENO'S PARADOXES AND MODERN SCIENCE

Zeno's Paradoxes have interested philosophers of all times (see [13], [14], [83], [73], [48], [84], [25] or [56] for historical background), although until the middle of the XIX century they were frequently considered as mere sophisms [13], [14], [72], [73]. From that time, and particularly along the XX century, they became the unending source of new philosophical, mathematical and physical discussions. Authors as Hegel [44], James [49], Russell [72], Whitehead [86], [87] or Bergson [9], [10] focused their attention on the challenging world of Zeno's paradoxes. At the beginning of the second half of the XX century the pioneering works of Black [11], Wisdom [88], Thomson [79], [80], and Benacerraf [8] introduced a new way of discussing the possibilities of performing an actual infinity of actions in a finite time (a performance involved in most of Zeno's paradoxes). I refer to Supertask Theory [69], [74]. In fact, infinity machines, or supermachines, are our modern Achilles substitutes. A supermachine is a theoretical devices supposedly capable of performing countably many actions in a finite interval of time. The possibilities of performing an uncountable infinity of actions were ruled out by P. Clark and S. Read [23], for which they made use of a Cantor's argument on the impossibility of dividing a real interval into uncountably many adjacent parts [20]. Although supertasks have also been examined from the perspective of nonstandard analysis ([58], [57], [1], [54]), as far as I know the possibilities to perform an hypertask along an hyperreal interval of time have not been discussed, although finite hyperreal intervals can be divided into hypercountably many successive infinitesimal intervals, the so called hyperfinite partitions ([77], [35], [50], [45], etc.). Supertask theory has finally turned its attention, particularly from the last decade of the XX century, towards the discussion of the physical plausibility of supertasks ([63], [65], [69],

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[73], [40], [42], [41]) as well as on the implications of supertasks in the physical world ([65], [66], [67], [31], [68], [61], [2], [3], [70]), including relativistic and quantum mechanics perspectives [85], [46], [29], [30], [61], [28], [74]

In the second half of the XX century, several solutions to some of Zeno's paradoxes have been proposed. Most of those solutions were found in the context of certain new branches of mathematics as Cantor's transfinite arithmetic, topology, measure theory [38], [39], [91], [40], [42], [41], and more recently internal set theory (a branch of nonstandard analysis) [58], [57]. It is also worth noting the solutions proposed by P. Lynds within a classical and quantum mechanics framework [52], [53]. Some of these solutions, however, have been contested [62], [1]. And in most of cases the proposed solutions do not explain where Zeno's arguments fail [62], [69]. Moreover, some of the proposed solutions gave rise to a new collection of problems so exciting as Zeno's paradoxes [73], [48] [74].

The four most famous Zeno's paradoxes are usually regarded as arguments against motion ([4], ([39], [43], [24], [73] etc.) be it performed in a continuous or in a discontinuous world. *Achilles and the Tortoise* and the *Dichotomy* in the continuous case, *the Stadium* and *the Arrow* in the discontinuous one. The paradoxes of the second case (together with the paradox of Plurality) are more difficult to solve, if a solution exists after all, particularly in a quantum spacetime framework. Most of the proposed solutions to Zeno's paradoxes are, in effect, solutions to the paradoxes of the first group or to the second one in a dense and continuous spacetime framework. This situation is very significant taking into account the increasing number of contemporary physical theories suggesting the quantum nature of spacetime, as for instance Superstring theory ([36], [37] [81], [32]), Loop Quantum Gravity ([75], [5] [76]), Quantum Computation Theory ([78], [6], [51]) or Black Hole Thermodynamics [6], [78]. Is at this quantum level where physics (the science of changes) will finally meet the problem of Change [7] whose insolvability probably motivated Zeno's' arguments? Is the problem of Change really inconsistent as some authors ([59], [60]) have defended? These are in fact two intriguing and still unsolved questions related to Zeno's arguments [62].

2. Zeno's paradoxes and ω^*-order

Not less intriguing, though for different reasons, is the fact that one immediately perceives when examining the contemporary discussions on Zeno's paradoxes. Surprisingly, the Axiom of Infinity is never involved in such discussions. Zeno's arguments have never been used to question the Axiom of Infinity, as if the existence of actual infinite totalities were beyond any doubt [33]. Grünbaum, for instance, proposed that if it were the case that from modern kinematics together

with the denseness postulate a false zenonian conclusion could be formally derived, then we would have to replace current kinematics by other mechanical theory [39, page 39]. Anything but questioning the hypothesis of the actual infinity from which the involved topological denseness derives. And this in spite of the lack of selfevidence of that hypothesis, which is even rejected by some schools of contemporary mathematics as constructivism, among whose precursors we find scholars as Newton, Fermat or Euler [55] and XX century thinkers of the intellectual stature of Poincaré [64] or Wittgenstein [89].

In the first half of the XIX century Bernard Bolzano, and in the second one Richard Dedekind, tried unsuccessfully[1] to prove the existence of infinite totalities [12], [27]. For his part, G. Cantor, the founder of transfinite mathematics, simply took it for granted the existence of such totalities. Thus, in §6 of his famous Beiträge (pp. 103-104 of the English translation) we can read:

> The first example of a transfinite set is given by the totality of finite cardinals.

although, as could be expected, he gave no proof of that existence. In accordance with his profound theological platonism [26], Cantor was firmly convinced of the actual existence of complete infinite totalities. He never explicitly declared the hypothetical nature of his infinitist assertions (at least not in his most relevant works on the transfinite [16], [17], [15], [21], [22]). He even tried to give a proof on the existence of actual infinities (quoted in [71], p. 3, from [18], p. 404):

> ... in truth the potential infinite has only a borrowed reality, insofar as potentially infinite concept always points towards a logically prior actually infinite concept whose existence it depends on.

Evidently this is not a formal proof but a personal belief. Cantor's infinite totality is evidently isomorph to the set $\mathbb{N}$ of natural numbers and then his implicit assumption on the existence of that complete totality is equivalent to our modern Axiom of Infinity. But convictions do not suffice in mathematics and we had finally to state the actual existence of complete infinite totalities by the expeditious way of axioms.

The (assumed) infinite totality of finite cardinals led Cantor to the essential notion of ω-order (Beiträge, p. 115 [19]):

> By ω we understand the type of a well ordered aggregate
>
> $$(e_1, e_2, \ldots, e_\nu, \ldots)$$
>
> in which
>
> $$e_\nu \prec e_{\nu+1}$$
>
> and where ν represents all finite cardinal numbers in turn.

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[1]Their respective proofs were compatible with the potential infinity.

Cantor then defined the notion of *fundamental series of ordinals* of which he proved the existence of a limit (Beiträge, Theorem §14 I). This limit plays a capital role in the proofs of the following 10 theorems in Beiträge §15 the last of which is the fundamental theorem K-15 stating that every ordinal of the second class (transfinite) is either the result of increasing by one the next smaller ordinal (ordinals of the first kind), or it is the limit of a fundamental increasing sequence of ordinals (ordinals of the second kind). Cantor construction of transfinite ordinals, from ω to the ϵ-numbers of the second number class, strongly depends on Theorem K-15. The imposing cantorian edifice was really founded on that theorem. And that theorem, in turn, depends on the hypothetical existence of a complete infinite totality: that of the finite cardinal numbers (Axiom of Infinity in modern terms), which is anything but selfevident.

In modern terms, we say a sequence is ω-ordered if it has a first element and each element has an immediate successor. Similarly, a sequence is ω^*-ordered if it has a last element and each element has an immediate predecessor. Evidently, both type of ordering are intimately related to Zeno's Dichotomies, although surprisingly, the analysis of Zeno's arguments as formal consequences of ω-order remains still undone. For some unknown reasons, it seems we are not interested in analyzing the consistency of the hypothetical existence of actual complete infinite totalities. And this in spite of the enormous problems the actual infinity poses to experimental sciences as physics (recall for example the problems of renormalization in elementary particle physics [34], [47], [36], [90], [37]). The discussion that follows is just oriented in that direction. Its main objective is to analyze Zeno's Dichotomies I and II from the perspective of both ω-order and ω^*-order .

3. The aleph-zero or zero dichotomy[2]

In what follows, and for the sake of clarity, I will consider a canonical version of the famous Achilles' race whose logical impossibility Zeno claimed. In fact, Achilles will be considered as a single point moving rightwards along the X axis, from point -1 to point 1, at a finite velocity v. In the place of the uncountable and densely ordered sequence of points within the real interval $[-1, 1]$ we will only consider the ω^*-ordered sequence of points:

$$\ldots, \frac{1}{2^4}, \frac{1}{2^3}, \frac{1}{2^2}, \frac{1}{2}, 1 \tag{1}$$

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[2]Although the usual way of reading $\aleph_0$ is aleph-null -it can also be read as aleph-zero- the original english translation by P. E. B. Jourdain of Cantor's Beiträge was "aleph-zero". Section 6 is entitled "The Smallest Transfinite Cardinal Number Aleph-Zero. The current English edition of Cantor's Beiträge is from 1955.

all of which Achilles must successively traverse in order to reach point
1 from the starting point -1. In fact, this denumerable sequence of
points (Z^*-points according to classical Vlastos' terminology [82]) is
not densely by successively ordered, which means that between any
two successive Z^*-points no other Z^*-point exists. In consequence, and
at a finite velocity, Z^*-points can only be traversed in a successive
way: one after the other. Assume now Achilles is just on point 0 at
the precise instant t_0. According to classic mechanics he will reach
point 1 just at $t_1 = t_0 + 1/v$. But before reaching his goal, he has
to successively traverse the controversial Z^*-points. We will focus our
attention just on the way Achilles performs such a traversal. For this,
let $f(t)$ be the number of Z^*-points Achilles has traversed at the precise
instant t, being t any instant within the closed real interval $[t_0, t_1]$. It
is quite clear that $f(t_0) = 0$ because at t_0 Achilles is just on point
0. For any other instant t in $[t_0, t_1]$ Achilles has already passed over
countably many Z^*-points, for if there were an instant t in $[t_0, t_1]$ at
which Achilles were passed only over a finite number $n > 0$ of Z^*-points,
these n Z^*-points would have to be the impossible firsts n points of an
ω^*-ordered sequence of points. So we can write:

$$f(t) = \begin{cases} 0 & \text{if } t = t_0 \\ \aleph_0 & \text{if } t_0 < t \leq t_1 \end{cases} \qquad (2)$$

Notice $f(t)$ is well defined for each t in $[t_0, t_1]$. Consequently, f maps
the real interval $[t_0, t_1]$ into the set of two elements $\{0, \aleph_0\}$. In this way
f defines a clair dichotomy, the *aleph-zero or zero dichotomy*, regarding
the numbers of Z^*-points Achilles has traversed when moving rightward
from -1 to 1 along the X axis. Accordingly, with respect to the number
of the traversed Z^*-points, Achilles can only exhibit two states:

   (1) State A_0: Achilles has traversed no Z^*-point.
   (2) State $A_{\aleph_0}$: Achilles has traversed aleph-zero Z^*-points.

Thus, Achilles directly becomes from having traversed no Z^*-point
(state A_0) to having traversed $\aleph_0$ of them (state $A_{\aleph_0}$). Finite inter-
mediate states, as A_n at which he would have traversed only a finite
number n of Z^*-points, simply do no exist. The set of states Achilles
exhibits with respect to the number of traversed Z-points is well defined
and has only two elements, namely A_0 and $A_{\aleph_0}$

   Let us now examine the transition from A_0 to $A_{\aleph_0}$ under the in-
evitable restriction of the above *aleph-zero or zero dichotomy*. The
topological successiveness of Z^*-points makes it impossible to traverse
them other than successively. And taking into account that between
any two successive Z^*-points a finite distance greater than 0 exists, to
traverse $\aleph_0$ Z^*-points -whatever they be- means to traverse a finite dis-
tance greater than 0. This traversal, at the finite Achilles' velocity v,

can only be accomplished by lasting a certain amount of time necessarily greater than 0. Achilles, therefore, has to expend a certain amount of time $\tau > 0$ in becoming $A_{\aleph_0}$ from A_0. The ω^*-ordering imposes this time τ has to be indeterminable, otherwise we would know the precise instant at which Achilles becomes $A_{\aleph_0}$ and, consequently, we would also know the precise Z^*-point point on which he reaches that state, and this is impossibly because in that case a natural number n would have to exist such that $n + 1 = \aleph_0$. The indeterminacy of τ means both the existence of more than one alternative and the impossibility to determine the actual alternative. Now then, indeterminable as it may be, τ must be greater than 0, and this inevitable requirement imposed by Achilles' finite velocity is incompatible with the aleph-zero or zero dichotomy. In fact, let τ be any real number greater than zero and assume the transition from A_0 to $A_{\aleph_0}$ lasts a time τ. Consider the real interval $(0, \tau)$, along this interval Achilles state cannot be neither A_0 nor $A_{\aleph_0}$. It cannot be A_0 because it that were the case the process of becoming $A_{\aleph_0}$ would not have begun, which is false because it is being performed. It cannot be $A_{\aleph_0}$ because in that case the process would have already finished, which is also false for it is still being performed. Now then, Achilles' state has to be either A_0 or $A_{\aleph_0}$ because it is well defined along the real interval $[0, 1]$ of which $(0, \tau)$ is a proper subinterval. Consequently, and being τ any real number, it is impossible for Achilles to become $A_{\aleph_0}$ from A_0 by lasting a time greater than zero. Notice this is not a question of indeterminacy but of impossibility: no real number greater than zero exists for the duration of Achilles' transition from A_0 to $A_{\aleph_0}$. He, therefore, has to become $A_{\aleph_0}$ from A_0 instantaneously. But this is impossible at his finite velocity v. He must, therefore, remain A_0. Or in other words, he cannot begin to move. Evidently, this conclusion is the same absurdity of Zeno's Dichotomy II, although in our case it has been directly derived from the topological successiveness of ω^*-order, which in turn derives from assuming the existence of complete denumerable totalities [19] (Axiom of infinity). To be complete (as the actual infinity requires) and uncompletable (because no last -first- element completes them) could be a contradictory attribute rather than a permissible eccentricity of both ω-ordered and ω^*-ordered sequences.

The above argument is confirmed by the following variant. Let us replace each Z^*-point with a mass Z^*-sensor capable of emitting a visible laser beam when it is activated by any mass passing over it. Assume the system of Z^*-sensors is regulated in such a way that each sensor emits its corresponding laser beam if, and only if, it is activated and no other laser beam is being emitted by other Z^*-sensor of the system. So only one laser beam can be being emitted by the system of Z^*-sensors: the one corresponding to the first activated Z^*-sensor, whatsoever it be. Assume Achilles performs his canonical race from point -1 to point

1. Will any laser beam being emitted at t_1? Evidently not, because it would have to be the impossible first Z^*-sensor of an ω^*-ordered sequence of Z^*-sensors. But on the other hand, why not? Is there any reason to explain the inevitable malfunctioning of Z^*-sensors system other than the inconsistency of assuming that it is possible to begin a sequence of discrete and successive actions without a first action to begin?

Let us now examine Zeno's Dichotomy I under the same canonical conditions of the above Dichotomy II. Consider again the real interval $[-1, 1]$ in the X axis. Let now $\langle z_i \rangle_{i \in \mathbb{N}}$ be the ω-ordered sequence of Z-points:

$$z_i = \frac{2^i - 1}{2^i}, \ \forall i \in \mathbb{N} \tag{3}$$

Achilles has to traverse in his race from point -1 to point 1. Assume also we remove from $[0, 1]$ all points except just Z-points (we would have a sort of *Zeno's powder*). In the place of a continuous race from point -1 to point 1, assume that Achilles is on point 0 just at instant t_0 and then he begins to step to z_1, to z_2, to z_3, ..., so that he is on each z_i just at t_i as a consequence of a Z-jump j_i, being t_i the i-th term of an ω-ordered sequence of instants $\langle t_i \rangle_{i \in \mathbb{N}}$ whose limit is t_b.

The one to one correspondence $f(t_i) = z_i$ proves[3] that at t_b Achilles has completed the ω-ordered sequence of Z-jumps $\langle j_i \rangle_{i \in \mathbb{N}}$ on the ω-ordered sequence of Z-points $\langle z_i \rangle_{i \in \mathbb{N}}$. Thus, at t_b Achilles has to be on a point $x \geq 1$ of the X axis. Otherwise, if he were on a Z-point z_i, only a finite number i of jumps would have been performed. We have now an uncomfortable asymmetry between the ω-ordered sequence of Z-jumps $\langle j_i \rangle_{i \in \mathbb{N}}$ and an $(\omega + 1)$-ordered sequence of points: the ω-ordered sequence of Z-points plus the last point x Achilles ends up his ω-ordered sequence of Z-jumps, i.e. the $(\omega + 1)$-ordered sequence $\langle \langle z_i \rangle_{i \in \mathbb{N}}, x \rangle$.

By definition Achilles is on each z_i at t_i as a consequence of the i-th Z-jump j_i. The one to one correspondence $f(j_i) = z_i$ proves that:

(1) No Z-jump j_i makes Achilles to reach point x.

(2) Achilles comes to point 1 from no Z-point.

But the only actions Achilles performs from t_0 is the ω-ordered sequence of Z-jumps $\langle j_i \rangle_{i \in \mathbb{N}}$ on the ω-ordered sequence of Z-points $\langle z_i \rangle_{i \in \mathbb{N}}$. So, Achilles can only come from a Z-point as a consequence of a Z-jump. How is then possible Achilles reaches point x at t_b if none of the performed Z-jumps places him there? At this point of the discussion, most

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[3]This is the way infinitists pretend to explain how an ω-ordered sequence of actions can be completed: by pairing off two endless sequences, the one of actions the other of instants at which the successive actions are carried out. Thus, in order to end an endless sequence of actions we only need to pair the endless sequence of actions with the endless sequence of instants as which they are performed, as if by pairing off two impossibilities a possibility could result.

infinitists claim that although Achilles comes to point x from no Z-point as a consequence of no Z-jump, it reaches that point at t_b as a consequence of having *completed* the (uncompletable) ω-ordered sequence of Z-jumps $\langle j_i \rangle_{i \in \mathbb{N}}$. As if the completion of the ω-ordered sequence of Z-jumps $\langle j_i \rangle_{i \in \mathbb{N}}$ were a place where one may come from (as well as an additional jump). But if the completion of the ω-ordered sequence of Z-jumps $\langle j_i \rangle_{i \in \mathbb{N}}$ means the completion of the ω-ordered sequence of Z-jumps $\langle s_i \rangle_{i \in \mathbb{N}}$, i.e. that each one of the countably many Z-jumps j_1, j_2, j_3, ..., *and only them,* have been performed, then it is quite clair that Achilles cannot reach point x at t_b. Simply because no Z-jump j_1, j_2, j_3, ... ends on point x. And if no Z-jump j_1, j_2, j_3, ..., end on point x and Achilles only performs Z-jumps, then he cannot end on point x either. On the other hand, if the completion of the ω-ordered sequence of Z-jumps $\langle j_i \rangle_{i \in \mathbb{N}}$ were an additional jump then we would have an (ω + 1)-ordered sequence of jumps rather that an ω-ordered one. But we have proved the ω-ordered sequence of Z-jumps $\langle j_i \rangle_{i \in \mathbb{N}}$ suffices to place Achilles on point x at t_b. It is therefore that ω-ordered sequence $\langle j_i \rangle_{i \in \mathbb{N}}$ which places and does not place Achilles on point x.

Achilles ends his ω-ordered sequence of Z-jumps on point x and this final position is unexplainable because no final jump places him there. And no final jump places him there because no final Z-jumps exists in the ω-ordered sequence of Z-jumps $\langle j_i \rangle_{i \in \mathbb{N}}$. The asymmetry is quite clair: there exists a last effect (to reach point x) but not a last jump causing it. Infinitist, therefore, have to make use of a mysterious last jump by converting the completion of an ω-ordered sequence of jumps in a subsequent additional jump different from all previously performed ones. But an ω-ordered sequence of jumps plus an additional last jump is not an ω-ordered sequence of jumps but an (ω + 1)-ordered one. Thus, this assumed additional jump does not solve the question, because Achilles reaches and does not reaches point x as a consequence of an ω-ordered (not of an (ω + 1)-ordered) sequence of jumps.

As in the case of Dichotomy II, assume that each Z-point z_i is provided with a mass Z-sensor, being the system of Z-sensors regulated in such a way that, once a Z-sensor is activated, it will be emitting its corresponding laser beam until other Z-sensor be activated. In consequence, once the system is activated there will always be a Z-beam being emitted: the one corresponding to the last activated Z-sensor. So, once activated, it is impossible to turn off the emission of Z-beams. Assume now Achilles performs an ω-ordered sequence of Z-jumps on the ω-ordered sequence of Z-sensorized Z-points. For the same reasons above, Achilles completes this ω-ordered sequence of Z-jumps at t_b. And now the question is: will any laser beam being emitted at t_b? According to the functioning of the Z-sensors system, once Achilles activates the first Z-sensor by Z-jumping on the first Z-point, it is impossible to turn off the emission of Z-beams. So, at t_b a Z-beam has to

be being emitted. Although, on the other hand, no Z-beam can be being emitted at t_b because, if Achilles has completed his uncompletable ω-ordered sequence of Z-jumps, that Z-beam would have to be being emitted by the impossible last Z-sensor of the ω-ordered sequence of Z-sensors. Thus, if Achilles has completed the uncompletable ω-ordered sequence of Z-jumps, a laser beam will and will not being emitted by the system of Z-sensors. This seems rather contradictory.

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