Performance Limitation of a Coupled Resonant Optical Waveguide Gyroscope

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***Abstract*—We demonstrate through theoretical analysis that unlike predicted by others, an unbiased coupled resonant optical waveguide (CROW) gyroscope made of N ring resonators has a response to a rotation rate Ω that is proportional to $(N\Omega)^2$, and hence its sensitivity to small rotation rates is vanishingly small. We further establish that when proper phase bias is applied to the CROW gyro, this response becomes proportional to $N\Omega$ and the sensitivity to small rotation rates is then considerably larger. However, even after optimizing the CROW parameters (N and the ring-to-ring coupling coefficient κ), the CROW gyro has about the same sensitivity as a conventional fiber optic gyroscope (FOG) with the same loop loss, detected power, and footprint. This maximum sensitivity is achieved for $N = 1$, i.e., when the CROW gyro resembles a resonant FOG. The only benefit of a CROW gyro is therefore that it requires a much shorter length of fiber, by a factor of about $1/(2\kappa)$, but at the expense of a stringent control of the rings' optical path lengths, as in a resonant FOG. Finally, we show that the slower apparent group velocity of light in a CROW gyro compared to a FOG is unrelated to this shorter length requirement.**

***Index Terms*—Coupled resonant optical waveguide (CROW), fiber optic gyroscope (FOG).**

I. Introduction

RECENT investigations have shown that slow light can have a profound impact on the optical properties of materials and systems. In particular, under certain conditions, the sensitivity of interferometric sensors can be in principle greatly enhanced by interrogating the interferometer with slow light [1]. This intriguing property has a number of physical origins, depending on the nature of the waveguide and of the perturbation applied to it. However, irrespective of the exact origin, the sensitivity of a number of different sensors has been shown to scale like the reciprocal of the group velocity. Since slow light can be characterized by an extremely large group index ($> 10^5$), its use can in principle improve the sensitivity of optical sensors by many orders of magnitude. This prospect has far-reaching implications for many applications.

As part of an on-going investigation of the specific conditions under which this sensitivity does and does not take place, we have carried out a detailed study of one particular sensor for which slow light has been recently claimed to be beneficial, which is a gyroscope based on coupled resonant optical waveguides (CROW).[2] This claim is particularly noteworthy because first the fiber-optic gyroscope (FOG) has been for many years the most successful commercial fiber sensors,[3] and second because developing an optical gyroscope with a sensitivity greater than possible with a conventional FOG is an appealing prospect that would find several important applications.

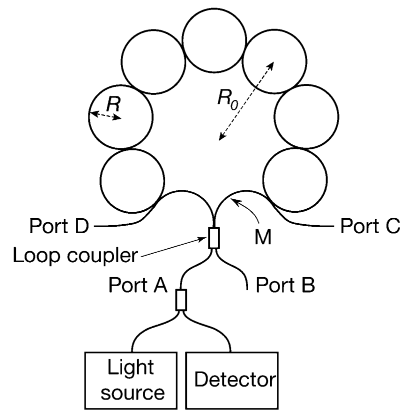

Fig. 1. A CROW gyro with $N = 7$ rings.

A diagram of the CROW gyro is shown in Fig. 1. It consists of a series of N low-loss rings ($N = 7$ in Fig. 1) optically coupled to each other with a power coupling ratio κ to form an open loop. This loop is closed with a 3-dB loop coupler to create a Sagnac interferometer. In the original reference where this sensor was first discussed,[2] an expression was presented for its sensitivity to rotation. This sensitivity was found to vary as $(N+1)^2$ and as $1/\kappa$. The conclusion that was drawn in [2] is that by using a large number of rings and weak coupling, a CROW gyro can be made considerably more sensitive than a conventional FOG. The term "slow light" was invoked in that original publication [2] presumably because the apparent group velocity of the signals traveling through these coupled resonators is greatly reduced [4] as the coupling coefficient κ is varied from 1 to very small values.

In this paper, we are concerned with assessing the best possible rotation sensitivity a CROW gyro can have compared to a conventional FOG, assuming similar gyroscope footprints. We show that although the sensitivity's $(N+1)^2$ dependence predicted in [2] is correct, it is only applicable when no external phase bias is applied between the gyro's counterpropagating signals. However, it is well known that in a conventional FOG with such zero phase bias, the response to first order in rotation rate Ω is zero [5]. Hence the power returning to the detector depends on the next higher (second) order term in the phase difference between the signals, i.e., on the *square* of the rotation rate. This configuration has therefore a very poor sensitivity to small rotation rates. We demonstrate that this is also true in

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a CROW gyro. A simple solution explored in this work is to supply a suitable phase bias to the CROW gyro and hence make the gyro signal proportional to Ω. This approach then raises a number of new questions. Can it be more sensitive than a conventional FOG or a resonant fiber optic gyroscope (RFOG), and under what conditions? If it can be more sensitive, what role plays the slow nature of the light traveling through its loop? We show that when a proper phase bias is added, the signal does become proportional to Ω rather than Ω^2. The CROW gyro is then more sensitive to small rotation rates, but its sensitivity is no longer proportional to $(N+1)^2$, but rather to $N+1$. As a result, the biased CROW gyro turns out to have a sensitivity to rotation comparable to that of a FOG with the same footprint and the same total propagation loss when $N=1$, and increasingly smaller relative sensitivity as N increases. The number of rings that maximizes the relative sensitivity of a CROW gyro is $N=1$, and in this configuration the CROW gyro looks sensibly like an RFOG, although this CROW gyro is operated differently than the RFOG typically is. The conclusion is that compared to a FOG, the CROW gyro has about the same sensitivity, and its only benefit is that it requires a shorter length of waveguide. However, it also has the same disadvantages over a FOG as an RFOG, namely stringent thermal and mechanical path-length stabilization requirements. This significant downside adds great engineering complexity and cost, and it certainly does not outweigh the length advantage. Finally, we demonstrate that unlike implied in the original publication,[2] the apparent lower group velocity of the light traveling through a CROW gyro has no bearing on its sensitivity. Unfortunately, the CROW gyro belongs to the class of sensors that are not enhanced by slow light.

II. Principle of the CROW Gyroscope

As in a conventional FOG, in a CROW gyro light is launched into the loop coupler, which splits it with equal power into two signals (see Fig. 1). One signal travels clockwise (cw) around the Sagnac loop, and the other one counterclockwise (ccw). For simplicity, we assume that the rings all have the same radius R, and that the power coupling coefficient between any two adjacent rings or between the rings and the leads is the same, and equal to κ. The conclusions of the following analysis are qualitatively the same when the dimension and/or the coupling coefficient vary from ring to ring. Note also that the exact shape of the loop has no bearing on the overall behavior of the sensor. For simplicity, we assume throughout that the rings are all centered on a circle of radius R_0, and that the leads connecting the rings to the 3-dB coupler are circular arcs with the same radius of curvature as the rings (see Fig. 1).

To understand the operating principle of the CROW gyro, consider first the limit of strong coupling ($\kappa = 1$). The cw light signal then travels about half way through the first ring to the coupler at the far end of the first ring, where it is *fully* coupled to the second ring. The same process takes place in each subsequent ring, until the signal reaches the 3-dB loop coupler. The light signal has therefore traveled around the loop in a "scalloped" pattern that encompasses a certain area B. The ccw signal undergoes the same process, except that it encounters the rings in the reverse order. So it follows the same optical path, in the opposite direction. Hence as in a conventional FOG, in the absence of rotation and nonreciprocal effects, the two signals experience the same phase shift as they travel around the CROW loop. Consequently, when they recombine at the 3-dB coupler, they interfere constructively into the input arm of the coupler, and all the light (minus what is lost to losses within the loop) is detected at the detector (see Fig. 1). When the loop is rotated at a rate Ω about its main axis (i.e., perpendicular to its plane), as a result of the Sagnac effect light traveling in the direction of the rotation undergoes a stronger phase shift than light traveling in the opposite direction. The two signals are no longer in phase, which translates into a change in the power received by the detector. Except for the scalloped shape of the path followed by the light around the loop, this particular configuration behaves exactly the same way as a conventional FOG. In particular, it has the exact same sensitivity to rotation as a FOG encircling the same area B.

Note that for the cw and ccw signals to return to the loop coupler instead of to port C and port D, respectively, which is undesirable, the number of rings must be odd. By switching the ports of one of the two injection couplers, a similar configuration is created that supports only an even number of rings. Both configurations have the same basic properties.

The CROW gyro starts differing from a FOG and becomes more interesting when $\kappa < 1$, especially when κ is very weak. Each ring in the Sagnac loop is then a high-finesse resonator. The signal frequency is selected to coincide with a resonance frequency of the (identical) rings at rest (zero rotation). When the gyro is at rest, each signal must thus travel multiple times around each ring, one ring after the other, before reaching the far end of the Sagnac loop and interfering with each other. As a result of these multiple passes, each signal accumulates a larger rotation-induced (Sagnac) phase shift in each and every ring than if it were traveling through each ring only once, and the differential phase shift is enhanced. This enhancement is expected to scale like the number of times each signal travels around each ring, i.e., like $1/\kappa$. For weak coupling ($\kappa \ll 1$), the increase in sensitivity can therefore be quite significant, as is the case in a resonant fiber optic gyroscope (RFOG) [5].

For the CROW gyro to work optimally, (1) all the rings must have a common resonance frequency at all times, (2) the frequency of the interrogating light must remain tuned to this common resonance frequency at rest, and (3) the linewidth of the light signal must be substantially narrower than the linewidth of the resonant modes of each resonator. These conditions require very tight control of the optical path length, index, and transverse dimensions of all N rings simultaneously, which constitutes a significant engineering challenge in practice. However, since the goal of this study is to investigate the validity of the claimed superiority of the ultimate theoretical sensitivity of this sensor,[2] and since similar conditions have been successfully met for the RFOG, in the remainder of this paper we assume that these conditions are satisfied.

III. Modeling the Sagnac Phase Shift in a CROW Gyro

To model the sensitivity to rotation of a CROW gyro, we used the transfer-matrix method detailed in [2]. The transfer-matrix method keeps track of the total phase each signal accumulates as it propagates from one coupler to the next inside the CROW.

The total phase consists of both a rotation-independent component and a rotation-dependent component due to the special-relativistic Fresnel drag experienced by a signal in a moving material. [2], [6] The transfer matrix of each element in the gyro (portion of a ring between couplers) was expressed as a function of the ring radius R, the effective index of the ring mode n, the power coupling coefficient between rings κ the rotation rate Ω, and the signal frequency ω. The matrix of the total gyro is then simply the product of these matrices. For a given input electric field coupled into the gyro loop, this final matrix provides a means for calculating the electric fields interfering at the output coupler. From these two fields, the rotation-induced Sagnac phase shift $\Delta\phi$ was then easily determined, for an arbitrarily number of rings N. The sensitivity was then calculated by inserting the Sagnac phase shift in the expression for the basic response of a Sagnac interferometer, which takes into account the phase bias of the interferometer.

The product of the transfer matrices was evaluated by one of two methods. In the first method, we used MATLAB to calculate this product numerically after assigning a value to each parameter. This approach works for any arbitrarily high number of rings.

The second method consisted in multiplying the transfer matrices symbolically using Mathematica, which yielded a closed-form analytical expression for the output electric field versus rotation rate. This approach allowed us to visualize the analytic dependence of the CROW gyro phase shift on the sensor parameters, in particular κ and N, which provides some guidance into the physics of this gyro. For example, for $N = 1$ and $N = 3$ this approach yielded the exact expressions for the electric field of the corotating output signal shown in (1) and (2) at the bottom of the page, where c is the speed of light in vacuum and $F = 2\pi R\omega/c^2$. The corresponding expressions for the counterrotating signal are the same, except that Ω is changed into - Ω. The total electric field at the coupler is the sum of the co- and counterrotating signals.

This analytical approach becomes increasingly unyielding for larger values of N, and it was therefore not pursued for values of N larger than 3. However, in the important limit of small rotation rates ($\Omega \ll \Omega_0/(N+1)$, where $\Omega_0 = \kappa c^2\omega^{-1}R^{-2}$), the phase of the electric fields given in (1)–(2) can be expanded in a Taylor series to first order in $\varepsilon = \Omega/(\Omega_0/(N+1))$. Comparison between these two expansions, one valid for $N = 1$ and the other for $N = 3$, suggests the following expression for the dependence on N (to first order in ε) of the Sagnac phase shift:

$$\Delta\phi_{\mathrm{CROW}} = \frac{4\pi R^2\omega\Omega}{c^2}\left(\frac{N+1}{2\kappa} + \frac{2\cot(\alpha/2)+\alpha-\pi}{\alpha}\right) \tag{3}$$

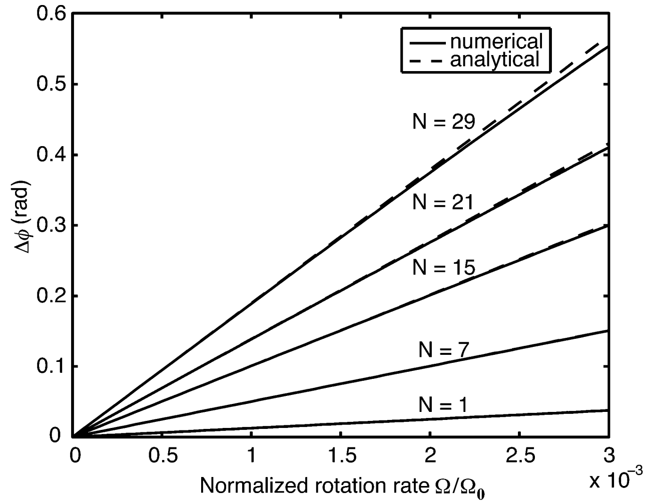

Fig. 2. Comparison of analytical (approximate) and numerical (exact) results for phase delay.

where $\alpha = 2\pi/(N+2)$ is the angle subtended by each ring from the center of the gyro loop. Note that (3) states that the phase delay is independent of the refractive index of the fiber. This is in strict agreement with the well-known fact established by Arditty and Lefèvre in 1981 using relativistic arguments that the response of a gyroscope is independent of the material index.[7]

This expression in (3) is exact for $N = 1$ and $N = 3$. To confirm its expected validity for higher values of N (to first order in ε), we plot in Fig. 2 the approximate phase shift predicted analytically by (3) versus the normalized rotation rate Ω/Ω_0 for $N = 1, 7, 15, 21$, and 29 while keeping R, ω, and κ constant. For comparison, we also plot the exact Sagnac phase shift for the same values of N, calculated numerically using the first (exact) method. Note that for all practical small rotation rates, the ratio Ω/Ω_0 is extremely small, and the agreement between the two models is extremely good. For example, for reasonable CROW parameters ($\kappa = 0.001$, $\lambda = 1.5$ μm, and $R = 1$ cm), $\Omega_0 = 716$ rad/s. For a rotation rate equal to Earth rate (conventional FOGs can routinely detect rotation rates three orders of magnitude smaller than Earth rate), $\Omega/\Omega_0 \approx 10^{-7}$. The maximum ratio Ω/Ω_0 plotted in Fig. 2 (0.003) therefore corresponds to an extremely large rotation rate (30 000 times Earth rate). Fig. 2 thus shows that even up to fairly high rotation rates, (the agreement improving for smaller rates), the agreement between the two models is exceedingly good, even for a large N. This agreement lends credence to the validity of this useful approximate analytical model.

One can investigate a number of useful limiting cases with (3). When $\kappa = 1$, the bracket becomes equal to a geometrical

$$E_{\mathrm{out}}^{\mathrm{co}}(N=1) = \frac{\kappa e^{iF(-cn(\alpha-3\pi)+3(\pi-\alpha)R\Omega-6R_0\Omega\cos(\alpha/2))/(2\pi)}}{e^{iFR\Omega}+(\kappa-1)e^{iFcn}} \tag{1}$$

$$E_{\mathrm{out}}^{\mathrm{co}}(N=3) = \frac{\kappa^2 e^{iF(-cn(\alpha-5\pi)+5R\Omega(\pi-\alpha)-10R_0\Omega\cos(\alpha/2))/(2\pi)}}{e^{i2FR\Omega}+(\kappa-1)(2e^{iF(cn+R\Omega)}-2e^{i2F(cn+R\Omega)}+e^{iF(3cn+R\Omega)}+e^{iF(cn+3R\Omega)}+(\kappa-1)e^{i2Fcn})} \tag{2}$$

form factor which, when multiplied by the πR^2 term in front of the bracket, yields the area B of the scalloped region traced by the signals through the coupled rings (see Fig. 1). Equation (3) then becomes

$$\lim_{\kappa\to 1}\Delta\phi_{\mathrm{CROW}} = \frac{4B\omega\Omega}{c^2} \tag{4}$$

which is exactly the expression of the Sagnac phase shift in a FOG of area B [5]. It shows that in this limit of strong coupling, as described with physical arguments earlier on, the CROW gyro has the same sensitivity as a FOG of same area. This result could not be predicted from the expression provided in [2] because only the term in $1/\kappa$ was retained there.

In the opposite limit of weak κ, for any value of N the second term in the bracket of (3) becomes negligible compared to the first term. The Sagnac phase shift then becomes

$$\lim_{\kappa\to 0}\Delta\phi_{\mathrm{CROW}} = \frac{4\pi R^2\omega\Omega}{c^2}\left(\frac{N+1}{2\kappa}\right). \tag{5}$$

The phase shift is now equal to that of a conventional FOG of area $(N+1)\pi R^2$, but enhanced by factor of $1/(2\kappa)$. This term originates entirely from light resonating around each of the N rings. As expected, it is proportional to area $A = \pi R^2$ of each ring. The effect of the resonant structures is therefore to increase the Sagnac phase shift in inverse proportion to the coupling strength, as in an RFOG. It is interesting to note that the Sagnac phase shift depends on the number of rings as $(N+1)$, instead of the expected N. Although this dependence is predicted by both our models as well as in [2], we do not have a physical explanation for it. However, as we shall see this detail has no impact on the conclusions of this study (if only because of large N the difference becomes vanishingly small).

To summarize, these two limits and the form of (3) indicate that in the CROW gyro two contributions are present: a phase shift independent of κ that depends on the overall loop area B, as in a FOG, and a resonant phase shift proportional to $1/\kappa$ and to the area of each ring A, as in an RFOG. Another important conclusion from (3) is that in order to optimize a CROW gyro with a given footprint, since the resonant term in the Sagnac phase shift is proportional to the second power of R but only the first power of $(N+1)$, it is best to increase the ring radius R, rather than the number of rings N. Hence, *for a given footprint* a CROW gyro has an optimum sensitivity when R is as large as possible, which is when $N = 1$.

We emphasize that the $(N+1)^2/\kappa^2$ dependence of the responsivity pointed out in [2] arises strictly from the choice of phase bias (zero), a point not specifically mentioned in [2]. In a conventional FOG with a zero phase bias, the detected power is proportional to cos $^2(\Delta\phi/2)$. Since $\Delta\phi$ is proportional to N_{FOG}, the number of turns in the sensing coil, the detected power scales like N_{FOG}^2. But it also scales like Ω^2. Consequently, the zero-bias sensitivity exceeds the $\pi/2$-bias sensitivity *only* for large enough rotation rates, a property that has unfortunately little practical utility. On the other hand, for very slow rotations, which is what most high-accuracy applications require, $(N_{\mathrm{FOG}}\Omega)^2 \ll N_{\mathrm{FOG}}\Omega$, and the sensitivity is extremely poor. Our simulations show that the same is true for the CROW gyro. To achieve maximum sensitivity, a nonzero bias point must be chosen.

IV. Numerical Analysis of CROW Gyro With Nonzero Bias

In practice, a CROW gyro could be phase biased in much the same manner as a FOG, namely by placing a phase modulator asymmetrically in the gyro loop, for example at point M in Fig. 1. This modulator would then be driven at the proper frequency of the loop, related to the time it takes either signal to go around the loop once (a complex but quantifiable function of the coupling ratio κ). Studying the effect of such a dynamic biasing scheme on the sensitivity of the CROW gyro would require propagating the two time-dependent counterpropagating signals through the N coupled rings, which would be extremely time-consuming. As a simpler alternative, we chose to bias the interferometer by subjecting it to a fixed rotation rate Ω_b. Although this approach is certainly not practical, it enables us to use the transfer-matrix formalism to quickly yet accurately study the effect of phase biasing on the sensitivity of a CROW gyro. In particular, it can tell us which bias rotation rate Ω_b (and hence which phase bias ϕ_b) maximizes the sensitivity to a perturbation $\delta\Omega$ in rotation rate. The gyro sensitivity is then given by $S = dP/d\Omega$, where P is the power measured by the detector (port A in Fig. 1).

Since in this work we are concerned primarily with comparing the performance of a CROW gyro to that of a conventional FOG, we must be careful not to unduly put the CROW at either an advantage or disadvantage. The three independent design parameters that affect the sensitivity of both types of gyros are (1) the length of waveguide (fiber or otherwise), (2) the waveguide loss per unit length, and (3) the diameter of the sensing loop (which by and large dictates the footprint of the packaged gyro). It would be unfair, for example, to compare a FOG with a 200-m loop coiled in a 10-cm diameter spool to a CROW gyro utilizing 1000 m of fiber spread on a loop of 10-m diameter, as the CROW would then have a much greater scale factor, and hence sensitivity.

So for fair comparison, in the following analysis we applied the following four conditions to the two types of gyros. First, we assumed that the same optical power P_0 is incident on the two gyros (port A in Fig. 1). Second, we assumed that the waveguides forming the rings have the same low loss as the typical conventional single-mode fiber in a gyro (~ 0.2 dB/km at 1550 nm). (FOGs typically use a polarization-maintaining fiber, which has a higher loss, but the exact value of the loss has no bearing on the result of the comparison.)

Third, we imposed that their respective effective lengths are equal to guarantee that signals suffer the same propagation loss when going around the sensing loop. In a conventional FOG, after propagating through N_{FOG} turns of radius R_{FOG} each signal is attenuated by a factor exp $(-\alpha L_{\mathrm{FOG}})$, where $L_{\mathrm{FOG}} = 2\pi N_{\mathrm{FOG}} R_{\mathrm{FOG}}$ is the total sensing loop length and α is the fiber loss. In a CROW gyro, after propagating through N rings of radius R, the signal is attenuated by a factor exp $(-\alpha L_{\mathrm{CROW}})$, where L_{CROW} is approximately equal to N times the length of one ring ($2\pi R$) multiplied by the number of times the signal goes around each loop ($\sim 1/(2\kappa)$, i.e., $L_{\mathrm{CROW}} \approx N\pi R/\kappa$)

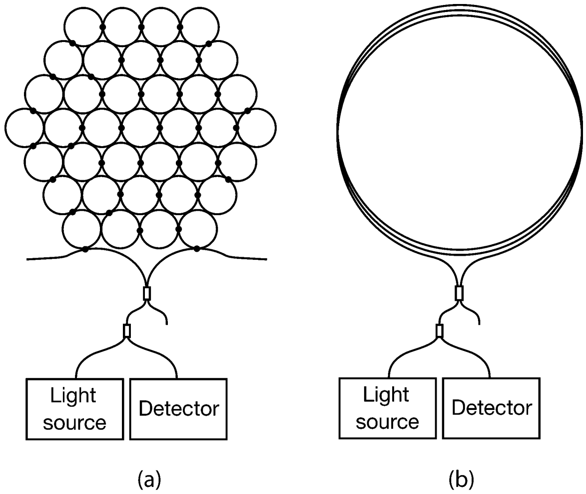

Fig. 3. (a) A CROW gyro with tightly-packed rings (dots show coupling points between rings) and (b) a FOG with equivalent footprint. In (a) dots indicate couplers.

[2]. The requirement for equal gyro losses therefore imposes $L_{\text{CROW}} = L_{\text{FOG}}$, i.e.

$$N_{\text{FOG}} R_{\text{FOG}} = \frac{NR}{2\kappa}. \tag{6a}$$

However, this approximate expression for L_{CROW} is accurate enough only for certain ranges of parameters (specifically, when N is large and κ is not too weak). To keep the comparison between the two gyroscopes as accurate as possible, instead of using this expression we computed L_{CROW} directly by calculating numerically the amount of power P_A exiting port A in the non-rotating gyro. L_{CROW} is then defined by $\exp(-\alpha L_{\text{CROW}}) = P_A/P_0$, where P_0 is the power incident on the first loop. Imposing $L_{\text{CROW}} = L_{\text{FOG}}$ then yielded a condition on N_{FOG} and R_{FOG} slightly different (and more accurate) than (6a).

Finally, the fourth condition is that we must compare two gyros with similar footprints. Reference to Fig. 1 shows that when N is reasonably large (more than a few rings), the footprint of a CROW gyro in which the rings are arranged approximately along a circle is mostly empty. Since in such a gyro the resonant component of the phase sensitivity is independent of the path followed by the rings [2], one can reduce the footprint of a CROW gyro without significantly affecting its sensitivity by arranging the rings along a different path that better utilizes this empty space. For example, as shown in Fig. 3(a) the string of rings can be coiled. As a result, either more rings can be packed in a circle of given radius, or a given number of rings will occupy a smaller footprint. Note than in doing so, the nonresonant component of the CROW's phase sensitivity (second term in (3)) is greatly reduced. However, for any reasonably small value of κ this component is negligible compared to the resonant component, so neglecting it does not unduly disfavor the CROW gyro.

For simplicity, to calculate the total area of a "coiled" CROW gyro of the type shown in Fig. 3(a), we assume that this area is simply equal to the sum of the area of the N rings, i.e., $N_{\text{CROW}}\pi R_{\text{CROW}}^2$. This approximation ignores the area of the interstitial regions between the rings, which artificially reduces the CROW gyro footprint and hence again favors the CROW gyro over the FOG. Note that this approximation is valid not just for the ring-packing arrangement shown in Fig. 3(a), but for any arrangement that fills the CROW gyro footprint fairly well.

Imposing equal footprint for this "coiled" CROW gyro and for the conventional FOG to which we compare it (see Fig. 3(b)) then yields the fourth condition:

$$N_{\text{CROW}}\pi R_{\text{CROW}}^2 = \pi R_{\text{FOG}}^2 \tag{6b}$$

In general, the Sagnac phase shift is proportional to the area around which light has traveled. For the conventional FOG, this phase shift is determined completely by the fiber loss as well as the device footprint. For a CROW gyro, the maximum distance that light can travel is still limited by the fiber loss. Thus, for a same device footprint, the maximal phase shift that a CROW gyro can have should be similar to the corresponding conventional FOG. Intuitively therefore, one should not expect an enhancement of absolute sensitivity in the CROW gyro system.

Based on the foregoing, we simulated numerically using the transfer matrix formalism the responsivity S versus phase bias of a few CROW gyros. These gyros all have the same coupling ratio ($\kappa = 0.001$), ring radius ($R = 5$ cm), and loss coefficient (0.2 dB/km) but different numbers of rings N. For each of these CROW gyros, we compared this calculated sensitivity to that of the "equivalent" FOG, namely the FOG with a sensing coil of N_{FOG} turns and radius R_{FOG} calculated from κN, and R by imposing $L_{\text{CROW}} = L_{\text{FOG}}$ and using (6b).

In the CROW gyro, as in a simple ring resonator, there exists a critical coupling value κ_{crit} which maximizes the power circulating in the individual rings. However, unlike in a simple ring resonator, the critical coupling value is not simply given by the single-pass resonator loss. Our simulations show that (1) the sensitivity of the CROW gyro is maximum for a κ value greater than κ_{crit}, and (2), no value of κ makes the CROW gyro more sensitive than the equivalent FOG. Hence, for simplicity and without loss of generality, in the following we investigate only the case of coupling greater than critical coupling.

The CROW gyro suffers from a power-loss mechanism not present in a conventional FOG, namely when its sensing loop is rotated, some of the power exits the loop at ports C and D (see Fig. 1). This is because the individual rings are on resonance when the loop is stationary. Under rotation, the rings are no longer on resonance, and these two ports transmit some (and equal) amount of power.

To illustrate this point, we plot in Fig. 4 the power exiting each port, normalized to the input power P_0, as a function of rotation rate for a CROW with the parameters cited above and $N = 1$. At $\Omega = 0$, the ring is probed on resonance and no power exits from ports C and D. Also, as in a classical FOG, all the power returns into port A, and none in the nonreciprocal port (port B). As Ω is increased, the power in port A decreases while the power in port B increases, as in a conventional fiber gyro. However, as the cw and ccw signals both slip increasingly off resonance, some of the power exits at ports C and D. This power leakage increases with Ω, until it is strong enough that the power in the non-reciprocal port B eventually decreases (see Fig. 4). At

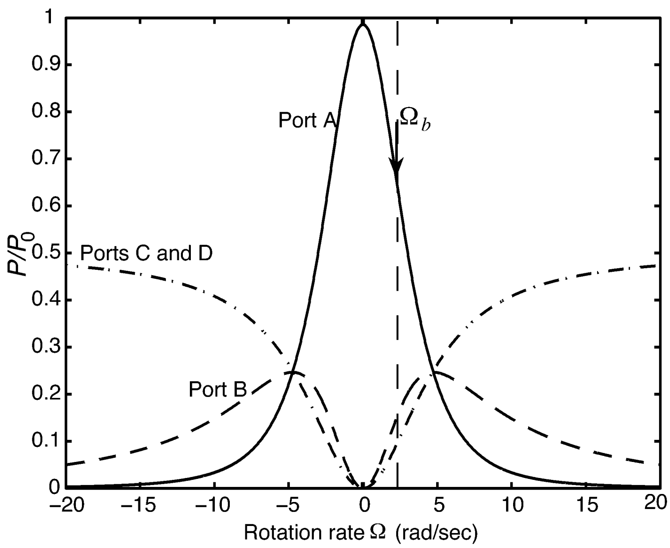

Fig. 4. Evolution of output power in the four ports of a CROW gyro with $N = 1$ ring.

large enough rotation rates, the power in port C (and D) reaches a plateau. For some extremely large Ω, well beyond the range of values covered by the figure, the next resonance frequency of the loop approaches the light frequency, and the same process takes place again, in reverse: power drains back out of ports C and D, until at this new resonance all the power is in port A. This figure shows that at some rotation rate, identified as Ω_b in Fig. 4, the dependence of the power in port A on Ω, and hence the sensitivity of the CROW gyro, is maximum. This confirms the existence of a phase bias that maximizes the response of a CROW gyro.

Fig. 5(a) shows the sensitivity computed for the same CROW gyro with $N = 1$, as well as the sensitivity of the equivalent conventional FOG, again interrogated with the same incident power. As expected from the foregoing discussion, the CROW sensitivity is maximum at Ω_b and it decreases on either side of this optimum value. The phase bias corresponding to this bias rotation rate is $\phi_b \approx 0.84$ rad. This value depends weakly on the strength of the coupling between rings. As κ is reduced from the value used in this example ($\kappa = 0.001$), the resonances narrow and hence Ω_b decreases. However, light also travels more times around each ring, so the actual differential phase shift due to Ω_b increases. Simulations shows that these two dependences cancel each other, as expected, so the optimum phase bias ϕ_b resulting from this Ω_b is essentially independent of coupling strength. It also depends weakly on the number of rings and on the ring radius. Fig. 5(a) also shows that when the CROW gyro is operated at its optimum bias ($\Omega_b \approx 2.1$ rad/s) its sensitivity is essentially the same (within 3%) as that of the equivalent FOG. The small difference in sensitivity is due to the different shapes of the transmission functions for the CROW and FOG, and it cannot be significantly enhanced by changing the CROW parameters.

Fig. 5(b) and (c) show the same curves for $N = 9$ and $N = 81$, respectively. As N increases, the maximum sensitivity of the equivalent FOG gyro increases. The reason is that the product

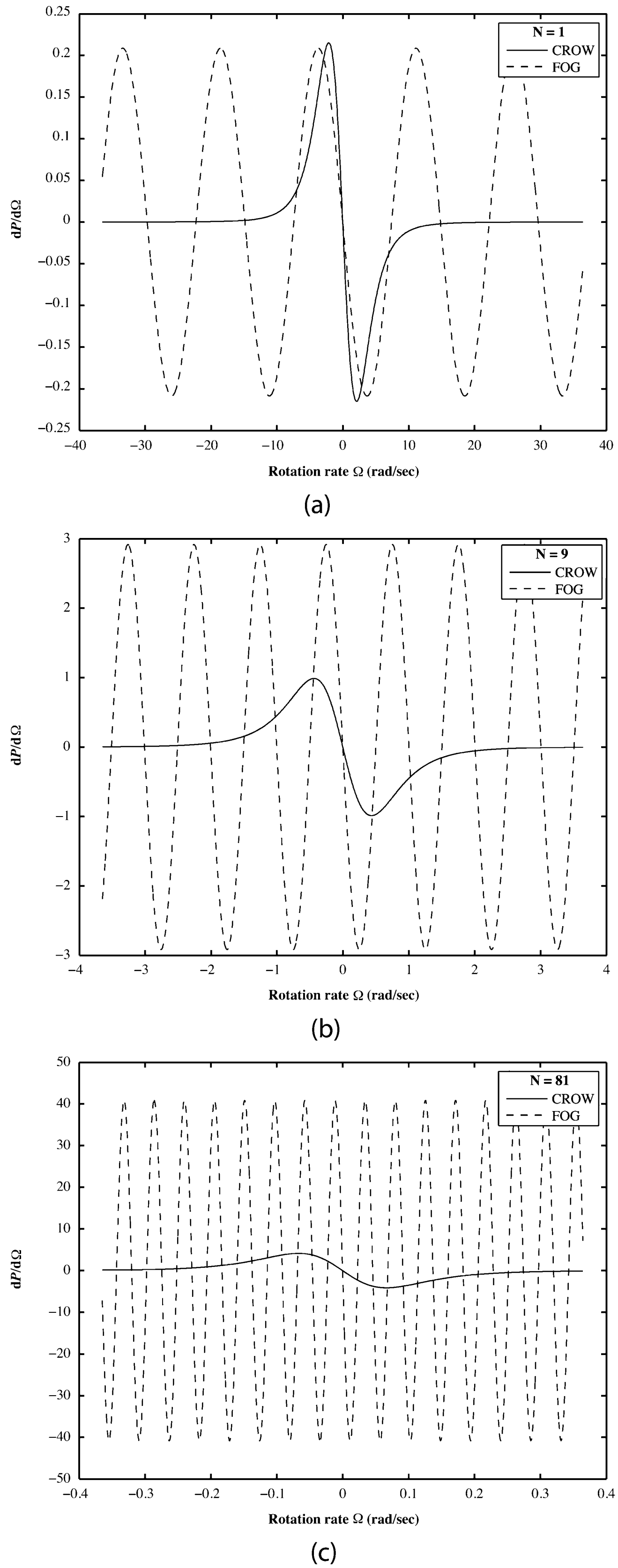

Fig. 5. Sensitivity of CROW gyro and equivalent FOG as a function of rotation rate for $N = 1, 9$, and 81.

$R_{\mathrm{FOG}} N_{\mathrm{FOG}}$ increases with N (see (6a)), hence the scale factor of the equivalent FOG, which relates the Sagnac phase shift (or S) to Ω and which is proportional to $R_{\mathrm{FOG}}^2 N_{\mathrm{FOG}}$,[5] increases. The sensitivity of the CROW gyro also increases with N, but it does so more slowly than the FOG, so the sensitivity of the CROW gyro relative to the FOG decreases. This is consistent with our earlier prediction that the sensitivity of the CROW gyro compares most favorably to that of the equivalent FOG for $N = 1$. Simulations show that this ratio of sensitivities is approximately constant for κ between $\sim 10^{-4}$ and $\sim 10^{-1}$. For $\kappa < 10^{-4}$ or $\kappa > 0.1$. this ratio decreases. As stated above, the ratio of maximum sensitivities (i.e., at the proper bias point) cannot be significantly enhanced by changing κ. For a given N, even when the coupling coefficient κ is optimized, the maximum sensitivity of the CROW gyro is never greater than the sensitivity of the equivalent FOG.

It is interesting to note that we could have approximately predicted the dependence of the ratio of FOG to CROW sensitivity from basic principles, using (3). In general, the maximum sensitivity of a gyroscope is proportional to the effective area it covers. For a FOG, this is simply $N_{\mathrm{FOG}}\pi R_{\mathrm{FOG}}^2$. From (3), we can see that the effective area covered by the CROW rings is approximately $N_{\mathrm{CROW}}\pi R_{\mathrm{CROW}}^2/(2\kappa)$ for large values of N. We ignore the first term in (3) here since it is the nonresonant term that depends on the overall path traced by the CROW. Using these definitions of effective area, as well as (6a), it is trivial to show that for the same loss, the equivalent FOG has an effective area (and hence a maximum sensitivity) $N_{\mathrm{CROW}}^{1/2}$ times greater than the CROW. Fig. 5 shows that this approximate expression works quite well. Intuitively, the FOG is more sensitive than the equivalent CROW because a signal traveling along a large loop (as in the FOG) accumulates more rotation-induced phase per unit length than a signal traveling along a small loop (as in the CROW).

These conclusions were drawn for a particular example, but they are independent of the choice of parameter values. The overall conclusion is that the CROW gyro offers no significant enhancement in small-rotation sensitivity compared to a conventional FOG.

V. Connection with Slow Light

The above analysis suggests that although the light travels through a CROW gyro with an apparent group velocity that is lower than in a non-resonant waveguide, slow light plays no role in the enhanced response of a CROW gyro. To illustrate this important point, consider the behavior of the CROW gyro shown in Fig. 1 when the radius of each ring approaches zero while keeping the overall area B covered by the loop of rings constant. In this case, the number of rings increases indefinitely, the total area covered by the individual ring resonators goes to zero, and the sensing loop converges to a circle of constant area B. It is then easy to show from (3) (and confirm with exact simulations) that the differential phase shift approaches $4B\omega\Omega/c^2$, which is precisely the differential phase shift of a FOG of area B. Yet in this CROW, the apparent group velocity of the light is much slower than in this FOG, in which light travels with a "normal" group velocity. Specifically, it is easy to show that this group velocity is roughly $c\kappa/(\pi n)$, independently of N. The two gyros have very different apparent group velocity yet the same phase sensitivity to rotation.

As was pointed out by Shahriar *et al.* [8] for this structure and a similar gyro proposed by Matsko *et al.*, [9], [10] the apparent group-velocity dependence of the sensitivity is more physically understood by the dependence of the sensitivity on the resonator finesse. The important point is that, just as in a conventional RFOG, the resonant part of the CROW gyro phase sensitivity depends on the total area covered by the ring resonators that make up the CROW and the number of times each signal travels around each ring (which is intimately connected to the finesse). The fact that for a CROW gyro (and the gyro considered by Matsko *et al.*) high finesse and low group velocity both occur when κ is small in no way implies that it is the slowness of the light that gives rise to the sensitivity of the gyro. We shall argue that other published slow light gyro structures, [11] in spite of many very interesting motion-induced effects on light propagation, in fact do not enhance the sensitivity for rotation detection either.

VI. Conclusion

The dependence of the sensitivity of a CROW gyro on the square of the number of rings reported earlier for a CROW gyro is applicable only when the gyro is biased at zero phase, in which case the sensitivity to small rotation rates is exceedingly small. To restore good sensitivity at low rotation rates, the CROW gyro must be biased. We showed using a transfer-matrix formalism that in this case, the differential Sagnac phase shift increases as $1/\kappa$, where κ is the coupling coefficient between rings. We derived a simple and useful expression for this phase shift as a function of the size and number of the rings and the coupling coefficient. Simulations show that the sensitivity of a CROW gyro relative to a conventional FOG with the same loop loss, detected power, and footprint is maximum with $N = 1$ ring, i.e., when the CROW gyro looks like a resonant FOG. The CROW gyro is therefore never significantly more (a few percent) sensitive to small rotation rates than a conventional FOG. Its only benefit is therefore that it requires a much shorter length of fiber (or waveguide), by a factor of about $1/(2\kappa)$. Thus, just like an RFOG, a CROW gyro may be superior for applications where reduced weight and/or volume are really critical. But this advantage comes at the expense of requiring a stringent control of the optical paths of a large number of high-finesse optical resonators, which would be extremely difficult to achieve in practice. Finally, we showed based on this analysis that although light propagates with a much slower apparent velocity in a CROW gyro than in a FOG, this slow character of the light is not responsible for the shorter length requirement.

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